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Existence of solution for nonlinear Volterra integral equations

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Abstract. We prove an existence theorem for a class of nonlinear Volterra integral equations.

Keywords. Nonlinear Volterra integral equation; deviating arguments.

1. Introduction

The theory of nonlinear Volterra integral equations with deviating arguments and functional integral equations have been studied by many authors [1, 2, 5, 7]. Banaś[4] has proved an existence theorem for functional integral equation and Balachandran[1] has proved an existence theorem for a nonlinear Volterra integral equation with deviating argument. In this paper we shall derive a set of sufficient conditions for the existence of a solution of nonlinear Volterra integral equations with the kernel containing a continuous operator.

2. Basic assumptions

Let $p(t)$ be a given continuous function defined on the interval $[0, \infty)$ and taking real positive values. Denote by $C_p = C([0, \infty), p(t): R^n)$ the set of all continuous functions from $[0, \infty)$ into R^n such that

$$\sup\{|x(t)|p(t): t \geqslant 0\} < \infty.$$

It has been proved [6] that C_p forms a real Banach space with regard to the norm

$$\|x\| = \sup\{|x(t)|p(t): t \geqslant 0\}.$$

If $x \in C_p$ then we will denote $\omega^T(x, \varepsilon)$ the usual modulus of continuity of x on the interval $[0, T]$ i.e.,

$$\omega^T(x, \varepsilon) = \sup\{|x(t) - x(s)|: |t - s| \leqslant \varepsilon, t, s \in [0, T]\}.$$

Our existence theorem is based on the following lemma.

Lemma 3. *Let E be a bounded set in the space C_p. If all functions belonging to E are equicontinuous on each interval* $[0, T]$ *and* $\lim_{T \to \infty} \sup\{|x(t)|p(t): t \geqslant T\} = 0$ *uniformly with respect to E, then E is relatively compact in C_p.*

Consider the nonlinear Volterra integral equation of the form

$$x(t) = G(x)g(t, x(h_1(t)). x(h_2(t)), \ldots, x(h_m(t))) + \int_0^t K(t, s, Tx(s))\,ds \qquad (1)$$

where x and K are n-vectors and G is a real-valued function and $T: R^n \to R^n$ is a continuous operator. Assume the following conditions:

(i). Let $\Delta = \{(t,s): 0 \leqslant s \leqslant t < \infty\}$.

The kernel $K: \Delta \times R^n \to R^n$ is continuous and there exist continuous functions $m: \Delta \to [0, \infty), a: [0, \infty) \to (0, \infty)$ $b: [0, \infty) \to [0, \infty)$ such that

$$|K(t, s, x)| \leqslant m(t, s) + a(t)b(s)|x|$$

for all $(t, s) \in \Delta$ and $x \in R^n$.

In order to formulate other assumptions let us define

$$L(t) = \int_0^t a(s)b(s)\,ds, \quad t \geqslant 0.$$

Take an arbitrary number $M > 0$ and consider the space C_p with $p(t) = [a(t)\exp(ML(t) + t)]^{-1}$.

(ii) there exists a constant $A > 0$ such that for any $t \in [0, \infty)$ the following inequality holds

$$\int_0^t m(t, s)\,ds \leqslant Aa(t)\exp[ML(t)]$$

(iii) for $i = 1, 2, \ldots, m$ the functions $h_i: [0, \infty) \to [0, \infty)$ are continuous, $h_i(t) \leqslant t$ for $t \geqslant 0$ and there exists a positive real number B_i such that $a(h_i(t)) \leqslant B_i a(t)$,

(iv) $G: C_p \to [0, \infty)$ is continuous and bounded.

Assume $|G(x)| \leqslant k_1$ where k_1 is a positive constant.

(v) the function $g: [0, \infty) \times R^{mn} \to R^n$ is continuous and satisfies the condition

$$|g(t, x_1, \ldots, x_m) - g(t, y_1, \ldots, y_m)| \leqslant \sum_{i=1}^m \alpha_i(t)|x_i - y_i|,$$

where $\alpha_i(t)$ is continuous and such that

$$\alpha_i(t) \leqslant \exp\{M[L(t) - L(h_i(t))]\} \text{ for } t \geqslant 0, \quad \text{for } i = 1, 2, \ldots, m$$

and $|g(t, 0, \ldots, 0)| \leqslant a(t)\exp(ML(t))$,

(vi) the function $T: R^n \to R^n$ is continuous and

$$|Tx(t)| \leqslant a(t)\int_0^t b(s)|x(s)|\,ds,$$

(vii) Assume that $k_1(1 + B) + A + (1/M^2) < 1$ where $B = \sum_{i=1}^m B_i$.

3. Existence theorem

Theorem. *Assume that the hypotheses (i) to (vii) hold; then eq. (1) has at least one solutions x in the space C_p such that $|x(t)| \leqslant a(t)\exp[ML(t)]$ for any $t \geqslant 0$.*

Proof. Define a transformation F in the space C_p by

$$(Fx)(t) = G(x)g(t, x(h_1(t)), \ldots, x(h_m(t))) + \int_0^t K(t, s, Tx(s))\,\mathrm{d}s. \tag{2}$$

From our assumptions we observe that $(Fx)(t)$ is continuous on the interval $[0, \infty)$. Define the set E in C_p by

$$E = \{x \in C_p : |x(t)| \leqslant a(t)\exp[ML(t)]\}.$$

Clearly E is nonempty, bounded, convex and closed in C_p. Now we prove that F maps the set E into itself. Take $x \in E$. Then from our assumptions we have

$$\begin{aligned}
|(Fx)(t)| &\leqslant |G(x)|\,|g(t, x(h_1(t)), \ldots, x(h_m(t)))| + \int_0^t |K(t, s, Tx(s))|\,\mathrm{d}s \\
&\leqslant k_1 |g(t, x(h_1(t)), \ldots, x(h_m(t))) - g(t, 0, \ldots, 0)| \\
&\quad + k_1 |g(t, 0, \ldots, 0)| + \int_0^t m(t, s)\,\mathrm{d}s + a(t)\int_0^t b(s)|Tx(s)|\,\mathrm{d}s \\
&\leqslant k_1 \sum_{i=1}^m \alpha_i(t)|x(h_i(t))| + k_1 a(t)\exp[ML(t)] + Aa(t)\exp[ML(t)] \\
&\quad + a(t)\int_0^t b(s)\left[a(s)\int_0^s b(u)|x(u)|\,\mathrm{d}u\right]\mathrm{d}s \\
&\leqslant k_1 \sum_{i=1}^m \exp[M[L(t) - L(h_i(t))]]\,a(h_i(t))\exp[ML(h_i(t))] \\
&\quad + k_1 a(t)\exp[ML(t)] + Aa(t)\exp[ML(t)] \\
&\quad + a(t)\int_0^t b(s)\left[a(s)\int_0^s b(u)a(u)\exp[ML(u)]\,\mathrm{d}u\right]\mathrm{d}s \\
&\leqslant k_1 Ba(t)\exp[ML(t)] + k_1 a(t)\exp[ML(t)] + Aa(t)\exp[ML(t)] \\
&\quad + (1/M)a(t)\int_0^t b(s)a(s)\exp[ML(s)]\,\mathrm{d}s \\
&\leqslant [k_1(1 + B) + A + (1/M^2)]a(t)\exp[ML(t)] \\
&< a(t)\exp[ML(t)]
\end{aligned}$$

which proves that $FE \subset E$.

Now we want to prove that F is continuous on the set E. In order to do this take $F = F_1 + F_2$, where

$$(F_1x)(t) = G(x)g(t, x(h_1(t)), \ldots, x(h_m(t))),$$

$$(F_2x)(t) = \int_0^t K(t, s, Tx(s))\,\mathrm{d}s.$$

We shall prove continuity of F_1, F_2 separately. Let us fix $\varepsilon > 0$ and take $x, y \in E$ such that $\|x - y\| \leqslant \varepsilon$. We have

$$\begin{aligned}
\|F_1x - F_1y\| &\leqslant k_1 \sup\{|g(t, x(h_1(t)), \ldots, x(h_m(t))) \\
&\quad - g(t, y(h_1(t)), \ldots, y(h_m(t)))| \cdot [a(t)\exp(ML(t)+t)]^{-1}\colon t \geqslant 0\} \\
&\quad + |G(x) - G(y)| \sup\{|g(t, y(h_1(t)), \ldots, y(h_m(t)))| \\
&\quad \times [a(t)\exp(ML(t)+t]^{-1}\colon t \geqslant 0\}. \\
&\leqslant k_1 \sum_{i=1}^{m} \sup\{\alpha_i(t)|x(h_i(t)) - y(h_i(t))|[a(t)\exp(ML(t)+t)]^{-1}\colon t \geqslant 0\} \\
&\quad + |G(x) - G(y)| \sup\{|g(t, y(h_1(t)), \ldots, y(h_m(t))) - g(t, 0, \ldots, 0)| \\
&\quad \times [a(t)\exp(ML(t)+t)]^{-1}\colon t \geqslant 0\} \\
&\quad + |G(x) - G(y)| \sup\{|g(t, 0, \ldots, 0)| \\
&\quad \times [a(t)\exp(ML(t)+t)]^{-1}\colon t \geqslant 0\}. \\
&\leqslant k_1 \sum_{i=1}^{m} B_i \sup\{|x(h_i(t)) - y(h_i(t))|[a(h_i(t))]^{-1} \\
&\quad \times \exp(M[L(t) - L(h_i(t))])\exp(-ML(t) - t)\colon t \geqslant 0\} \\
&\quad + |G(x) - G(y)| \sum_{i=1}^{m} \sup\{\alpha_i(t)|y(h_i(t))| \\
&\quad \times [a(t)\exp(ML(t)+t)]^{-1}\colon t \geqslant 0\} \\
&\quad + |G(x) - G(y)| \sup\{\exp(-t)\colon t \geqslant 0\} \\
&\leqslant k_1 \sum_{i=1}^{m} B_i \sup\{|x(h_i(t)) - y(h_i(t))| \\
&\quad \times [a(h_i(t))\exp(ML(h_i(t)) + h_i(t))]^{-1}\colon t \geqslant 0\} \\
&\quad + |G(x) - G(y)| \sum_{i=1}^{m} \sup\{a(h_i(t))\exp(ML(t)) \\
&\quad \times [a(t)\exp(ML(t)+t)]^{-1}\colon t \geqslant 0\} + |G(x) - G(y)| \\
&\leqslant k_1 B\|x - y\| + (B+1)|G(x) - G(y)|.
\end{aligned}$$

this implies that F_1 is continuous in view of (iv).

Now we prove that F_2 is continuous on the set E. For this let us fix $\varepsilon > 0$ and x, $y \in E$ such that $\|x - y\| \leqslant \varepsilon$. Further, take an arbitrary fixed $T > 0$. In view of (i) the function $K(t, s, x)$ is uniformly continuous on

$$[0, T] \times [0, T] \times [-r(T), r(T)]^n$$

where $r(T)=\max\{a(s)\exp(ML(s))\colon s\in[0,T]\}$. Thus, we have for $t\in[0,T]$

$$\begin{aligned}|(F_2x)(t)-(F_2y)(t)| &\leqslant \int_0^t |K(t,s,Tx(s))-K(t,s,Ty(s))|\,\mathrm{d}s\\ &\leqslant \beta(\varepsilon)\end{aligned} \tag{3}$$

where $\beta(\varepsilon)$ is some continuous function such that $\lim_{\varepsilon\to 0}\beta(\varepsilon)=0$. Further, let us take $t\geqslant T$. Then we have

$$\begin{aligned}|(F_2x)(t)-(F_2y)(t)| &\leqslant |(F_2x)(t)|+|(F_2y)(t)|\\ &\leqslant 2a(t)\exp(ML(t)).\end{aligned}$$

Consequently

$$|(F_2x)(t)-(F_2y)(t)|p(t)\leqslant 2\exp(-t).$$

Hence for sufficiently large T we have

$$|(F_2x)(t)-(F_2y)(t)|p(t)\leqslant \varepsilon. \tag{4}$$

By (3) and (4) we get that F_2 is continuous on the set E. Hence $F=F_1+F_2$ is continuous on E.

Now we prove that FE is relatively compact. For every $x\in E$ we have $Fx\in E$ which gives $|(Fx)(t)|p(t)\leqslant\exp(-t)$. Hence $\lim_{T\to\infty}\sup\{|(Fx)(t)|p(t)\colon t\geqslant T\}=0$ uniformly with respect to $x\in E$.

Furthermore, let us fix $\varepsilon>0$, $T>0$; t, $s\in[0,T]$ such that $|t-s|\leqslant\varepsilon$. Then for $x\in E$, we have

$$\begin{aligned}|(Fx)(t)-(Fx)(s)| &\leqslant |G(x)|\,|g(t,x(h_1(t)),\dots,x(h_m(t)))\\ &\quad -g(s,x(h_1(s)),\dots,x(h_m(s)))|\\ &\quad +\left|\int_0^t K(t,u,Tx(u))\,\mathrm{d}u-\int_0^s K(s,u,Tx(u))\,\mathrm{d}u\right|\\ &\leqslant k_1\omega^T(g,\varepsilon)+\left|\int_0^t K(t,u,Tx(u))\,\mathrm{d}u-\int_0^s K(t,u,Tx(u))\,\mathrm{d}u\right|\\ &\quad +\left|\int_0^s K(t,u,Tx(u))\,\mathrm{d}u-\int_0^s K(s,u,Tx(u))\,\mathrm{d}u\right|\\ &\leqslant k_1\omega^T(g,\varepsilon)+\int_s^t |K(t,u,Tx(u))|\,\mathrm{d}u\\ &\quad +\int_0^s |K(t,u,Tx(u))-K(s,u,Tx(u))|\,\mathrm{d}u\\ &\leqslant k_1\omega^T(g,\varepsilon)+\varepsilon\max\{m(t,u)+a(t)b(u)|Tx(u)|\colon 0\leqslant u\leqslant t\leqslant T\}\\ &\quad +T\omega^T(K,\varepsilon)\end{aligned}$$

which tends to zero as $\varepsilon\to 0$. Thus FE is equicontinuous on $[0,T]$.

Therefore from the lemma FE is relatively compact. Thus the Schauder fixed point theorem guarantees that F has a fixed point $x \in E$ such that $(Fx)(t) = x(t)$. Hence the theorem.

References

[1] Balachandran K, Existence of solution for nonlinear Volterra Integral equation with deviating argument, *J. Math. Phys. Sci.* **23** (1989) 201–205

[2] Banaś J, An existence theorem for nonlinear Volterra integral equation with deviating argument, *Rend. Circ. Math. Palermo*, Series 2, **35** (1986) 82–89

[3] Banaś J and Goebel K, Measure of Noncompactness in Banach Spaces, (New York: Marcel Dekkar Inc) (1980)

[4] Banaś J, On solutions of some functional integral equations, *Bull. Inst. Math. Acad. Sin.*, **12** (1984) 23–30

[5] Burton T A, Volterra Integral and Differential Equations, (New York: Academic Press) (1983)

[6] Corduņeanu C, Integral Equations and Stability of Feedback Systems, (New York: Academic Press) (1973)

[7] Lakshmikantham V and Leela S, Differential and Integral inequalities, (New York: Academic Press) (1969) Vol. 1