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Anderson model with decaying randomness existence of extended states

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Abstract. We study the Anderson model with decaying randomness in $v \geqslant 3$ dimensions and show that there is absolutely continuous spectrum in $[-2v, 2v]$. The distribution of the potentials is assumed to have finite variance and the coupling constants decay at infinity at a rate $\alpha > 1$.

Keywords. Anderson model; decaying randomness; extended states.

1. Introduction

In the last two decades scattering theory of finite particle quantum systems has been well developed and as an offshoot several techniques were developed for the study of spectral properties of self-adjoint operators. Trace class theory is one such tool that was effectively used in a beautiful way by Simon and Spencer [18] recently to show the absence of absolute continuous spectrum for unbounded potentials in one-dimensional discrete systems.

The spectral properties of the Anderson-like systems have proved elusive in higher dimensions in the low disorder regime. While the expected results are that for small disorder there is some absolute continuous spectrum near zero energy and pure point spectrum elsewhere, only the pure point nature of the spectrum was proved in the last few years, after a powerful estimate on the Green function was obtained by Fröhlich and Spencer [9], Simon and Wolff [19], Fröhlich *et al* [8], Delyon *et al* [6] and Carmona *et al* [3]. The only higher dimensional result is from Kunz and Souillard [10] who exhibited Anderson – Mott transition on the Bethe lattice, which in some sense represents the ∞-dimensional case.

In the one-dimensional model rich theory exists. There are excellent reviews of these results in Cycone *et al* [4], Carmona [2] and Spencer [20] and we refer to them for a complete set of references for the one-dimensional theory. The consequences arising from the presence of absolute continuous spectrum was investigated by Kotani [11] in continuous models and Simon [16] for the discrete case. On the other hand Simon [17] and Delyon *et al* [7] studies the spectral properties for various decaying random potentials in one-dimension. They obtain various spectral types along with information on the decay rate of eigenfunctions. Similar studies were done by Kotani and Ushiroya [12] for the continuous models. In addition to these results Deift and Simon [5] study the absolutely continuous spectrum for some almost periodic potentials and Claude–Allen Pillet [13] studied the absolute continuous spectrum for time dependent potentials in the discrete and continuous cases for $\nu \geqslant 3$ dimensions and showed asymptotic completeness for a class of potentials.

In this paper we study the time evolution of regions in the configuration space $l^2(\mathbb{Z}^\nu)$, $\nu \geqslant 3$ and show the presence of absolute continuous spectrum in the Anderson-like models with the *coupling constants decaying at infinity at a rate bigger than* 1. Our strategy in this paper is to exploit the fact, well known in scattering theory of finite particle quantum systems, that existence of wave operators $\Omega(A, B)$ for two self-adjoint operators A and B will show that $\sigma_{ac}(B) \subset \sigma_{ac}(A)$, by the intertwinning property of the wave operators. Our paper consists mainly of two sections, in both of them the coupling constants are assumed to decay at infinity at a rate $\alpha > 1$. In the first we present a proof for potentials with bounded support where we do not really assume anything about the randomness in the potentials (other than that the support of their distribution is bounded). In the next section we consider independent random variables and allow their distribution to have unbounded support but finite variance.

Throughout this paper we follow the convention that if $n \in \mathbb{Z}^\nu$, then $|n|$ denotes the norm in $\mathbb{Z}^\nu$ namely, $|n| = (n_1^2 + \cdots + n_\nu^2)^{1/2}$. For a subset S of $\mathbb{R}^\nu$ we denote by $F(S)$ the orthogonal projection in $l^2(\mathbb{Z}^\nu)$ corresponding to $S \cap \mathbb{Z}^\nu$.

2. Bounded potentials

We consider the operator $H^\omega = \Delta + V^\omega$ on $l^2(\mathbb{Z}^\nu)$, $\nu \geqslant 3$, where we define the discrete Laplacian Δ by

$$(\Delta u)(n) = \sum_{|i-n|=1} u(i). \tag{1}$$

Then the operator Δ is bounded self-adjoint and has purely absolutely continuous spectrum $[-2\nu, 2\nu]$. We consider the potentials V^ω as multiplication operators and the corresponding functions are required to satisfy the following condition, where $q^\omega(n)$ is bounded measurable real valued function, on some probability space, $(\Omega, \mathfrak{B}, d\mu)$ for each $n \in \mathbb{Z}^\nu$.

$$V^\omega(n) = a_n q^\omega(n), \quad \text{with } |a_n| \leqslant \frac{\text{const.}}{(1+|n|)^\alpha}, \alpha > 1 \text{ as } |n| \to \infty. \tag{2}$$

Under the above conditions on the potentials, H^ω will be a bounded self-adjoint operator on $l^2(\mathbb{Z}^\nu)$. For the following theorem we note that by Stone's theorem the one parameter group $\exp(iAs)$, generated by a self-adjoint operator A, is strongly differentiable in s.

Theorem 2.1. *Let $\nu \geqslant 3$ and let $V^\omega(n)$ be of the form given in* (2). *Then the wave operators*

$$\Omega_+ = \lim_{t \to +\infty} \exp(iH^\omega t)\exp(-i\Delta t) \text{ exist } \forall \omega \in \Omega.$$

Proof. To prove the theorem it suffices to show that $\exp(iH^\omega t)\exp(-i\Delta t)f$ *is strongly cauchy for any* $f \in l^2(\mathbb{Z}^\nu)$. Since $\{\delta_n, n \in \mathbb{Z}^\nu\}$ forms an orthonormal basis for $l^2(\mathbb{Z}^\nu)$, finite linear combinations of these vectors are dense in $l^2(\mathbb{Z}^\nu)$, therefore by density arguments without loss of generality we may take $f = \delta_k$. We will further take $f = \delta_0$, since we are going to be interested in large time behaviour in the following, it will be clear that no generality will be lost by doing so. Then by the fundamental theorem of calculus,

$$\begin{aligned}
&\| \exp(iH^\omega t)\exp(-i\Delta t) - \exp(iH^\omega u)\exp(-i\Delta u)\delta_0 \| \\
&= \left\| \int_u^t \mathrm{d}s \exp(iH^\omega s)(iH^\omega - i\Delta)\exp(-is\Delta)\delta_0 \right\| \\
&\leqslant \int_u^t \mathrm{d}s \left\| \exp(iH^\omega s)\, V^\omega \exp(-i\Delta s)\delta_0 \right\|. \qquad (3)
\end{aligned}$$

Now,

$$\begin{aligned}
\| \exp(iH^\omega s)\, V^\omega \exp(-i\Delta s)\delta_0 \| \leqslant{} & \| V^\omega F(|n| \geqslant \beta s)\exp(-i\Delta s)\delta_0 \| \\
& + \| V^\omega F(|n| \leqslant \beta s)\exp(-i\Delta s)\delta_0 \|
\end{aligned}$$

for some $\beta > 0$ sufficiently small compared to 1. Our conditions (2) on the potentials immediately show that,

$$\| V^\omega F(|n| \geqslant \beta s)\exp(-i\Delta s)\delta_0 \| \leqslant \frac{C}{(1+s)^\alpha} \text{ independent of } \omega \qquad (4)$$

as long as β is strictly positive. Now,

$$\begin{aligned}
&\| V^\omega F(|n| \leqslant \beta s)\exp(-i\Delta s)\delta_0 \|^2 \qquad (5)\\
&= \sum_{|n| \leqslant \beta s} V^\omega(n)^2 \overline{\langle \delta_n, \exp(-i\Delta s)\delta_0 \rangle} \langle \delta_n, \exp(-i\Delta s)\delta_0 \rangle.
\end{aligned}$$

Clearly by Proposition A.4

$$\langle \delta_n, \exp(-i\Delta s)\delta_0 \rangle = \prod_{i=1}^{\nu} (-i)^{n_i}(J_{n_i}(2s)) = \prod_{i=1}^{\nu} (i)^{n_i}(J_{-n_i}(2s)),$$

$n = (n_1, \ldots, n_\nu)$, which gives us the estimate

$$\begin{aligned}
&\| V^\omega F(|n| \leqslant \beta s)\exp(-i\Delta s)\delta_0 \|^2 \\
&\leqslant \sum_{i=1}^{\nu} C_1 \sum_{0 < n_i \leqslant \beta s} \frac{C}{(1+|n|)^{2\alpha}} \prod_{i=1}^{\nu} |J_{n_i}(s)|^2, \ \alpha > 1.
\end{aligned}$$

Now using Proposition A.3 we see that

$$\| V^\omega F(|n| \leqslant \beta s)\exp(-i\Delta s)\delta_0 \|^2 \leqslant \frac{C}{(1+s)^{2+\varepsilon}}\{1 + 0(1)\} \quad \text{for some } \varepsilon > 0.$$

Once we have this estimate, using inequalities (3), (4) and (5) we can immediately conclude that

$$\lim_{u,t \to \infty} \| (\exp(itH^\omega)\exp(-i\Delta t) - \exp(iuH^\omega)\exp(-i\Delta u))\delta_0 \| \to 0.$$

#

Theorem 2.2. *Let $H^\omega = \Delta + V^\omega$, with V^ω satisfying condition* (2). *Then,* $\sigma_{ac}(H^\omega) = [-2\nu, 2\nu]$ *for each* ω.

Proof. Since the wave operator exists for the (H^ω, Δ) pair by Theorem 2.1, applying Theorem A.1 shows that

$$\sigma_{ac}(H^\omega) \supset \sigma_{ac}(\Delta) = [-2\nu, 2\nu].$$

On the other hand since $\forall n$, $V^\omega(n)$ is bounded, and goes to zero as $|n| \to \infty$, V^ω is a compact operator on $l^2(\mathbb{Z}^\nu)$. Hence H^ω has the same essential spectrum as Δ by Weyl criterion (see Reed and Simon [14]). Therefore $[-2\nu, 2\nu] = \sigma_{ess}(H^\omega)$. But $\sigma_{ess}(H^\omega) \supset \sigma_{ac}(H^\omega)$, hence proving the theorem. #

3. Potentials with finite variance

In this section we consider potentials V^ω acting as multiplication by the functions $V^\omega(n) = a_n q^\omega(n)$, which are independent random variables satisfying the following conditions, $\{a_n\}_{n\in\mathbb{Z}^\nu}$ is a real valued sequence such that

$$|a_n| \leq \frac{C}{(1+|n|)^\alpha}, \quad \alpha > 1 \tag{6}$$

and $\{q^\omega(n)\}_{n\in\mathbb{Z}^\nu}$ are i.i.d. random variables, distributed according to a probability measure $d\mu$ satisfying

$$\int x \, d\mu(x) = 0 \quad \text{and} \quad \int x^2 \, d\mu(x) = \sigma^2 < +\infty. \tag{7a}$$

For a special case, we also consider $d\mu$ satisfying

$$\int_{\mathbb{R}} |x|^p \, d\mu < +\infty \quad \text{for } p > 2. \tag{7b}$$

In the following we consider potentials V^ω given as multiplication by $V^\omega(n)$ on $l^2(\mathbb{Z}^\nu)$. Then the operator $H^\omega = \Delta + V^\omega$ is self-adjoint on $l^2(\mathbb{Z}^\nu)$ with the set of finite linear combinations of $\{\delta_k\}$, $k\in\mathbb{Z}^\nu$ forming a core for the operator.

Remark 3.1. At this stage let us say that the conditions (7a) would suffice for existence of absolutely continuous spectrum. We take the later condition to say more about the spectrum when the distribution has higher order moments (as in the case of Gaussian for example). We also note here that $d\mu$ need not have a density with respect to the Lebesgue measure and is allowed to be sum of point measures in which case the integral in the definition of the mean etc., (in conditions (7a) and (7b) is replaced by a sum.

Before we present the main theorem of this section we state and prove the following technical lemma.

Lemma 3.2. *For any $t \in \mathbb{R}^+$ and $k \in \mathbb{Z}^\nu$, both fixed,*

$$V^\omega \exp(-i\Delta t)\delta_k \in l^2(\mathbb{Z}^\nu) \text{ a.e. } \omega$$

whenever V^ω satisfies condition (7a)

Proof. Let us recall that $(V^\omega \delta_m) = V^\omega(m)\delta_m \forall m \in \mathbb{Z}^\nu$ and $V^\omega(m) = a_m q^\omega(m)$, $\{q^\omega(m)\}_{m\in\mathbb{Z}^\nu}$ i.i.d. random variables. So consider the events

$$\Delta_m = \{\omega : |q^\omega(m)| > (1+|m|)^{2(\nu+\varepsilon)}\},\ m \in \mathbb{Z}^\nu.$$

Then we have by Cauchy–Schwarz inequality,

$$\text{Prob}(\Delta_m) = \int_{\Delta_m} \frac{1}{x} \cdot x \, d\mu(x) \leqslant \frac{\sigma}{(1+|m|)^{\nu+\varepsilon}}, \forall m \in \mathbb{Z}^\nu.$$

Hence,

$$\sum_{m\in\mathbb{Z}^\nu} \text{Prob}(\Delta_m) \leqslant \sum_{m\in\mathbb{Z}^\nu} \frac{\sigma^2}{(1+|m|)^{\nu+\varepsilon}} < +\infty.$$

Therefore by Borel–Cantelli lemma only finitely many events Δ_m can occur. Therefore $\Omega_0 = \Omega \backslash (\cup \Delta_m)$ has full measure. Now, consider the set $S \subset l^2(\mathbb{Z}^\nu)$ of finite linear linear combination of $\delta_n, n \in \mathbb{Z}^\nu$. S is a dense linear space in $l^2(\mathbb{Z}^\nu)$ and on this set V^ω is well defined, so for $f \in S$, let us look at

$$\sup_{\{f\in S, \|f\|=1\}} |\langle V^\omega f, \exp(-it\Delta)\delta_k \rangle|^2 \quad \omega \in \Omega_0.$$

If this is finite then the lemma follows. To show this let us take $f = \Sigma \alpha_i \delta_i$, where the sum ranges over a finite set in $\mathbb{Z}^\nu$. Then we have,

$$|\langle V^\omega f, \exp(-it\Delta)\delta_k \rangle|^2 \leqslant \left| \sum \alpha_i \langle V^\omega \delta_i, \exp(-it\Delta)\delta_k \rangle \right|^2$$

By Cauchy–Schwarz inequality we have,

$$\begin{aligned} |\langle V^\omega f, \exp(-it\Delta)\delta_k \rangle|^2 &\leqslant \left(\sum \alpha_i^2 \right) \left(\sum_{i\in\mathbb{Z}^\nu} |V^\omega(i)|^2 |\langle \delta_i, \exp(-it\Delta)\delta_k \rangle|^2 \right) \\ &= \|f\|^2 \left[\left(\sum_{i\in S^\omega} |V^\omega(i)|^2 |\langle \delta_i, \exp(-it\Delta)\delta_k \rangle|^2 \right) \right. \\ &\quad \left. + \left(\sum_{i\notin S^\omega} |V^\omega(i)|^2 |\langle \delta_i, \exp(-it\Delta)\delta_k \rangle|^2 \right) \right]. \end{aligned} \tag{8}$$

Since S^ω is a finite set in $\mathbb{Z}^\nu$, the first term in the above inequality is bounded by some finite C_ω for each $\omega \in \Omega_0$. The second term is bounded by

$$\sum_{i\in\mathbb{Z}^\nu\backslash S^\omega} (1+|i|)^{\nu+\varepsilon} \quad |\langle \delta_i, \exp(-it\Delta)\delta_k \rangle|^2,$$

which is finite for each fixed t and k, since there is exponential decay in i for $|\langle\delta_i, \exp(-it\Delta)\delta_k\rangle|$, as $|i| \to \infty$, by proposition A.2. #

Lemma 3.3. *Consider a $k \in \mathbb{Z}^\nu$ fixed. Then, we have*

$$\int \| V^\omega \exp(-i\Delta t)\delta_k \| \, dt < \infty \text{ a.e. } \omega,$$

whenever V^ω satisfies condition (7a).

Proof. Let us consider the random variable,

$$\| V^\omega \exp(-i\Delta t)\delta_k \|^2$$

which is well defined a.e. ω by Lemma 3.2 and note that it suffices to by Fubini and Cauchy–Schwarz inequality that

$$\mathbb{E}\int dt \{ \| V^\omega \exp(-i\Delta t)\delta_k \| \} \leqslant \int dt \, \mathbb{E}\{ \| V^\omega \exp(-i\Delta t)\delta_k \|^2 \} < \infty,$$

to get the almost everywhere statement mentioned above. Now by Fubini we have,

$$\mathbb{E}\{ \| V^\omega \exp(-i\Delta t)\delta_k \|^2 \} = \| V_\sigma \exp(-i\Delta t)\, \delta_k \|^2,$$

where V_σ is bounded operator on $l^2(\mathbb{Z}^\nu)$ given by,

$$(V_\sigma u)(m) = a_m \{ \mathbb{E}(q^\omega(m))^2 \}^{1/2} u(m),$$

and a_m is as in condition (6). Now the required estimate is obvious by proceeding as in the proof of Theorem (2.1). #

We now state the main theorem of this section, namely.

Theorem 3.4. *Consider the operator $H^\omega = \Delta + V^\omega$ on $l^2(\mathbb{Z}^\nu)$, $\nu \geqslant 3$. Then the following are valid.*

(i) *If V^ω satisfies condition* (7a), *then $[-2\nu, 2\nu] \subseteq \sigma_{ac}(H^\omega)$.*
(ii) *If V^ω satisfies condition* (7b) *in addition to* (7a) *with $p\alpha > \nu$, then*

$$\sigma_{ac}(H^\omega) = [-2\nu, 2\nu]$$

Proof.

(i) The proof of this theorem follows from Proposition A.1, if we show that the strong limits,

$$\operatorname*{slim}_{t\to\infty} \exp(itH^\omega)\exp(-it\Delta) \text{ exist.}$$

To show this we need only show that, (as in Theorem 2.1)

$$\lim_{u,t\to\infty} \| (\exp(itH^\omega)\exp(-it\Delta) - \exp(iuH^\omega)\exp(-iu\Delta))\delta_0 \| \to 0.$$

Now by the fundamental theorem of calculus and Lemma 3.2 we have the inequality,

$$\|(\exp(itH^{\omega})\exp(-it\Delta)-\exp(iuH^{\omega})\exp(-iu\Delta))\delta_0\| \leqslant \int_u^t \mathrm{d}s \,\|V^{\omega}\exp(-is\Delta)\delta_0\| \quad \text{a.e.}\, \omega$$

from which using Lemma 3.3 the result follows.

(ii) If $\alpha p > \nu$, then we already know by (i) $\sigma_{ac}(H^{\omega}) \supset [-2\nu, 2\nu]$. Now consider the sequence $|V^{\omega}(n)|^p, n \in \mathbb{Z}^{\nu}$. This sequence goes to zero as $|n| \to \infty$ almost everywhere, since

$$\mathbb{E}\left\{\sum |V^{\omega}(n)|^p\right\} \leqslant \sum \mathbb{E}\{|V^{\omega}(n)|^p\} \leqslant \sum \frac{\mathbb{E}\{|q^{\omega}(n)|^p\}}{(1+|n|)^{\alpha p}} < \infty$$

by conditions (7b). Hence the sequence $V^{\omega}(n)$ also goes to zero almost everywhere as $|n| \to \infty$, showing that V^{ω} is a compact operator almost every ω. The rest of the result then follows from Weyl's theorem. #

Remark 3.5. In case (*i*) of the above theorem we cannot say what happens outside $[-2\nu, 2\nu]$ at the moment. We note here that according to Simon's analysis in [17] where he explains Pearson's intuition, the quantity $\Sigma\, a_n^2$ being finite or not plays a role in the presence of continuous spectrum in one dimension. We note here that we actually have $\Sigma\, a_n^2 = \infty$. We also note that the condition (7a) does not mean that $V^{\omega}(n) \sim O(a_n)$ a.e. as $|n| \to \infty$.

4. Conclusion

In this paper we have presented a time dependent method for studying the absolutely continuous spectrum of some Anderson models in higher dimensions. The proofs show that it is sufficient to study the free evolution of the random potential in bounded, though increasing in time, regions in the configuration space. Our results show that the time dependent study of the disordered system is a powerful tool, for handling the absolutely continuous spectrum. It is also quite clear that these techniques require the dimension to be bigger than 2, otherwise the free evolution does not have integrable decay in time, a fact crucial for the method to work. To this extent it seems that the role of higher dimension in producing absolutely continuous spectrum in the Anderson model is clear. The small energy condition is also clear from the method. Though we run the risk of being speculative, we would like to mention that heuristically, the sum

$$\sum_{|n| \leqslant \beta s} V^{\omega}(n)\langle \delta_n, \exp(-is\Delta)\delta_0 \rangle$$

converges in distribution, as $s \to \infty$, being the sum of independent mean zero random variables. When the disorder goes to zero, most of the terms in the sum are very small, since the potential is small with large probability in that case, so it might be possible to do a scaling analysis similar to the one done by Fröhlich–Spencer to control the growth of $\Sigma\, V^{\omega}(n)^2\, (1+s)^{-\nu+2} F(|n| \leqslant \beta s)$, in s, as the disorder goes to zero. This would be sufficient to show the existence of absolutely continuous spectrum in the case of homogeneous potentials.

Though we still cannot handle the case of homogeneous potentials, i.e. no decay in the coupling constants, it is still worth investigating if these methods can yield absolutely continuous spectrum in that case also as also the case when the coupling constants decay slower than the rate we considered here. Since the intuition for the techniques employed here is borrowed from scattering theory, it is reasonable to conjecture that: *If the coupling constants decay at infinity at a rate* $\alpha > 0$ *then* $\sigma_{ac}(H^{\omega}) \supset [-2v, 2v]$ *when* $v \geqslant 3$.

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Appendix

PROPOSITION A.1.

If A and B are two self-adjoint operators on a Hilbert space with B bounded and suppose the limits $\mathrm{slim}_{t\to\infty} \exp(iAs)\exp(-iBs)E_{ac}(B)$ exist. Then $\sigma_{ac}(B) \subseteq \sigma_{ac}(A)$.

In the above proposition we take B bounded since it is enough for our purposes. This is a standard result in scattering theory and the proof involves the fact that the limits mentioned above intertwie the operators A and B. The reader is referred to (Proposition 1, XI·3) Reed and Simon [14] or (Proposition 5.21) Amrein *et al* [1] for a proof.

PROPOSITION A.2.

Suppose A is a bounded self-adjoint operator on $l^2(\mathbb{Z}^v)$ *such that* $\langle \delta_{k'}, A\delta_k \rangle = 0$ *whenever* $|k - k'| > 1$ *for a fixed* 1, *then for any fixed* $n' \in \mathbb{Z}^v$, *there exist* $\alpha_1 > 0$ *and* $C_2 > 1$ *such that*

$$|\langle \delta_n, \exp(iAs)\delta_{n'} \rangle \leqslant C_1(n')\exp(-\alpha_1|n|), \quad \text{as } |n| \to \infty$$

whenever $n \in \mathbb{Z}^v$ *satisfies* $|n - n'| > C_2(\|A\| s \cdot l + 1)$.

Proof. Notice that since $|\langle \delta_k, A\delta_{k'} \rangle| = 0$ if $|k - k'| > 1$,

$$\langle \delta_n, A^m \delta_{n'} \rangle \equiv 0 \text{ whenever } |n - n'| > m \cdot 1.$$

Hence, when $[|n - n'|] = M \cdot l + 1$, we have

$$\langle \delta_n, \exp(iAs)\delta_{n'} \rangle = \left\langle \delta_n, \left\{ \exp(iAs) - \sum_{k=0}^{M} \frac{(iAs)^k}{k!} \right\} \delta_{n'} \right\rangle.$$

Using the spectral theorem for bounded self-adjoint operators and the inequality

$$|\exp(i\lambda s) - \sum_{k=0}^{M} \frac{(i\lambda s)^k}{k!}| \leqslant C_1 \frac{|\lambda s|^M}{M!}$$

for any positive integer M and the Sterling formula

$$M! \simeq \sqrt{2\pi M}\cdot M^M \exp(-M), \quad \text{as } M \to \infty$$

for the factorial of a large integer we see that

$$\begin{aligned}
&|\langle \delta_n, \exp(iAs)\delta_{n'}\rangle| \\
&\leqslant C_1 \ \mathrm{Sup}\ \{M^{-1/2}\exp(M\{\ln|\lambda s|\}\exp(-M\ln M)\exp(M)\} \\
&= C_1 \exp(M\ln)\left(\left\{\frac{\|A\|\,|s|}{M}\right\}\right)M^{-1/2}.
\end{aligned}$$

Now taking $|n-n'| > \exp\cdot\|A\|s\cdot 1 + 2$, we see that $\ln(\|A\|\,|s|\cdot l\cdot\exp)/([|n-n'|]-1) < 0$. Since l is a fixed finite number, we get,

$$|\langle \delta_n, \exp(iAs)\delta_0\rangle| \leqslant C_1 \exp(-\alpha_1|n|)$$

as claimed in the proposition. #

In the following $J_k(r)$ will denote the Bessel function of integral order k and argument r.

PROPOSITION A.3.

Let $\alpha > 1$, $\beta_1 \ll 1$ be given and consider s sufficiently large. Then $\exists \varepsilon > 0$ such that for $n \in \mathbb{Z}^\nu$,

$$\sum_{|n|\leqslant\beta_1 s}\prod_{i=1}^{\nu}|J_{n_i}(s)|^2\frac{1}{(1+|n|)^{2\alpha}} \leqslant \frac{c}{(1+s)^{2+\varepsilon}}\{1+\mathrm{o}(1)\}$$

Proof. Note that $J_{-k}(s) = (-1)^k J_k(s)$ for $k \in \mathbb{Z}^+$. Hence it suffices to consider $n_i \leqslant \beta_1 s$ $\forall i = 1,\ldots\nu$. Now the asymptotic expansions for Bessel function (cf. Watson [21] Section 8.12) show that for $k/s \ll 1$,

$$|J_k(s)| \leqslant \frac{C}{\sqrt{(2\pi s)}}\{1+\mathrm{o}(1)\}.$$

Therefore a crude estimate shows that for any $\beta_1 \ll 1$ and $n \in \mathbb{Z}^\nu$,

$$\begin{aligned}
&\sum_{|n|\leqslant\beta_1 s}\prod_{i=1}|J_{n_i}(s)|^2\frac{1}{(1+|n|)^{2\alpha}} \leqslant \frac{C(1+\mathrm{o}(1))}{(1+s)^\nu}\sum_{|n|\leqslant\beta_1 s}\frac{1}{(1+|n|)^{2\alpha}} \\
&\leqslant \frac{C_1(1+\mathrm{o}(1))}{(1+s)^\nu}\cdot\sum_{i=1}^{\nu}\sum_{\substack{n_i\in\mathbb{Z}\\0<n_i\leqslant\beta_1 s}}\frac{1}{(1+n_i)^{2\alpha}}. \\
&\leqslant \frac{c_2(1+\mathrm{o}(1))}{(1+s)^{2+\varepsilon}}.
\end{aligned}$$

since $\alpha > 1$, the last step follows for a suitable $\varepsilon > 0$.

PROPOSITION A.4.

Let n, $m\in\mathbb{Z}^\nu$ and Δ as in (1), then we have

$$\langle\delta_n, \exp(-i\Delta s)\delta_m\rangle = \prod_{i=1}^{\nu}(-i)^{(n-m)_i}J_{(n-m)_i}(2s)$$

where $J_k(s)$ is the Bessel function of integral order k.

Proof. First note that the Fourier transform U gives the equality

$$(U\Delta U^{-1}u)(k) = \left(2\sum_{i=1}^{\nu}\cos k_i\right)u(k).$$

Hence a simple computation shows that,

$$\langle\delta_n, \exp(-i\Delta s)\delta_m\rangle = \frac{1}{(2\pi)^\nu}\int \mathrm{d}k \exp(i(n-m)\cdot k)\exp\left(-i2s\sum_{i=1}^{\nu}\cos k_i\right).$$

Since $\mathrm{d}k = \prod_{i=1}^{\nu}\mathrm{d}k_i, k_i\in[-\pi,\pi]$, we have

$$\langle\delta_n, \exp(-i\Delta s)\delta_m\rangle = \prod_{i=1}^{\nu}\left\{\frac{1}{(2\pi)}\int_{-\pi}^{\pi}\mathrm{d}k_i \exp(i(n-m)_i k_i)\exp(-i2s\cos k_i)\right\}.$$

From the definition of Bessel functions and the above equality the proposition follows. #

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