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Effect of a viscous fluid flow past a spherical gas bubble on the growth of its radius

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Abstract. A nearly spherical gas bubble expands adiabatically in a viscous incompressible fluid flowing past it. The Rayleigh-Plesset formula for the growth of the bubble radius is modified due to the flow of the viscous fluid.

Keywords. Rayleigh-Plesset formula; gas bubble.

1. Introduction

Rayleigh ([4]) obtained the basic differential equation, for the growth of the radius $R(t)$ with time t, of a gas bubble that expands adiabatically in an incompressible and inviscid fluid. Plesset ([6]) modified Rayleigh's equation taking into account the viscosity of the fluid surrounding the gas bubble.

The modification of Rayleigh's formula for the growth of the radius $R(t)$ with time t, due to the flow of an inviscid, incompressible fluid past a nearly spherical gas bubble, has been recently obtained ([3]). In this note we obtain the modification of Rayleigh's formula taking into account the viscosity, as well as the flow, of a fluid past the expanding gas bubble. Our result is thus a modification of the Rayleigh-Plesset formula, which is obtained when one considers the flow of a viscous fluid past an expanding gas bubble.

Ceschia and Nabergoj [2] have considered the motion of a nearly spherical gas bubble in a viscous liquid using the Lagrangian procedure and obtained the Rayleigh-Plesset formula. Adopting the same procedure, we shall give a straightforward derivation of the modification of Rayleigh-Plesset formula due to the flow of the viscous fluid surrounding the gas bubble. In this straightforward derivation we neglect the deviation of the gas bubble shape from the spherical form, although, as in the inviscid case [3], in the case of the viscous flow past an expanding gas bubble also, the shape of a gas bubble always deviates from the spherical form. Such a deviation is however small, when the radius of the bubble is sufficiently small [1]. The straightforward derivation will thus give us accurate result for a gas bubble of sufficiently small radius, rising freely in a viscous fluid.

2. Analysis of the problem

We consider an external, incompressible and viscous fluid of density ρ and coefficient of viscosity μ flowing past a spherical gas bubble expanding adibatically. The initial radius of the bubble is R_0. The fluid outside the bubble flows past it and has a uniform velocity U at a large distance from it.

As we consider the fluid flowing past the bubble to have a small viscosity, we take the flow to be irrotational, the fluid velocity being derived from a velocity potential function ϕ. Such an irrotational flow will give us a true description of the flow throughout the flow region except in the boundary layer ([2]) surrounding the bubble surface. The dissipation function F([4] p. 581) is given by

$$F = \frac{\mu}{2}\int (\text{curl}\,\mathbf{u})^2\,\mathrm{d}v + \frac{\mu}{2}\int \nabla \mathbf{u}^2\cdot\mathrm{d}\mathbf{s} - \mu\int \mathbf{u}\times\text{curl}\,\mathbf{u}\cdot\mathrm{d}\mathbf{s} \tag{1}$$

where the first term on the right side of (1) is a volume integral taken over the whole of the fluid region outside the bubble, and the other terms are surface integrals taken over the surface area A of the bubble surface. The direction of dS is along outward normal to the surface. As argued by Ceshia and Naborgoj, since the contribution of the transport of vorticity to the dissipation function F given by (1) can be shown to be small, the existence of the boundary layer in the present problem may be neglected ([1]).

The velocity potential function in the liquid flowing past the bubble, in a spherical polar system of coordinates in which the centre of the bubble is taken to be the origin and at rest, is given by

$$\phi = U\cos\theta(r + R^3/(2r^2)) + R^2\dot{R}/r. \tag{2}$$

In (2), the term $U\cos\theta(r + R^3/(2r^2))$ represents the velocity potential of the fluid flow past a rigid sphere of radius R when the flow at infinity is of uniform velocity $-U$ along the axis of symmetry and the term $R^2\dot{R}/r$ represents the contribution due to the change in $R(t)$ with time t. In our analysis we find it more convenient to use a coordinate system in which the fluid at infinity is at rest, so that the potential functions now takes the form [7]

$$\phi = UR^3\cos\theta/(2r^2) + \dot{R}R^2/r. \tag{3}$$

The kinetic energy of the fluid flow outside the bubble is given by

$$T = 1/2\rho\int \mathbf{u}^2\,\mathrm{d}v \tag{4}$$

the integration in (4) is throughout the entire region outside the gas bubble. Since the fluid velocity $\mathbf{u}$ is given by

$$\mathbf{u} = -\nabla\phi. \tag{5}$$

we have from (4), in view of (3), the result

$$T = \pi\rho R^3(2\dot{R}^2 + U^2/3). \tag{6}$$

T, as given by (6), will be taken to be the kinetic energy of the whole system, since, in calculating the kinetic energy we shall neglect the contribution of the motion of the gas and the vapour in the bubble.

The potential energy of the system is

$$V = -\int (p_i - p_o)\,\mathrm{d}v + \sigma \int_A \mathrm{d}A \tag{7}$$

where v is the volume of the gas bubble and $p_e(t)$ is the pressure exerted by the liquid at a large distance from the bubble, and p_i is pressure inside the bubble and σ is the surface tension acting on the bubble surface. As shown by Ceschia and Nabergoj [2], we can write

$$V = \frac{4\pi R^3}{3}\left\{p_e(t) - \frac{p_{g0}}{(\gamma-1)}(R_0/R)^{3\gamma} + 3\sigma/R\right\}, \tag{8}$$

where we neglect the vapour pressure, and p_{g_0} is the initial gas pressure in the bubble when its radius is R_0. γ is the ratio of the two specific heats of the gas.

Since curl $\mathbf{u} = 0$ for the flow considered by us, the dissipation function F, in view of (1), (3) and (5), becomes

$$F = 2\pi\mu R(4\dot{R}^2 + 3U^2). \tag{9}$$

Defining the Lagrangian function L as

$$L = T - V, \tag{10}$$

and taking $R(t)$ and z, where z is the height of the centre of the bubble above a fixed horizontal plane, as the only independent coordinates in our problem, the Lagrangian equation becomes ([5])

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{R}}\right) - \frac{\partial L}{\partial R} = -\frac{\partial F}{\partial \dot{R}} \tag{11}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial \dot{z}} = -\frac{\partial F}{\partial \dot{z}}. \tag{12}$$

Equation (11), in view of (6), and (8)–(10), finally gives us

$$R\ddot{R} + 3\dot{R}^2/2 - U^2/4 + \frac{1}{\rho}\left\{p_e - p_{g0}(R_0/R)^{3\gamma} + \frac{2\sigma}{R} + 4\mu R/R\right\} = 0 \tag{13}$$

while (12) similarly gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(UR^3) = -18\mu RU/\rho, \tag{14}$$

we have used the result $\dot{z} = U$ in deriving (13) and (14).

Equation (13) gives us the modification of Rayleigh's equation for the growth of $R(t)$, the radius of a spherical gas bubble expanding adiabatically in fluid, when we take into account the viscosity ($\mu \neq 0$) and the flow ($U \neq 0$) of the fluid surrounding the bubble. Equation (13) gives us Rayleigh–Plesset equation ([6]) if we neglect the fluid flow ($U = 0$). If we take $\mu = 0$ in (13), we obtain the modification ([3]) of Rayleigh's original formula due to the flow of the inviscid fluid past the bubble.

In the absence of viscosity ($\mu = 0$), (14) shows that

$$UR^3 = k, \tag{15}$$

where k is a constant. Equation (15) has a simple interpretation. As the bubble rises in the liquid, and if the liquid displaced by the bubble were to move with the velocity of the bubble, momentum of the displaced bubble will be conserved.

Finally, in the absence of viscosity ($\mu = 0$), (13), in view of (15), takes the form

$$\ddot{R} + 3\dot{R}^2/(2R) - \frac{k^2}{4R^7} + \frac{1}{\rho R}\left\{p_e - p_{g_0}(R_0/R)^{3\gamma} + \frac{2\sigma}{R}\right\} = 0. \tag{16}$$

Equation (16) shows, as it is apparent from the term $-k^2/4R^7$, the effect of fluid flow on the growth of bubble radius will be expected to be significant when the radius of the bubble is small.

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