Proc. Indian Acad. Sci. (Math. Sci.), Vol. 100, No. 3, December 1990, pp. 259–273.

Köthe spaces and topological algebra with bases

SUBHASH J BHATT and G M DEHERI

Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar 388 120, India

MS received 6 February 1989; revised 23 March 1990

Abstract. Nuclear Köthe sequence space $\lambda(P)$, its crossdual $\lambda(P)^{\times}$ and their non-nuclear variants are examined as topological algebras. Modelling on them, a general theory of nuclear topological algebras with orthogonal basis is developed. As a by-product, abstract characterizations of sequence algebras ℓ^{∞} and c_0 are obtained. In a topological algebra set-up, an abstract Grothendieck-Pietsch nuclearity criterion is developed.

Keywords. Köthe sequence spaces; G_1- and G_{∞}-conditions; cross-dual of a sequence space; nuclear locally convex space; Schauder basis; equicontinuous and absolute basis; unconditional convergence; locally convex algebra; orthogonal basis; Grothendieck-Pietsch nuclearity criteria.

1. Introduction

Let ω denote the vector space of all sequences of complex numbers, A subset $P\subset\omega$ is a *power set* [15, Ch. 6] if (i) for each $a=(a_n)\in P$, $a_n\geqslant 0$, (ii) for each $a=(a_n)$, each $b=(b_n)$ in P, there exists a $c=(c_n)$ in P satisfying $a_n\leqslant c_n$, $b_n\leqslant c_n$ for all n, (iii) for each $n\in\mathbb{N}$, there exists $a\in P$ such that $a_n>0$. Further, P satisfies G_1-*property* (respectively G_{∞}-*property*) [12] if (iv) for each $a\in P$, $a_{n+1}\leqslant a_n$ (respectively $a_n\leqslant a_{n+1}$) for all n; (v) for each $a\in P$, there exists a $d\in P$ satisfying $a_n\leqslant d_n^2$ (respectively $a_n^2\leqslant d_n$) for all n. Given a power set P, *Köthe sequence space* $\lambda(P)=\{b=(b_n)\in\omega\,|\,p_a(b)=\Sigma_{n\geqslant 1}|b_n|a_n<\infty$ for all $a\in P\}$ is a complete locally convex space with the Köthe topology τ_P, defined by seminorms $\Gamma_P=\{p_a: a\in P\}$ and having Schauder basis $e_n=(\delta_{nm})$, δ_{nm} being the Kronecker delta [12], [13]. For a G_1 (respectively G_{∞}) power set P, $\lambda(P)$ is denoted by $\lambda_1(P)$ (respectively $\lambda_{\infty}(P)$). Note that $\lambda(P)$ is a very general construction that includes very many standard sequence spaces as particular cases [16]. The space $\lambda(P)$ occupies a central position in the theory of locally convex space with basis ([14], [12], [15]); in fact, a complete locally convex space with an absolute basis has to be $\lambda(P)$ [15, Th. 10.1.4].

Now consider ω as a unital *algebra with pointwise multiplication and pointwise complex conjugation as the involution. It happens that each of $\lambda_1(P)$ and $\lambda_{\infty}(P)$ (assumed nuclear) is a *subalgebra of ω which, with topology τ_P, is a locally convex algebra. The present paper is aimed at understanding these topological algebras and examining their relevance in (and impact on) the general axiomatics of functional analysis. It so turns out that $\lambda(P)$ also occupies a central position in the theory of topological algebras with orthogonal basis initiated by Husain and his colleagues [7]–[10].

In §2, after discussing the basic properties of the topological algebra $(\lambda_1(P), \tau_P)$, it is shown in Theorem 2.4 that a sequentially complete nuclear (as a locally convex space) topological algebra A with jointly continuous multiplication and having an equicontinuous orthogonal basis is homeomorphically isomorphic to a (semi) G_1-Köthe algebra $\lambda(P)$. This provides a topological algebra analogue of [15, Th. 10.1.4] which implies that the space $\lambda(P)$ comprise all complete nuclear spaces with equicontinuous basis. This also generalizes a characterization, due to Husain and Watson [8], of the algebra ω (pointwise convergence) as the only complete unital locally m-convex algebra with orthogonal basis. The relation between nuclearity of A and existence of identity in A is examined in Corollary 2.6, as well as the relevance of identity in above quoted result of Husain and Watson is also discussed. Even in the absence of validity of open mapping theorem, it turns out that an orthogonal basis in a topological algebra is Schauder. In fact, in Proposition 2.1, we prove a much stronger assertion improving [5, Th. 1.1]. This leads to a characterization of isomorphisms of B_0-algebras with bases. It is shown, in Theorem 2.13, that under fairly general conditions, a sequentially complete topological algebra with an orthogonal basis is a countable direct sum of simple algebras, each homeomorphically isomorphic to the field ϕ of complex numbers. This improves a structure theorem for Banach algebras with basis proved in [10].

Section 3 contains abstract characterizations of Banach algebras ℓ^∞ and c_0. A unital dual Banach algebra A, with predual A_*, which, for the topology $\sigma(A, A_*)$, (i) is sequentially complete, and (ii) admits an equicontinuous orthogonal basis, is dual algebra isomorphic to ℓ^∞. Further, such an A is a uniform (Banach) algebra iff this isomorphism is an isometry. On the other hand, a uniform Banach algebra with an orthogonal basis is isometrically isomorphic to c_0.

In §4, nuclearity of a topological algebra with basis is characterized in terms of ring theoretic structure. Recall the Grothendieck-Pietsch nuclearity criteria [12], [15] viz $(\lambda(P), \tau_P)$ is nuclear iff for each $a\in P$, there exists $ab\in P$ and $au\in\ell^1$ such that $a_n \leqslant u_n b_n$ for all n iff τ_P is determined by the family of seminorms $\{q_a | a\in P\}$, $q_a(x) = \sup_n |x_n| a_n$. Further, $(\lambda_1(p), \tau_P)$ is nuclear iff $P \subset \ell^1$; and $(\lambda_\infty(P), \tau_P)$ is nuclear iff there is a $u\in P$ such that $(1/u_n)\in\ell^1$. With a view of obtaining an abstract version of this for a topological algebra A with an orthogonal basis (e_n), we consider, for $1 \leqslant p \leqslant \infty$, $K_p(A) = \{x\in A | (e_n^*(xy))\in\ell^p \text{ for all } y\in A\}$ and $J(A) = \{x\in A | (e_n^*(xy))\in c_0 \text{ for all } y\in A\}$ with natural topologies τ_A^p defined by the seminorms $p_y^p(x) = \| e_n^*(xy) \|_p$, $y\in A$. Note that $J(A) \subset K_\infty(A)$ and $J(A)$ carries the relative topology. As in [9], each $K_p(A)$ and $J(A)$ are dense ideals in A satisfying, for $1 \leqslant p \leqslant \infty$, $K_1(A) \subset K_p(A) \subset J(A) \subset K_\infty(A)$. We show that if A is a B_0-algebra with identity having (e_n) unconditional, then A is nuclear iff $K_1(A) = J(A) = K_\infty(A)$ topologically as well.

In §5, we have briefly discussed a number of examples of topological algebras with orthogonal basis, to impress upon the usefulness of algebra point of view in the theory of sequence spaces and to illustrate the general theory. The paper ends with concluding remarks, wherein we discuss three aspects of topological algebras with basis viz the problem of automatic continuity of multiplicative functionals, uniqueness of topology and non-existence of orthogonal basis in an abstract nuclear B_0-algebra.

By a *(locally convex) topological algebra A* is meant a Hausdorff (locally convex) topological vector space with topology t which is also a linear associative algebra over complex scalars such that the multiplication is separately continuous, i.e. for each x, the linear maps $L_x y = xy$ $R_x y = yx$ are continuous. A basis (e_n) in A is *orthogonal* if $e_n e_m = \delta_{nm} e_m$ for all n, m. For $x \in A$, the expansion $x = \Sigma_1^\infty \alpha_n e_n$ leads to coefficient functionals $e_n^*(x) = \alpha_n$ which are multiplicative if (e_n) is orthogonal. A nuclear algebra is a topological algebra that is a nuclear locally convex space. A topological algebra with an orthogonal basis is always commutative [8]. We refer to [14], [21] for topological algebras; [15], [19] for locally convex spaces; [12], [13], [16] for sequence spaces; [15] for linear topological aspects of basis theory; and [7], [8], [9], [10] for orthogonal basis. $\ell^p (1 \leqslant p \leqslant \infty)$, c, c_0 denote the usual Banach sequence spaces.

2. $\lambda_1(P)$ and topological algebras with basis

Defining property (v) of G_1-condition implies that $\lambda_1(P)$ is closed under pointwise multiplication and $(\lambda_1(P), \tau_p)$ is a locally convex *algebra with jointly continuous multiplication having $(\delta_{nm})_{n=0}^\infty$ as an orthogonal Schauder basis. Further, it is easily seen that $\lambda_1(P)$ admits identity (which has to be constant sequence $\{1\}$ denoted by 1) iff $(\lambda_1(P), \tau_P)$ is nuclear; in which case, it is a generalized B^*-algebra in the sense of [1], [3] containing the C^*-sequence algebra $(\ell^\infty, \|\cdot\|_\infty)$ continuously and densely embedded in it. In this section, we aim to incorporate this observation in general theory of topological algebras with basis.

We begin with the following 'automatic continuity' result that improves [8, Prop. 3.1] and [5, Th. 1.1] pushing them in the final form leading to a variant in a more general framework. Our arguments are modifications of those in [5] wherein part (i) of the following is proved under the additional assumption that multiplication in A is jointly continuous. This is a strong requirement for non-m-convex non-metrizable algebras.

Theorem 2.1. *If A is a topological vectorspace that is an algebra;*

(i) *If A is a topological algebra, then every orthogonal basis in A is Schauder.*
(ii) *If multiplication in A is sequentially separately continuous (i.e. for a sequence (x_n) in A, $x_n \to 0$ implies $x_n y \to 0$, $yx_n \to 0$ for all y), then every orthogonal basis (e_n) in A is b-Schauder (i.e. each coefficient functional e_n^* is bounded in the sense that it maps a bounded set to a bounded set).*

Proof. Let (e_n) be an orthogonal basis in A. Let $n \in \mathbb{N}$ be fixed orthogonality of (e_n) applied to the expansion $x = \Sigma_1^\infty e_n^*(x) e_n$ implies that $e_n x = e_n^*(x) e_n = x e_n$ for all $x \in A$. Choose a balanced o-neighbourhood U such that $e_n \notin U$. Let $r = \inf\{d > 0 | e_n \in dU\}$. Then $r > 1$.

(i) Let (x_α) be a net in A such that $\lim_\alpha x_\alpha = 0$. Hence $\lim_\alpha x_\alpha e_n = 0$. Given $\delta > 0$, there exists an α_0 such that $e_n^*(x_\alpha) e_n = x_\alpha e_n \in \delta U$ for all $\alpha \geqslant \alpha_0$. As U is balanced, $|e_n^*(x_\alpha)| e_n \in \delta U$ for $\alpha \geqslant \alpha_0$. Hence by definition of $r, |e_n^*(x_\alpha)|^{-1} \delta \geqslant r > 1$, and so $|e_n^*(x_\alpha)| < \delta$ for all $\alpha \geqslant \alpha_0$. Thus $\lim_\alpha e_n^*(x_\alpha) = 0$.

(ii) It is sufficient to show that e_n^* maps a bounded sequence (x_k) to a bounded sequence. For any sequence $r_k \to \infty$, $r_k > 0$, $x_k / r_k \to 0$. Hence $e_n x_k / r_k \to 0$. Thus $(e_n^*(x_k) e_n)_{k=1}^\infty$ is bounded and for all $k, e_n^*(x_k) e_n \in \lambda U$ for some $\lambda = \lambda(n, U) > 0$. Again by definition of $r, |e_n^*(x_k)| \leqslant r/\lambda$ for all k. This completes the proof.

It follows, from the continuity of the injective map $x \in A \to (e_n^*(x)) \in \omega$ (pointwise convergence) that a topological algebra A with an orthogonal basis carries an lmc topology coarser than the given topology. This makes spectral theory of A tractable; e.g. (e_n^*) constitute the Gelfand space $\chi(A)$ of A viz, the space of all continuous multiplicative functionals on A with weak *topology exactly as in [5, Corollary 1.3] in view of Theorem 2.1; and if A is also a division algebra, then A is isomorphic to scalar field $\mathbb{C}$ (Gelfand-Mazur Theorem). The following consequence of Theorem 2.1 can be regarded as a first step towards classifying topological algebras with orthogonal basis.

COROLLARY 2.2.

Let A and B be topological algebras with orthogonal bases (e_n) and (f_n) respectively.

(a) *If $\phi: A \to B$ is a homeomorphic isomorphism of A onto B, then there exists a permutation σ of $\mathbb{N}$ (the set of natural numbers) satisfying*

> (C_1): "*(e_n) and $(f_{\sigma^{-1}(n)})$ are equivalent (in the sense that for any sequence of scalars (x_n), $\Sigma x_n e_n$ converges in A iff $\Sigma x_n f_{\sigma^{-1}(n)}$ convergences in B*",

and ϕ is of the form $\phi = T_\sigma$ where $T_\sigma(\Sigma x_n e_n) = \Sigma x_{\sigma(n)} f_n$.

(b) *For any permutation σ of $\mathbb{N}$ satisfying (C_1), T_σ defines an isomorphism of A onto B. Further, if A and B are F-algebras (i.e. complete metric algebras), then T_σ is a homeomorphism.*

Proof. (a) By Theorem 2.1, (e_n) and (f_n) are Schauder bases, and as noted above $\chi(A) \approx \mathbb{N} \approx \chi(B)$. The conjugate map between the topological duals $\phi^*: B^* \to A^*$, $\phi^* f = f \circ \phi$ is a bijective linear map that is $\sigma(A^*, A) - \sigma(B^*, B)$ continuous. Further, ϕ being an isomorphism, $\phi^*(\chi(B)) \subset \chi(A)$. Hence there is a permutation $\sigma: \mathbb{N} \to \mathbb{N}$ such that $\phi^*(f_n^*) = e_{\sigma(n)}^*$. Thus for all x in A, $f_n^*(\phi(x)) = e_{\sigma(n)}^*(x) = x_{\sigma(n)}$; and so $\phi(\Sigma x_n e_n) = \Sigma_n x_{\sigma(n)} f_n = \Sigma x_n f_{\sigma^{-1}(n)}$ giving (a).

(b) Let σ be a permutation of $\mathbb{N}$ satisfying (C_1). Then for any $x = \Sigma x_n e_n$ in A, $\Sigma x_n f_{\sigma^{-1}(n)} = \Sigma x_{\sigma(n)} f_n$ converges in B, thus defining $\phi(x)$ in B. This defines a linear map $\phi: A \to B$ which preserves the ring structure, as (e_n) and (f_n) are orthogonal. Condition (C_1) also implies that ϕ is bijective. By Theorem 2.1, the linear maps

$$T_n: A \to B,\ T_n(x) = \sum_{k=1}^{n} e_k^*(x) f_{\sigma^{-1}(k)}$$

are continuous, and $\phi(x) = \lim_{n \to \infty} T_n x$ (pointwise) for all $x \in A$. Hence if A and B are F-algebras, then ϕ is continuous; and hence is a homeomorphism by open mapping theorem.

Remarks 2.3. (a) In the absence of metrizability, condition (C_1) is not sufficient to make T_σ a homeomorphism. This is seen in the case of $A = (\ell^1, \|\cdot\|_1)$, $B = (\ell^1, \eta(\ell^1, \ell^\infty))$; both are topological algebras with $e_n = (\delta_{nm})$ as orthogonal basis. Here the normal topology $\eta(\ell^1, \ell^\infty)$ on ℓ^1 is defined by seminorms $p_x(y) = \Sigma |y_i x_i|$, $x \in \ell^\infty$.

(b) Now suppose A and B are locally convex. Let $S(A)$ (resp. $S(B)$) be a family of continuous seminorms determining the topology of A (resp. B). Given a homeomorphic isomorphism $\phi: A \to B$, let σ be the permutation of $\mathbb{N}$ defined by Corollary 2.2 (a) so that $\phi = T_\sigma$. A consequence of continuity of ϕ and that of ϕ^{-1} is that σ satisfies

(C_2): (i) Given $q \in S(B)$, there exists a $p \in S(A)$ and a $k > 0$ such that $q(f_{\sigma^{-1}(n)}) \leqslant k p(e_n)$ for all n.
(ii) Given a $p \in S(A)$, there exists a $q \in S(B)$ and a $k' > 0$ such that $p(e_n) \leqslant k' q(f_{\sigma^{-1}(n)})$ for all n.

The following shows that this suffices in the nuclear case. Note that in a B_0-space, a Schauder basis is equicontinuous [15, Th. 10.1.2].

PROPOSITION.

Let A and B be nuclear locally convex algebras with equicontinuous orthogonal bases (e_n) and (f_n) respectively. The isomorphism $T_\sigma: A \to B$ defined by a permutation σ of $\mathbb{N}$ satisfying (C_1) and (C_2) is a homeomorphism.

Indeed, due to nuclearity, (e_n) is an absolute basis [15, Theorem 10.2.1] so that given $p \in S(A)$, there is a $p' \in S(A)$ such that $\Sigma e_n^*(x) p(e_n) \leqslant p'(x)$ for all x. Hence for a $q \in S(B)$, (C_2) (i) gives, for $x \in A$,

$$q(T_n x) \leqslant \sum_{1 \leqslant l}^{n} e^*(x) q(f_{\sigma^{-1}(l)}) \leqslant k \sum_{1}^{n} e_l^*(x) p(e_l) \leqslant k p'(x).$$

Thus (T_n) is an equicontinuous family. This makes ϕ continuous. Similarly C_2 (ii) gives continuity of ϕ^{-1}.

Now it is well known [15, Th. 10.1.4] that a complete locally convex space with an absolute basis is linearly homeomorphic to a Köthe space $(\lambda(P), \tau_P)$. Further, by the Basis Theorem [15, Th. 10.2.1], an equicontinuous basis in a nuclear space is absolute, with the result, a complete nuclear space with an equicontinuous basis has to be a Köthe space. The following supplements this by exploiting the added ring theoretic structure on $\lambda_1(P)$. We shall call a power set P with property (v) of §1 semi-G_1-set and $\lambda_1^0(P)$ denotes a semi-G_1-Köthe space.

Theorem 2.4. *Let (A, t) be a topological algebra satisfying the following*

(i) *A is locally convex*

(ii) *A has jointly continuous multiplication*

(iii) *A admits an orthogonal absolute basis (e_n). Then there exists a homeomorphic isomorphism $\phi: (A, t) \to (\lambda_1^0(P), \tau_P)$ from A onto a dense subalgebra of some $\lambda_1^0(P)$. Further, (a) if A is a *algebra with continuous involution and (e_n) is hermitian (i) (i.e. each e_n a hermitian element), then $\phi(A)$ is a *subalgebra of $\lambda_1^0(P)$ and ϕ is a *isomorphism; (b) if A is sequentially complete, then ϕ is onto.*

Proof. By (ii), given a o-neighbourhood U, there exists a o-neighbourhood V such that $VV \subset U$. Additionally using (i), the topology of A is determined by a family $\Gamma = \{p_\alpha | \alpha \in \Delta\}$ of seminorms satisfying: For each $\alpha \in \Delta$, there exists a $\beta \in \Delta$ such that

$$p_\alpha(xy) \leqslant p_\beta(x) p_\beta(y) \quad (x, y \text{ in } A). \tag{1}$$

Let (e_n) be an orthogonal absolute basis. For each $\alpha \in \Delta$, let $a^{(\alpha)} = (a_n^{(\alpha)}) \in \omega$, $a_n^{(\alpha)} = p_\alpha(e_n)$. By (2) and Hausdorff property of A, $P = \{a^{(\alpha)} | \alpha \in \Delta\}$ is a power set. The Köthe topology

τ_P on $\lambda(P)$ is determined by $\Gamma_P = \{q_\alpha | \alpha \in \Delta\}$, $q_\alpha(b) = \Sigma_1^\infty |b_n| a_n^{(\alpha)} < \infty$ for all $\alpha \in \Delta$. Expanding each $x \in A$ as $x = \Sigma_1^\infty e_n^*(x) e_n$, orthogonality of (e_n) implies that each e_n^* is a multiplicative linear functional, which is continuous by Theorem 2.1. As (e_n) is absolute, for each α, there is a β satisfying, for all x,

$$\sum_{n=1}^{\infty} |e_n^*(x)| p_\alpha(e_n) \leqslant p_\beta(x). \tag{2}$$

This with continuity of p_α, implies that $p_\alpha(x) \leqslant \Sigma |e_n^*(x)| a_n^{(\alpha)} \leqslant p_\beta(x) < \infty$ showing $(e_n^*(x)) \in \lambda(P)$. Thus, there is defined linear map $\phi: A \to \lambda(P)$, $\phi(x) = (e_n^*(x))$ satisfying: Given $\alpha \in \Delta$, there is a $\beta \in \Delta$ such that $p_\alpha(x) \leqslant q_\alpha(\phi(x)) \leqslant p_\beta(x)$. Thus ϕ is one–one and is a homeomorphism into. Orthogonality of (e_n) has the following consequences;

1° Taking $x = y = e_n$ in (1), given α, there is a β such that $a_n^{(\alpha)} = p_\alpha(e_n) = p_\alpha(e_n^2) \leqslant (p_\beta(e_n))^2 = (a_n^{(\beta)})^2$. Thus, P satisfies semi-G_1-property and $(\lambda(P), \tau_P)$ is a complete locally convex topological algebra.
2° For $x = \Sigma_1^\infty e_n^*(x) e_n$, $y = \Sigma_1^\infty e_n^*(y) e_n$; $xy = \Sigma_{n,k} e_n^*(x) e_k^*(y) e_n e_k = \Sigma_n e_n^*(x) e_n^*(y) e_n$ giving $\phi(xy) = \phi(x)\phi(y)$.

(a) If $x \to x^*$ is the continuous involution of A with each e_n hermitian, then for an $x \in A, x^* = (\Sigma e_n^*(x) e_n)^* = \Sigma \overline{e_n^*(x)} e_n$, $\phi(x^*) = \phi(x)^*$.

(b) Let A be sequentially complete. Let $u = (u_n) \in \lambda(P)$ so that for each α, $\Sigma_1^\infty p_\alpha(u_n e_n) = \Sigma_1^\infty |u_n| a_n^{(\alpha)} < \infty$. Hence taking $x^{(p)} = \Sigma_1^p u_k e_k$, for $n > m$ and for each α, $p_\alpha(x^{(n)} - x^{(m)}) \leqslant \Sigma_{m+1}^n |u_k| p_\alpha(e_k) \to 0$. Thus $(x^{(n)})$ is Cauchy, hence $x^{(n)} \to x \in A$ and $x = \Sigma u_n e_n$ showing that ϕ is onto. This completes the proof.

The following is immediate from [18, Th. 2.17] and [15, Th. 10.1.2]. A *B_0-algebra* is a complete metrizable locally convex algebra. A *locally m-convex algebra* [14] is a topological algebra whose topology is determined by a separating family of submultiplicative seminorms.

COROLLARY 2.5.

A nuclear B_0-algebra with an orthogonal basis is homeomorphically isomorphic to a semi-G_1 Köthe algebra $(\lambda_1^0(P), \tau_P)$ for a countable P.

It is shown in [8, p. 246] that an infinite dimensional normed algebra with orthogonal basis cannot have identity. The following describes some connections between the existence of identity and nuclearity.

COROLLARY 2.6.

Let A be a locally convex algebra having an orthogonal basis (e_n). Let multiplication in A be jointly continuous.

(1) *Let (e_n) be an absolute basis. If A has identity, then A is nuclear.*

(2) *If A has identity, then (e_n) is bounded.*

(3) *If A is sequentially complete and (e_n) is bounded and equicontinuous, then A has identity.*

Proof. (1) *Let* e be the identity of A. Then $e = \Sigma_1^\infty e_n$, $\phi(e) \in \lambda_1^0(P)$ and for each $\alpha, \Sigma a_n^{(\alpha)} = q_\alpha(\phi(e)) < \infty$. Thus $P \subset \ell^1$, $\ell^\infty \subset \lambda_1^0(P)$. This also implies $\phi(e) = 1$ (constant sequence 1) and by the proof of Theorem 2.4; for each α, there is a β such that

$a_n^{(\alpha)}/a_n^{(\beta)} \leqslant a_n^{(\beta)}$, $(a_n^{(\beta)})\in\ell^1$. Hence $(\lambda_1^0(P),\tau_P)$ is nuclear. Therefore $\phi(A)$, and hence A, is nuclear.

(2) $e=\Sigma e_n$ implies $e_n\to 0$, hence $(p_\alpha(e_n))$ is bounded for each continuous seminorm p_α on A.

(3) In the notations of proof of Theorem 2.4, for each α there is a scalar $M_\alpha>0$ satisfying $p_\alpha(e_n)\leqslant M_\alpha$ for all n. Now A is nuclear, (e_n) is equicontinuous, hence (e_n) is absolute. Thus $\lambda_1^0(P)$ is nuclear. Hence for each α, there is a β such that $p_\alpha(e_n)/M_\beta\leqslant p_\alpha(e_n)/p_\beta(e_n)$ and $\{p_\alpha(e_n)/p_\beta(e_n)\}\in\ell^1$. Then $P\subset\ell^1$, $\ell^\infty\subset\lambda_1^0(P)$, hence $\lambda_1^0(P)$, and so A has identity.

Example 2.7. Converse of Corollary 2.6 (1) does not hold. Consider the G_∞-Köthe space $(\lambda_\infty(P),\tau_P)$ assumed nuclear. There is a $d\in P$ such that $\{1/d_n\}\in\ell^1$. Given x,y in $\lambda_\infty(P)$, $p_a(xy)=\Sigma_1^\infty|x_ny_n|a_n\leqslant\|1/d\|_1p_a(x)p_a(y)$ showing $\lambda_\infty(P)$ to be a locally convex algebra having absolute basis $e_n=(\delta_{nm})$ and with jointly continuous multiplication. Now if $1\in\lambda_\infty(P)$, then $P\subset\ell^1$, hence for all $a\in P$, $a_n\to 0$ contradicting $a_n\leqslant a_{n+1}$ for all n. Thus $\lambda_\infty(P)$ cannot have identity. In fact, $\lambda_\infty(P)\subset\ell^1$. Taking P countable $\lambda_\infty(P)$ can be chosen a B_0-algebra. Thus, in view of Corollary 2.6. (3), in a nuclear B_0-algebra, (e_n) need not be automatically bounded, though it is equicontinuous [15, Th. 10.1.2]; and nuclear B_0-algebra with an orthogonal basis need not be unital.

Example 2.8. In Corollary 2.6 (1), the hypothesis that (e_n) is absolute cannot be omitted even if A is assumed metrizable. Let $A=\{x=(x_n)\in\omega|\lim_n x_n n^{-1/2^k}=0$ for all $k=1,2,3,\ldots\}$. A is a B_0-algebra with topology defined by the seminorms $p_k(x)=\sup_n|x_n|n^{-1/2^k}$ having identity 1 and admitting orthogonal unconditional basis $e_n=(\delta_{nm})$. The algebra A is not nuclear (in view of Theorem 4.2, since $K(A)\neq J(A)$ as shown in [9]) and (e_n) fails to be an absolute basis (because, otherwise Theorem 2.2 applies) though it is equicontinuous.

Similarly, the hypothesis that A has identity can also not be omitted. Consider for a G_1-power set P, the Köthe sequence algebra $(\lambda_1(P),\tau_P)$. Then $e_n=(\delta_{nm})$ is an absolute basis, and $\lambda_1(P)$ is nuclear iff $\lambda_1(P)$ has identity.

COROLLARY 2.9. [8].

Let A be a complete locally m-convex algebra with an orthogonal basis (e_n). If A has identity e, then A is metrizable and is homeomorphically isomorphic to (ω,t), t being the topology of pointwise convergence.

Proof. By [8, Th. 1.3], an lmc algebra with an orthogonal basis having identity is metrizable. Thus (e_n) is equicontinuous. If $\Gamma=(p_k)_{k=1}^\infty$ is a sequence of submultiplicative seminorms determine the topology on A, then $p_k(e_n)\leqslant p_k(e_n)^2$ gives $p_k(e_n)\geqslant 1$ or $p_k(e_n)=0$. But $e=\Sigma_{n=1}^\infty e_n$ implies $p_k(e_n)\to 0$ as $n\to\infty$. Thus, for each k, $p_k(e_n)=0$ for all $n\geqslant$ some N_k. This makes (e_n) an absolute basis. By Theorem 2.4, $A\approx\lambda_1^0(P)$ and $P=\varphi$, the space of sequences with only finitely many non-zero terms. From this, the assertion follows immediately.

The following two particular cases of Köthe algebra $\lambda(P)$ reveal the importance of the existence of identity in the above Corollary.

Example 2.10. In the absence of identity, an lmc algebra with an orthogonal basis need not be metrizable. Consider the sequence algebra $\varphi = \{x \in \omega | x_n \neq 0$ only for finitely many $n\}$. Let $\varphi_m = \{x \in \varphi | x_k = 0$ for $k \geqslant m+1\}$, a finite dimensional ideal in φ for each m, and $\varphi = \cup_{m=1}^{\infty} \varphi_m$. Let τ be the strongest locally convex linear topology on φ making each of the embeddings $\varphi_m \to \varphi$ continuous. It follows from [20, pp. 195–196] that (φ, τ) is an lmc algebra and τ is the finest lmc topology making each of above embeddings continuous. Also, $e_n = (\delta_{nm})$ is an orthogonal basis. A similar phenomena is also exhibited by the algebra $A = \cup_{1 \leqslant p < \infty} \ell^p$, each ℓ^p carries the relative to topology from ℓ^{∞}. The topology τ is not metrizable.

Example 2.11. An lmc algebra with an orthogonal basis, even if non-normed Frechet, need not be the whole of ω. This is exhibited by the sequence algebra $B = \cap_{1 < p \leqslant \infty} \ell^p$ with the topology defined by $\|\cdot\|_p$: $p = 1, 2, 3, \ldots, \infty$. It is shown in [5, Th. 2.1] that an lmc algebra with orthogonal basis and having identity carries via the mapping φ of Theorem 2.3, the topology of pointwise convergence. This example, as well as the algebra $L^{\omega}(T)$ discussed in §5.5 show that this need not be the case in the absence of identity.

Remark 2.12. There is another interesting characterization of ω. A *σ-C*-algebra* is complete metrizable lmc algebra that is an inverse limit of C^*-algebras.

PROPOSITION [2]

A commutative σ-C-algebra with identity that is a nuclear space admits an orthogonal basis and is homeomorphically isomorphic to* ω.

The following was proved for Banach algebras in [9]. It reveals 'sequential nature' of a topological algebra with an orthogonal basis.

Theorem 2.13. *Let* (A, t) *be a locally convex algebra with an orthogonal basis* (e_n). *Consider the following*

(i) *A is sequentially complete with jointly continuous multiplication and* (e_n) *is absolute and bounded*
(ii) *A is locally m-convex*
(iii) *A is a* B_0*-algebra.*

Let A satisfy at least one of the above conditions. Then A is a countable direct sum of simple topological algebras Ae_n, *each of which is homeomorphically isomorphic to* $\mathbb{C}$.

Proof. As in [9, Th. 4.4, Prop. 4.5], each Ae_i is a closed minimal ideal, $A = \Sigma_1^{\infty} Ae_i$, each $M_n = \ker e_n^*$ is a closed maximal ideal and for each n, ψ: $(A/M_n, t_q) \to (Ae_n, t)$, $\psi(x + M_n) = xe_n$ is a continuous isomorphism from A/M_n onto Ae_n. Here t_q = quotient topology on A/M_n and t = relative topology from A on Ae_n.

(a) Assume (i). In notations of proof of Theorem 2.4, consider the homeomorphic isomorphism ϕ: $(A, t) \to (\lambda_1^0(P), \tau_P)$. Since (e_n) is bounded, $P \subset \ell^{\infty}$, hence $\ell^1 \subset \lambda_1^0(P)$, in fact, ℓ^1 is sequentially dense in $\lambda_1^0(P)$. Then the subalgebra $B = \phi^{-1}(\ell^1)$ of A, which is a Banach algebra with norm $\|x\| = \|\phi(x)\|_1$, is sequentially dense in (A, t). By Theorem 2.1, $e_n^*/B \neq 0$. Being a multiplicative functional, e_n^* is $\|\cdot\|$ continuous on B, hence ker $(e_n^*/B) = M_n \cap B$ is a normclosed maximal regular ideal. By Gelfand-Mazur Theorem $B/(M_n \cap B)$ is isomorphic to $\mathbb{C}$. Since B is sequentially dense in (A, t),

$B/(M_n \cap B)$ is sequentially t_q-dense in A/M_n. Thus A/M_n is $\mathbb{C}$; $t_q = t$ via ψ and the conclusion follows.

(b) Assume (ii) or (iii). $(A/M_n, t_q)$ is a division algebra that is either a B_0-algebra or is lmc. Since Gelfand-Mazur Theorem holds in those two classes of algebras ([14], [21]); $(A/M_n, t_q)$, hence (Ae_n, t), is $\mathbb{C}$. This completes the proof.

3. Abstract characterizations of ℓ^∞ and c_0

A *dual Banach algebra* $(A, \|\cdot\|)$ is a Banach algebra that is dual, as a Banach space, of some Banach space $(A_*, \|\cdot\|)$ called its predual. A dual algebra isomorphism $\psi: A \to B$ from a dual Banach algebra A to a dual Banach algebra B is an algebra isomorphism that is $\|\cdot\| - \|\cdot\|$ homeomorphism as well as $\sigma(A, A_*) - \sigma(B, B_*)$ homeomorphism. The Banach algebra $\ell^\infty (= \lambda(P)$ with $P = \{a \in \ell^1 / a_n \geqslant 0$ for all $n\})$ is a dual Banach algebra with 1 having predual $(\ell^1, \|\cdot\|_1)$. Further, $(\ell^\infty, \sigma(\ell^\infty, \ell^1))$ is a sequentially complete topological algebra having orthogonal basis $e_n = (\delta_{nm})_1^\infty$.

Theorem 3.1. *Let A be a dual Banach algebra with identity e and predual A_*. If (i) $(A, \sigma(A, A_*))$ is sequentially complete, and (ii) $(A, \sigma(A, A_*))$ admits an equicontinuous orthogonal basis (e_n), then there exists a dual algebra isomorphism ϕ from A onto ℓ^∞.*

Proof. Define $\phi: A \to \omega$ as $\phi(x) = (e_n^*(x))$. Each e_n^* being multiplicative, it is $\|\cdot\|$ continuous, and for all x, $|e_n^*(x)| \leqslant \|x\|$. Thus $\phi(x) \in \ell^\infty$, $\phi(A) \subset \ell^\infty$ and $\|\phi(x)\|_\infty \leqslant \|x\|$. As in the proof of Theorem 2.4, ϕ is one–one, linear and is homeomorphism.

We show that ϕ is $\sigma(A, A_*) - \sigma(\ell^\infty, \ell^1)$ continuous. Let $g = (g_n) \in \ell^1$, regarded as a linear functional on ℓ^∞. Thus $g \circ \phi \in A_*$. Indeed, for all $x \in A$ $|(g \circ \phi)(x)| = |\Sigma < (\phi(x))_n, g_n > | \leqslant \Sigma |g_n|\ |e_n^*(x)|$. By equicontinuity of (e_n) in $\sigma(A, A_*)$, given $\varepsilon > 0$, there is a o-neighbourhood U in $\sigma(A, A_*)$ satisfying, for all n, $|e_n^*(x)| \leqslant \varepsilon$. Thus for all $x \in U$, $|(g \circ \phi)(x)| \leqslant \varepsilon \|g\|_1$, giving $g \circ \phi \in A_*$. Thus if $x \to 0$ in $\sigma(A, A_*)$ (so that $f(x) \to 0$ for all $f \in A_*$), $g(\phi(x)) \to 0$ for all $g \in \ell^1$ giving $\phi(x) \to 0$ in $\sigma(\ell^\infty, \ell^1)$.

Now $e = \Sigma_1^\infty e_n$ in $\sigma(A, A_*)$. Hence for all $f \in A_*$, $(f(e_n) \in \ell^1$ and $f(1) = \Sigma_1^\infty f(e_n)$. This we use to show that ϕ is onto ℓ^∞. Let $y \in \ell^\infty$, $y = (y_n)$. Let $e'_n = (\delta_{nm})$, $y^{(k)} = \Sigma_1^k y_n e'_n$. Then for all $f \in A_*$, $k \geqslant m$, $|f(x^{(k)} - x^{(m)})| \leqslant |\Sigma_{m+1}^\infty |y_n|\ |f(e_n)| \leqslant \|y\|_\infty\ |\Sigma_{m+1}^k f(e_n)| \to 0$ as $k, m \to \infty$. Thus $(x^{(k)})$ is $\sigma(A, A_*)$ Cauchy, hence $x^{(k)} \to x$ in A in $\sigma(A, A_*)$. Therefore $\phi(x) = \lim_k \phi(x^{(k)})$ (in $\sigma(\ell^\infty, \ell^1)) = y$.

It follows, by the open mapping theorem, that ϕ is $\|\cdot\| - \|\cdot\|_\infty$ homeomorphism. To show that ϕ is $\sigma(\ell^\infty, \ell^1) - \sigma(A, A_*)$ continuous, it suffices to show $\phi^*(\ell^1) = A_*, \varphi^*: (\ell^\infty)^* \to A^*$ being the conjugate of ϕ between the respective norm duals, $\phi^*(g) = g \circ \phi$. We have already shown $\phi^*(\ell^1) \subset A_*$. Let $f \in A_*$, let $y_n = f(e_n)$. Then $y = (y_n) \in \ell^1$ and for all n, $\langle f, e_n \rangle = y_n = \langle y_n, e'_n \rangle = \langle y, \phi(e_n) \rangle = \langle \phi^*(y), e_n \rangle$, and so, for all $x \in A$, $\langle f, x \rangle = \langle f, \Sigma_1^\infty x_n e_n \rangle = \Sigma_1^\infty x_n \langle f, e_n \rangle = \Sigma_1^\infty x_n y_n = \langle \phi^*(y), x \rangle$. Thus $f = \phi^*(y) \in \phi^*(\ell^1)$ implying $\phi^*(\ell^1) = A_*$. This completes the proof.

A *uB-algebra* (uniform Banach algebra) is a Banach algebra $(A, \|\cdot\|)$ with $\|x^2\| = \|x\|^2$ for all $x \in A$.

COROLLARY 3.2.

Let A be a dual uB-algebra with identity e satisfying (i) and (ii) of Theorem 3.1. Then A is isometrically dual algebra isomorphic to ℓ^∞.

Proof. If $r(\cdot)$ denotes spectral radius of an element, then for all $x \in A$, $\|x\| = \limsup_{n\to\infty} \|x^n\|^{1/n} = r(x) = r(\phi(x))$ (by Theorem 3.1) $= \|\phi(x)\|_{\infty}$, ℓ^{∞} being a C^*-algebra.

Theorem 3.3. *A uB-algebra $(A, \|\cdot\|)$ with an orthogonal basis (e_n) is isometrically isomorphic to the Banach algebra c_0.*

Proof. A cannot have identity by [8, p. 346], and it is well known that the uniform condition on A forces A to be commutative. Gelfand theory provides a homomorphism $\phi: A \to C_0(\chi(A))$ of A into a subalgebra of the supnorm Banach algebra $C_0(\chi(A))$ of continuous functions on the (locally compact) Gelfand space $\chi(A)$ vanishing at infinity. Further, by the uniform condition, the Gelfand map ϕ is an isometry. Thus ϕ is a closed subalgebra of $C_0(\chi(A))$. But by [8], $\chi(A) = \mathbb{N}$. Hence $C_0(\chi(A)) = c_0$. As A has an orthogonal basis, the sequence algebra φ (sequences with only finitely many non-zero terms), which is already dense in c_0, is dense in $\phi(A)$. Thus $\phi(A) = c_0$, and the proof is complete.

A *Hilbert algebra A* with a biorthonormal basis (e_n) is a Hilbert space A which is also a Banach algebra with orthogonal basis (e_n) such that (e_n) is an orthonormal family. It can be shown, as above, that a Hilbert algebra with a biorthonormal basis is isometrically isomorphic to ℓ^2.

4. A characterization of nuclearity: Abstract Grothendieck-Pietsch criterion

DEFINITION 4.1.

Let A be a topological algebra with an orthogonal basis (e_n).

(a) (i) For $1 \leqslant p < \infty$, let $K_p(A) = \{x \in A \mid (e_n^*(xy)) \in l^p \text{ for all } y \in A\}$; (ii) $J(A) = \{x \in A \mid e_n^*(xy) \to 0$ as $n \to \infty$ for all $y \in A\}$ (iii) $J_c(A) = \{x \in A \mid e_n^*(xy)$ is convergent for all $y \in A\}$.
(b) The natural topologies on these subsets of A are defined as follows (i) For $1 \leqslant p < \infty, \tau_A^p$ is the locally convex topology on $K_p(A)$ defined by semi-norms $\{p_y^p \mid y \in A\}$, $p_y^p(x) = (\Sigma |e_n^*(xy)|^p)^{1/p}$.

On $K_\infty(A)$, the topology τ_A^∞ is defined by seminorms $p_y^\infty(x) = \sup|e_n^*(xy)|, y \in A$; (ii) On $J(A)$ and $J_c(A)$, we consider the relative topology from $(K_\infty(A), \tau_A^\infty)$.

The sets $K_1(A)$ and $J(A)$ are considered in [9] wherein the problem of equality $K_1(A) = J(A)$ is discussed. The following is essentially due to [9]. Our presentation clarifies configurations of $K_p(A)$, $J_c(A)$ and $J(A)$ in A; also part (*e*) refines [9, Prop. 6].

PROPOSITION 4.1.

Let A be a topological algebra with an orthogonal basis (e_n).

(a) *Each of $K_p(A)$, $1 \leqslant p < \infty$, $J(A)$ and $K_\infty(A)$ is an ideal in A.*
(b) *$K_1(A)$ is the intersection of dense maximal ideals in A if (e_n) is unconditional*
(c) *For $1 \leqslant p < \infty$, $\overline{K_p(A)} = \overline{J(A)} = \overline{J_c(A)} = A$*
(d) *Let A have identity e and (e_n) be unconditional. Via the mapping $\phi: x \in A \to (e_n^*(x)) \in \omega$ which is one-one, $K_p(A)$ is a subalgebra of $\ell^p (1 \leqslant p \leqslant \infty)$, $J_c(A)$ is a subalgebra of c_0 and $J(A)$ is a subalgebra of ℓ^∞. The horizontal and vertical single arrows in the following define continuous embedding when the spaces involved carry their natural*

topologies.

$$\begin{array}{ccccccccccc}
\ell^1 & \xrightarrow{id} & \ell^p & \longrightarrow & c_0 & \longrightarrow & c & \longrightarrow & \ell^\infty & \cdots & \\
\phi\uparrow & & \phi\uparrow & & \phi\uparrow & & \phi\uparrow & & \phi\uparrow & & \phi^{-1} \\
K_1(A) & \underset{id}{\rightarrow} & K_p(A) & \rightarrow & J(A) & \rightarrow & J_c(A) & \rightarrow & K_\infty(A) & \cdots\cdots & \rightarrow A
\end{array}$$

(e) *Let A have identity e.*

(i) *If there exists an invertible element x in A such that $(e_n^*(x))\in\ell^p (1 \leqslant p < \infty)$, then $K_p(A) = J(A) = J_c(A) = K_\infty(A)$.*

(ii) *If there exists an invertible element x in A such that $e_n^*(x)\rightarrow 0$, then $J(A) = J_c(A) = K_\infty(A)$.*

(f) *Let A be a B_0-algebra. Then its dual A^* is linearly homeomorphic to $K_1(A)$.*

The algebra A of Example 2.8, as shown in [9], is such that $K_1(A) \neq J(A)$. Incidentally, this A is not nuclear. On the other hand, it is also noted in [9] that for the Frechet algebras ω and $H(D)$ (holomorphic functions on open unit disc with Hadamard product—see Example 5.7) [8], $K_1(A) = J(A)$; and both ω and $H(D)$ are nuclear. The following explains this.

Theorem 4.2. *Let A be a B_0-algebra with identity e and having an orthogonal basis (e_n).*
(a) *If A is nuclear, then for $1 \leqslant p < \infty$, $K_1(A) = K_p(A) = J(A) = J_c(A) = K_\infty(A)$ algebraically as well as topologically.*
(b) *Let (e_n) be unconditional, if $K_1(A) = J(A)$ algebraically as well as topologically, then A is nuclear.*

Proof. A being B_0, multiplication in A is jointly continuous and (e_n) is equicontinuous. The nuclearity implies [15, Th. 10.21] that (e_n) is absolute. By Theorem 2.4, $\phi(x) = (e_n^*(x))$ establishes a homeomorphic isomorphism from A onto the semi-G_1-Köthe algebra $\lambda_1^0(P)$ (denoted by $\lambda_1(P)$ only) with $P = \{(p_\alpha(e_n))|\alpha\in\mathbb{N}\}$, $\Gamma = \{p_\alpha|\alpha\in\mathbb{N}\}$ being a countable family of seminorms determining the topology on A. Further, since A has identity, $P \subset \ell^1$ and so $\lambda_1(P)^\times \subset \ell^1 \subset \ell^\infty \subset \lambda_1(P)$. Since each e_n^* is multiplicative, $K_1(A) = \{x\in A|\Sigma|e_n^*(x)|\,|e_n^*(y)| < \infty \text{ for all } y\in A\}$. This, with $\phi(A) = \lambda_1(P)$ implies that $K_1(A) \subset \phi^{-1}(\lambda_1(P))$. We claim that $\phi(K_1(A)) = \lambda_1(P)^\times$. Indeed, let $b = (b_n)\in\lambda_1(P)^\times$, $\Sigma|b_n|\,|e_n^*(y)| < \infty$ for all $y\in A$. Then $b\in\ell^1 \subset \ell^\infty$. As (e_n) is absolute, it is unconditional, hence a bounded multiplier basis, with the result, as noted in [17, Lemma 1], given $c = (c_n)\in\ell^\infty$, there exists an $x\in A$ such that $x = \Sigma_1^\infty c_n x_n$, hence $e_n^*(x) = c_n$ for all n. Thus, $b = \phi(x)$ for some $x\in A$. This implies $x\in K_1(A)$ and $(K_1(A), \tau_A^1)$ is homeomorphically isomorphic to $(\lambda_1(P)^\times, \tau_P^\times)$.

Now as A is nuclear and B_0, $(\lambda_1(P), \tau_P)$ is also nuclear and B_0. But then, as is well known, $(\lambda_1(P)^\times, \tau_P^\times)$, and hence $(K_1(A), \tau_A^1)$ is also nuclear. Therefore by the Grothendieck-Pietsch nuclearity criteria the topology τ_A^1 on $K_1(A)$ is also determined by the seminorms $p_y^\infty(x) = \sup\{|e_n^*(x)\,e_n^*(y)|\}$, $y\in A$. Thus via ϕ and ϕ^{-1}, $K_1(A) = \{u = (u_n)\in\omega|(|u_n|e_n^*(x))\in\ell^\infty\} \supset K_\infty(A) \supset J(A) \supset K_1(A)$. This proves (a).

(b) Assume that $K_1(A) = J(A)$ topologically also. Since A has identity $e = \Sigma_1^\infty e_n$, we have $(p_\alpha(e_n))\in c_0$ for all α. As (e_n) is unconditional, there is a $z\in A$ such that $e_n^*(z) = p_\alpha(e_n)$.

Further for any $x \in A$, $x = \Sigma_1^\infty e_n^*(x)e_n$ implies that $|e_n^*(xz)| = |e_n^*(x)|p_\alpha(e_n) = p_\alpha(e_n^*(x)e_n) \to 0$ as $n \to \infty$. Thus $(p_\alpha(e_n)) = (e_n^*(z)) \in J(A) = K_1(A) \subset \ell^1$ via ϕ by Proposition 4.1. Now since (e_n) is equicontinuous, given α, there is a β and a constant $k = k_\alpha$ such that $|e_n^*(x)|p_\alpha(e_n) \leqslant kp_\beta(x)$ for all $x \in A$ [15, p. 172]. This gives, for all $x \in A$, $\Sigma|e_n^*(x)|p_\alpha(e_n) \leqslant kp_\beta(x)\Sigma p_\alpha(e_n) = kMp_\beta(x)$ (say) showing that (e_n) is absolute. It follows, by Theorem 2.1, that ϕ establishes a homeomorphic isomorphism between A and $\lambda_1(P)$, $P = \{(p_\alpha(e_n)) | \alpha \in N\}$. But then $(K_1(A), \tau_A^1)$ is homeomorphically isomorphic to $(\lambda_1(P))^\times, \tau_P^\times)$. Now the assumption implies that τ_A^1 is also determined by the seminorms $\{p_y^\infty | y \in A\}$. It follows that $(K_1(A), \tau_A^1)$ and so $(\lambda_1(P)^\times, \tau_P^\times)$, is nuclear. Metrizability of $\lambda_1(P)$ implies that $(\lambda_1(P), \tau_P)$, and so A, is nuclear. This completes the proof.

5. Examples

In this section, we point out several examples of topological algebras with basis. Our point of view is that almost all sequence spaces discussed in the literature are sequence algebras, which via Theorem 2.4, provide framework for various convolution algebras of functions and distributions. Throughout, P denotes a power set. We omit easy proofs.

(5.1) Consider the cross-dual $\lambda(P)^\times = \{x \in \omega | (x_n y_n) \in \ell^1$ for all $y \in \lambda(P)\}$ with the cross-dual topology $\tau_P^\times$ determined by seminorms $\{p_x | x \in \lambda(P)\}$, $p_x(y) = \|xy\|_1$. If $\lambda_1(P)$ is nuclear, then $(\lambda_1(P)^\times, \tau_p^\times)$ is a topological *algebra without identity with orthogonal basis $e_n = (\delta_{nm})$ and $\lambda_1(P)^\times \subset \ell^\infty$. If $\lambda_\infty(P)$ is nuclear, then $(\lambda_\infty(P)^\times, \tau_P^\times)$ is also a topological *algebra with identity containing ℓ^∞ as a dense *subalgebra—in fact, it is a GB^*-algebra. By [4, p. 91], for a barrelled $\lambda(P)$, the dual $\lambda(P)^* = \lambda(P)^\times = \{x \in \omega |$ there exists $a \in P$ and $0 < \rho_x < \infty$ with $|x_n| \leqslant \rho_x a_n$ for all $n\} = \cup \{N_a | a \in P\}$, where $N_a = \{x \in \lambda(P)^\times | \sup(|x_n|/a_n) < \infty\}$, is a Banach space with norm $\rho_a(x) = \inf\{\lambda > 0 | |x_n| \leqslant \lambda a_n$ for all $n\}$. Let ρ be the strongest locally convex linear topology making each embedding $id_a: (N_a, \rho_a) \to \lambda(P)^\times$ continuous. For a nuclear $\lambda_1(P)$, N_a turns out to be an ideal; and for a with $a_n \leqslant 1$ for all n, (N_a, ρ_a) is a Banach *algebra. Using [20], other useful topologies can also be defined on $\lambda(P)^\times$.

(5.2) We consider non-nuclear variants of Köthe sequence algebras. For $1 \leqslant p \leqslant \infty$, let $\ell^p(P) = \{x \in \omega | (x_n a_n) \in \ell^p$ for all $a \in P\}$ with topology τ_P^p defined by semi-norms $\{p_a^p | a \in P\}$, $p_a^p(x) = \|xa\|_p$. Then $(\ell^p(P), \tau_P^p)$ is a complete topological *algebra with jointly continuous multiplication; and for $1 \leqslant p < \infty$, $e_n = (\delta_{nm})$ is an equicontinuous orthogonal basis for $\ell^p(P)$. If $P \subset \ell^\infty$, then $\ell^p(P)$ contains ℓ^p as a dense *subalgebra, $\ell^p(P)$ has identity iff $P \subset \ell^p$ iff $\ell^p(P) = \ell^\infty(P)$, and $\ell^\infty(P)$ is a GB^*-algebra. In general, $\ell^p(P)$ is not a GB^*-algebra, though it is symmetric. Two important closed subalgebras of $\ell^\infty(P)$ are $c_0(P) = \{x \in \omega | (x_n a_n) \in c_0$ for all $a \in P\}$ and $c(P) = \{x \in \omega | (x_n a_n) \in c$ for all $a \in P\}$.

In [5], a topological algebra A with orthogonal basis (e_n) is called a *ϕ-algebra* if there exists a family F of sequences with non-negative entries defining a topology on $\phi(A) = \{(e_n^*(x)) | x \in A\}$ by seminorms $p_a((e_n^*(x))) = \sup_n |a_n e_n^*(x)|$, $a \in F$, such that the map $\phi: x \to \phi(x) = (e_n^*(x)) \in \phi(A)$ is a homeomorphism. It follows from Grothendieck-Pietsch nuclearity criteria that nuclear Köthe algebras $\lambda_1(P)$ and $\lambda_\infty(P)$ supply large classes of ϕ-algebras. As shown in Example 2.8, nuclear $\lambda_\infty(P)$ is a ϕ-algebra without identity, and $\lambda_1(P)$ is nuclear iff it has identity in which case, it is a ϕ-algebra. It follows from previous paragraph, that for $p \neq 1$ and for P countable, if $\ell^p(P)$ has identity, then it is a ϕ-algebra.

(5.3) Let P satisfy G_1-condition. For $1 \leqslant p < \infty$, let $\lambda^p(P) = \{x \in \omega | (x_n a_n^{1/p}) \in \ell^p$ for all $a \in P\}$ with topology τ^p defined by seminorms $\{p_{a,p} | a \in P\}, p_{a,p}(x) = \|xa^{1/p}\|_p$. Then $(\lambda^p(P), \tau^p)$ too is a topological algebra with $e_n = (\delta_{nm})$ as an orthogonal basis. Similarly the sequence spaces $\lambda^d(P) = \{x \in \omega | \sup_n |x_n a_n|^{1/n} < \infty$ for all $a \in P\}$ and $\lambda^\delta(P) = \{x \in \omega | (x_n a_n)^{1/n} \to 0$ for all $a \in P\}$ are topological algebras with topology defined by seminorms $\{p_{a,d} | a \in P\}$, $p_{a,d}(x) = \sup_n |x_n a_n|^{1/n}$.

(5.4) The space $s = \{x \in \omega | (n^k x_n) \in \ell^\infty$ for all $k = 1, 2, \ldots\}$ of rapidly decreasing sequences and its cross-dual $s^\times = s' = \{x \in \omega | \sup_n |x_n| n^{-m} < \infty$ for some $m\}$ of all tempered sequences are subalgebras of ω with $s \subset \ell^1 \subset \ell^\infty \subset s'$. In fact, with $P = \{(n+1)^k | k \geqslant 1\}$ a G_∞-power set, $\lambda(P) = s$ [12, p. 291], and $\lambda(P)^\times = s'$. By [12, Pro. 7.20], for an arbitrary G_1-power set P, $(\lambda_1(P), \tau_P)$ is nuclear iff $P \subset s$. Thus a nuclear $\lambda_1(P)$ always contains s' as a subalgebra. In particular, s' is a nuclear algebra with an orthogonal basis.

(5.5) For the unit circle T, the Lebesgue space $L^p(T)$, $1 < p < \infty$ is a convolution Banach algebra with orthogonal basis $e_n(t) = t^n, n \geqslant 1$ [8]. Hence the space $L^\omega(T) = \cap_{2 \leqslant p < \infty} L^p(T) = \cap_{1 < p < \infty} L^p(T) \neq L^\infty(T)$ is a convolution lmc B_0-algebra without identity in the topology of L^p-convergence for each $p = 2, 3, 4, \ldots$ having orthogonal basis (t^n). The algebra $L^\omega(T)$ is not nuclear; otherwise, the basis (e_n), which is already equicontinuous, will be absolute; and Corollary 2.6 (1) would imply that $L^\omega(T)$ has identity.

(5.6) The Frechet space $C^\infty(T)$ of all infinitely differentiable functions on T with the topology τ defined by the seminorms $p_n(f) = \sup_{t \in T} \left[\sum_{k=0}^{n} \frac{|f^{(k)}(t)|}{k!} \right]$ is a convolution *algebra with involution $f \to f^*$, $f^*(z) = \overline{f(\bar{z})}$. $(C^\infty(T), \tau)$ is a nuclear lmc B_0-algebra without identity having orthogonal basis (e_n), $e_n(x) = e^{inx}$, and is homeomorphically isomorphic to the sequence algebra (s, τ_P). Its dual $D(T)$ consisting of all distributions of T is a convolution *algebra haivng Dirac delta δ as identity and involution $u \to u^*$, $\langle u^*, f \rangle = \overline{\langle u, f^* \rangle}$ $(f \in C^\infty(T))$. With the weak topology $\sigma = \sigma(D(T), C^\infty(T))$, $D(T)$ is a locally convex GB^*-algebra with orthogonal basis (e_n), $e_n(x) = e^{inx}$, and is homeomorphically isomorphic to the sequence algebra s'.

(5.7) Let D be open unit disc. The F-space $H(D)$ of homomorphic functions on D with compact open topology is a nuclear B_0-algebra with Hadamard product

$$f * g(x) = \frac{1}{2\pi i} \int_{|z|=r} f(z) g(xz^{-1}) z^{-1} \mathrm{d}z, x \in D, |x| < r < 1,$$

with identity $e(z) = (1-z)^{-1}$ having orthogonal basis $e_n(z) = z^n, n \geqslant 1$ [8]; and hence is isomorphic to some $(\lambda_1^0(P), \tau_P)$. Its subalgebra $H^\omega(D) = \cap_{2 \leqslant p < \infty} H^p(D) \neq H^\infty(D)$ [17, Ch. 17, Ex. 10] is a Frechet algebra with topology defined by seminorms $f \to \|f\|_p$,

$2 \leqslant p < \infty$. It has orthogonal basis (z^n), since each $H^p(D)$, $1 < p < \infty$ is known to be a Banach algebra with basis (z^n) [8]. The F-space E of entire functions with compact open topology is, with Hadamard product, a nuclear B_0-algebra having orthogonal basis (z^n) and is isomorphic to $\lambda(P)$ with $P = \{(k^n) | k = 1, 2, \ldots\}$ [16]. Similarly the algebra A_0 of complex functions holomorphic at 0 is isomorphic to the cross-dual $\lambda(P)^\times$.

6. Concluding remarks

(6.1) Theorem 2.1 suggests the problem: Is a B_0-topological algebra A with an orthogonal basis (e_n) functionally continuous in the sense that every multiplicative linear functional on A is continuous (and hence of the form e_n^*)? There exists non-metrizable topological algebras with orthogonal basis admitting discontinuous multiplicative functionals; e.g. $(\ell^\infty, \sigma(\ell^\infty, \ell^1))$. This problem is a variant of Michael problem [14]. Note that if such A is an lmc B_0-algebra, then A is functionally continuous [22]. If A is a sequentially complete nuclear algebra with 1 and with jointly continuous multiplication, then by Theorem 2.4 and the first paragraph in §2, there is an involution on A making A a GB^*-algebra, so that every multiplicative functional f is positive. Then by [3, §8], f is bounded. Thus if A is a nuclear B_0-algebra with identity, then A is functionally continuous. Another large class of functionally continuous topological algebras with basis is supplied by the following.

For a countable G_1-power set P satisfying $0 \neq a_n \leqslant 1$ for all n, assume $\lambda_1(P)$ to be B_0 and nuclear. Let τ' be the algebraic inductive limit topology on $\lambda_1(P)^\times$, in the sense of [20], defined by the embedding id_a: $(N_a, \|\cdot\|_\infty) \to \lambda_1(P)^\times$, $\|\cdot\|_\infty$ being the l^∞-norm on N_a (defined in Example 5.1). Then $(N_a, \|\cdot\|_\infty)$ is a Q-algebra (i.e. quasi-regular elements form an open set). It follows from [20, Prop. 1] that $(\lambda_1(P)^\times, \tau')$ is a functionally continuous lmc algebra. Since $\tau' \leqslant \tau_P^\times$, $(\lambda_1(P)^\times, \tau_P^\times)$ is functionally continuous.

(6.2) A problem closely related with above is: let (A, τ) be a B_0-algebra with orthogonal basis (e_n). Let τ' be a topology on A making (A, τ') a B_0-algebra. Is $\tau = \tau'$? Note that a topological algebra with orthogonal basis is semi-simple [8]. Hence by standard Banach algebra theory, if both τ and τ' are normable, then $\tau = \tau'$. More generally, if τ is also lmc, then (A, τ) is functionally continuous; and hence by [14, Th. 14.2], $\tau = \tau'$ if τ' is also lmc. Also, assume that each e_n^* is τ'-continuous; in particular, τ' is functionally continuous, or (e_n) is a basis for (A, τ') also. Then via the map $\phi(x) = (e_n^*(x))$ which is one–one, τ and τ' are FK-topologies on (the sequence space) A. It follows from [18, Ch. 3, Th. 2.1], that $\tau = \tau'$.

(6.3) A commutative separable semisimple nuclear B_0-algebra A need not necessarily admit an orthogonal basis, even if A is lmc and having identity element. Consider the nuclear F-space $H(D)$ of all holomorphic functions on open unit disc D with compact open topology. With pointwise multiplication, $H(D)$ is an lmc B_0-algebra with identity. It is separable by Example (5.7), and is known to be semisimple. If it admits an orthogonal basis (e_n), then by [5, Cor. 1.3] its Gelfand space is (e_n^*), with the result by [14], each $f \in H(D)$ must have a countable spectrum $\sigma(f) = \{e_n^*(f)\}$. On the other hand, $\sigma(f)$ is, in general, uncountable; since it contains range of f. Thus it would be interesting to characterize topological algebras that admit orthogonal basis.

Acknowledgement

We are thankful to Prof. M H Vasavada for discussions, Prof. Taqdir Husain for making available reprints of his papers and the referee for some suggestions.

References

[1] Allan G R, On a class of locally convex algebras, *Proc. London Math. Soc.* **17** (1967) 91–114
[2] Bhatt S J and Karia D J, On an intrinsic characterization of pro-C^*-algebras (communicated)
[3] Dixen P G, Generalized B^*-algebras, *Proc. London Math. Soc.* **23** (1970) 639–715
[4] Dubinsky E and Ramanujan M S, On λ-nuclearity, *Mem. Am. Math. Soc.* **128** (1972)
[5] Helaly S El and Husain T, Orthogonal bases are Schauder and a characterization of ϕ-algebras, *Pacific J. Math.* **132** (1988) 265–275
[6] Guicherdet, *Special topics in topological algebras*, (New York: Gordon and Beach Science Publishers) (1971)
[7] Husain T and Liang J, Multiplicative functionals on Frechet algebras with basis, *Can. J. Math.* **29** (1977) 270–276
[8] Husain T and Watson S, Topological algebras with orthogonal Schauder basis, *Pacific J. Math.* **91** (1980) 339–347
[9] Husain T and Watson S, Algebras with unconditional orthogonal basis, *Proc. Am. Math. Soc.* **79** (1980) 539–545
[10] Husain T, Orthogonal primitive idempotents and Banach algebras isomorphic to l_2, *Pacific J. Math.* **117** (1985) 313–327
[11] Husain T, Topological algebra isomorphic to l_1 or $H(D)$, *Panjab Uni. Math. J.* (Lahore, Pakistan) XIV-XV (1981–82) 1–14
[12] Kamthan P K and Gupta M, *Sequence space and Series* (New York and Basel: Marcel Dekker Inc.) (1981)
[13] Köthe G, *Topological vector spaces I* (Berlin, Heidelberg, New York: Springer Verlag) (1969)
[14] Michael E, Locally multiplicatively convex topological algebras, *Mem. Am. Math. Soc.* **10** (1952)
[15] Pietsch A, *Nuclear locally convex spaces* (Berlin, Heidelberg, New York: Springer Verlag) (1972)
[16] Ruckle W H, *Sequence spaces* (Pitman Adv. Publ. Co.) (1981)
[17] Rudin W, *Real and complex analysis* (McGraw Hill) (1972)
[18] Rudin W, *Functional analysis* (McGraw Hill) (1973)
[19] Schaefer H H, *Topological vector spaces* (McMillan) (1964)
[20] Warner S, Inductive limits of normed algebras, *Trans. Am. Math. Soc.* **82** (1956) 190–216
[21] Zelazko W, *Selected topics in topological algebras* (Aarhus Univ. Lect. Notes in Math.) 31 (1971)
[22] Zelazko W, Functional continuity of commutative m-convex B_0-algebra with countable maximal ideal space, *Colloq. Math.* **51** (1987) 395–399