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Riesz means for the sublaplacian on the Heisenberg group

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Abstract. The uniform boundedness of the Riesz means for the sublaplacian on the Heisenberg group H^n is considered. It is proved that S_R^α are uniformly bounded on $L^p(H^n)$ for $1 \leqslant p \leqslant 2$ provided $\alpha > \alpha(p) = (2n+1)[(1/p)-(1/2)]$.

Keywords. Fourier transform; representations; sublaplacian; Riesz means; projections.

1. Introduction

Consider the Heisenberg group H^n which is a Lie group whose underlying manifold is $\mathbb{R} \times C^n$ with coordinates (t, z). Its Lie algebra h^n is generated by the left invariant vector fields

$$T = \frac{\partial}{\partial t}, \quad Z_j = \frac{\partial}{\partial z_j} + i\bar{z}_j T, \quad \bar{Z}_j = \frac{\partial}{\partial \bar{z}_j} - iz_j T. \tag{1}$$

Let $L = -\frac{1}{2}\sum_{j=1}^n (Z_j\bar{Z}_j + \bar{Z}_j Z_j)$ be the sublaplacian on H^n. Then L is a formally nonnegative hypoelliptic differential operator which has a unique self adjoint extension to $L^2(H^n)$. Let $E(s)$ denote the spectral resolution of this extension which we normalize so as to become left continuous. $E(s)$ then becomes a right convolution operator with a C^∞ kernel.

In [5] Mauceri has studied the boundedness properties of the Riesz means S_R^α. These are defined for $\operatorname{Re}\alpha > 0$ by the equation

$$S_R^\alpha f = \int_0^R \left(1 - \frac{s}{R}\right)^\alpha \mathrm{d}E(s) f. \tag{2}$$

Let $\alpha(p) = (2n+1)[(1/p)-(1/2)]$ for $1 \leqslant p \leqslant 2$. Then Mauceri has proved, among other things, the following theorem.

Theorem. *Assume that $1 \leqslant p \leqslant 2$ and $\alpha > \alpha(p)$. Then the uniform estimates*

$$\| S_R^\alpha f \|_{L^p(H^n)} \leqslant C \| f \|_{L^p(H^n)} \tag{3}$$

hold for $f \in L^p(H^n)$.

Mauceri proved this theorem by getting L^p norm estimates for the kernel of the Riesz means. Here we propose to give a different proof of the above theorem using a pointwise estimate for the kernel.

Our proof follows an idea of Fefferman-Stein. In [1], in connection with a similar problem for the Laplacian on $\mathbb{R}^n$, it was shown that L^p-L^2 restriction properties of the Fourier transform imply the boundedness of the Riesz means. Later in [6] Chris Sogge has adapted the same techniques to prove the uniform boundedness of the Riesz means associated with self adjoint elliptic operators on a compact manifold. There it was proved that if we have good pointwise estimates of the kernel of S_R^α for large α and proper L^p-L^2 bounds for the projection operators associated with the Riesz means under consideration then we can get uniform L^p estimates for S_R^α itself. The same idea was used by this author in [9] to prove certain multiplier theorem for the Weyl transform.

Thus the plan of the paper is as follows. After collecting all preliminary materials in §2, we will get a good pointwise estimate for the kernel of S_R^α when α is an even integer in §3. Finally in §4 we take a partition of unity and write S_R^α as an infinite sum. For each piece there are certain 'projections' associated with it and for those projections we prove L^p-L^2 bounds. These bounds and the kernel estimate will then be used to prove that each piece is bounded on $L^p(H^n)$. The proof of the theorem will be completed by summing a geometric series.

2. Preliminaries

The main reference for this section is [2]. See also [5]. The $(2n+1)$ dimensional Heisenberg group is the Lie group whose underlying manifold is $\mathbb{R}\times C^n$. The group structure is given by

$$(s,z)\cdot(t\cdot\zeta)=(s+t+2\,\mathrm{Im}\,z\cdot\bar{\zeta},\ z+\zeta).$$

The Haar measure on H^n is simply the Lebesgue measure $\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}\bar{z}$ on $\mathbb{R}\times C^n$. If $w=(s,z)$ the homogeneous norm $|w|$ of w is defined by $|w|^4=s^2+|z|^4$. The homogeneous dimension of H^n is $Q=2n+2$.

Next we recall the definition of the Fourier transform on the Heisenberg group. The infinite dimensional representations of H^n are parametrized by $\mathbb{R}\backslash\{0\}$. If $\lambda\neq 0$ all the representations π_λ can be realized on the same Hilbert space, namely, $L^2(\mathbb{R}^n)$. For $f\in L^2(\mathbb{R}^n)$, $\pi_\lambda(s,z)$ is the operator defined by

$$\pi_\lambda(s,z)f(\xi)=\exp(i\lambda s)\exp(i2\lambda(2\xi-x)\cdot y)f(\xi-x) \tag{4}$$

where $z=x+iy\in C^n$ and $\xi\in\mathbb{R}^n$. The Fourier transform $\hat{f}$ of a function $f\in L^1(H^n)$ is then the operator valued function

$$\hat{f}(\lambda)=\int_{H^n} f(w)\pi_\lambda(w)\,\mathrm{d}w. \tag{5}$$

We have the following Plancherel and inversion formulas:

$$\|f\|_2^2=c_n\int\|\hat{f}(\lambda)\|_{\mathrm{HS}}^2|\lambda|^n\,\mathrm{d}\lambda \tag{6}$$

$$f(w)=\int\mathrm{tr}\,(\pi_\lambda(w)^*\hat{f}(\lambda))|\lambda|^n\,\mathrm{d}\lambda \tag{7}$$

where tr is the canonical semifinite trace and $\|\cdot\|_{HS}$ is the Hilbert–Schmidt norm.

For a bounded operator S on $L^2(\mathbb{R}^n)$ let us define $|S|_p$ by

$$|S|_p = (\operatorname{tr}(S^*S)^{p/2})^{1/p}, \quad p < \infty$$
$$|S|_\infty = \text{operator norm of } S.$$

Let $\mathscr{L}_p$ denote the Banach space of weakly measurable operator valued functions $F: \mathbb{R}\backslash\{0\} \to B(L^2(\mathbb{R}^n))$ such that

$$\|F\|_{\mathscr{L}_p} = \left\{ C_n \int |F(\lambda)|_p^p |\lambda|^n \, d\lambda \right\}^{1/p} < \infty, \quad \text{for } p < \infty$$
$$\|F\|_{\mathscr{L}_\infty} = \operatorname*{ess\cdot sup}_{\lambda \neq 0} |F(\lambda)|_\infty < \infty.$$

The Plancherel formula allows us to extend the Fourier transform as an isometry from $L^2(H^n)$ onto $\mathscr{L}_2$. More generally the following version of the Hausdorff-Young inequality holds (see [3]). Let $\mathscr{I}$ denote the inverse Fourier transform defined on $\mathscr{L}_1$ by

$$\mathscr{I}F(w) = \int \operatorname{tr}(\pi_\lambda(w)^* F(\lambda)) |\lambda|^n \, d\lambda \tag{8}$$

and then extend $\mathscr{I}$ to $\mathscr{L}_p$ by interpolation for $1 < p < 2$. Then $\mathscr{I}$ maps $\mathscr{L}_p$, $1 \leqslant p \leqslant 2$ into $L^q(H^n)$, $(1/p) + (1/q) = 1$ and we have

$$\|\mathscr{I}F\|_q \leqslant C \|F\|_{\mathscr{L}_p}. \tag{9}$$

For each $\lambda \neq 0$ we can select an orthonormal basis for $L^2(\mathbb{R}^n)$. Let $\Phi_\alpha^\lambda(x) = (2|\lambda|^{1/2})^{n/2} \Phi(2|\lambda|^{1/2}x)$ where Φ_α are the n-dimensional Hermite functions. Let $P_N(\lambda)$ denote the projection onto the subspace spanned by $\{\Phi_\alpha^\lambda : |\alpha| = N\}$. For a function $f(s,z) = f(s,|z|)$ which is called a zonal function, the Fourier transform reduces to the Laguerre transform and is given by

$$\hat{f}(\lambda) = \sum_{N=0}^{\infty} R_N(\lambda, f) P_N(\lambda) \tag{10}$$

where $R_N(\lambda, f)$ are defined by

$$R_N(\lambda, f) = C_n \frac{N!}{(N+n-1)!} \int_0^\infty \tilde{f}(\lambda, r) L_N^{n-1}(2|\lambda| r^2) \times \exp(-|\lambda| r^2) r^{2n-1} \, dr. \tag{11}$$

Here $\tilde{f}(\lambda, r)$ stands for the Euclidean Fourier transform and L_N^{n-1} are the Laguerre polynomials of type $n-1$. The inversion formula becomes

$$f(s,r) = \frac{2^{n-1}}{\pi^{n+1}} \int \exp(-i\lambda s) \left(\sum_{N=0}^{\infty} R_N(\lambda, f) L_N^{n-1}(2|\lambda| r^2) \exp(-|\lambda| r^2) \right) |\lambda|^n \, d\lambda. \tag{12}$$

The Plancherel and the Hausdorff-Young theorem will then read as follows:

$$\|f\|_2^2=\frac{2^{n-1}}{\pi^{n+1}}\int\sum_{N=0}^{\infty}|R_N(\lambda,f)|^2\frac{(N+n-1)!}{N!}|\lambda|^n\,\mathrm{d}\lambda \tag{13}$$

$$\|f\|_q\leqslant C\left(\int\sum_{N=0}^{\infty}|R_N(\lambda,f)|^p\frac{(N+n-1)!}{N!}|\lambda|^n\,\mathrm{d}\lambda\right)^{1/p}. \tag{14}$$

The Hausdorff–Young inequality in the above form will be used to prove the pointwise estimates for the kernel $S_R^\alpha(w)$ of the Riesz means.

Finally the following facts are also useful in the sequel. Each representation π_λ determines a Lie algebra representation $\mathrm{d}\pi_\lambda$. The operators $\mathrm{d}\pi_\lambda(Z_j)$ and $\mathrm{d}\pi_\lambda(\bar{Z}_j)$ are closable operators. If $W_j(\lambda)$ and $\bar{W}_j(\lambda)$ denote the closures then we have

$$W_j(\lambda)\Phi_\alpha^\lambda=(2|\lambda|(\alpha_j+1))^{1/2}\Phi_{\alpha+e_j}^\lambda$$
$$\bar{W}_j(\lambda)\Phi_\alpha^\lambda=(2|\lambda|\alpha_j)^{1/2}\Phi_{\alpha-e_j}^\lambda$$

where e_j are the the coordinate vectors in $\mathbb{R}^n$. Let $H_j(\lambda)=-\frac{1}{2}(W_j(\lambda)\bar{W}_j(\lambda)+\bar{W}_j(\lambda)W_j(\lambda))$ and $H(\lambda)=\sum_{j=1}^n H_j(\lambda)$. Then $H(\lambda)$ is the closure of $\mathrm{d}\pi_\lambda(L)$ and has the spectral resolution

$$H(\lambda)=\sum_{N=0}^{\infty}(2N+n)|\lambda|P_N(\lambda).$$

More generally one has

$$\mathrm{d}\pi_\lambda(\phi(L))=\sum_{N=0}^{\infty}\phi((2N+n)|\lambda|)P_N(\lambda) \tag{15}$$

for any reasonable function ϕ.

3. Pointwise estimates for the Riesz kernel

The aim of this section is to prove the following estimate for the Riesz kernel.

Theorem 3.1. *Assume that $\alpha>2m$ where m is a positive integer. Then the kernel $S_R^\alpha(w)$ of the Riesz means satisfies the following estimate*

$$|S_R^\alpha(w)|\leqslant CR^{Q/2}(1+R^{1/2}|w|)^{-2m}. \tag{16}$$

Recall that $S_R^\alpha f=\phi_R^\alpha(L)f$ where $\phi_R^\alpha(\lambda)=(1-(\lambda/R))_+^\alpha$. Therefore, $S_R^\alpha f=S_R^\alpha * f$ where the Fourier transform of the kernel S_R^α is given by the equation

$$(S_R^\alpha)\hat{}(\lambda)=\sum_{N=0}^{\infty}\left(1-\frac{(2N+n)|\lambda|}{R}\right)_+^\alpha P_N(\lambda). \tag{17}$$

If we apply the inversion formula (12) we get

$$S_R^\alpha(w)=\int\exp(-i\lambda s)$$

$$\times\left\{\sum_{N=0}^{\infty}\left(1-\frac{(2N+n)|\lambda|}{R}\right)_+^{\alpha} L_N^{n-1}(2|\lambda|r^2)\exp(-|\lambda|r^2)\right\}|\lambda|^n\,d\lambda \quad (18)$$

where we have set $w=(s,z)$, $r=|z|$. From the above expression it is clear that $S_R^\alpha(s,z)=R^{Q/2}S_1^\alpha(Rs,R^{1/2}z)$ and hence it is enough to estimate $S_1^\alpha(w)$.

Since $S_1^\alpha(w)$ is the inverse Fourier transform of the operator valued function $(1-H(\lambda))_+^\alpha$, Hausdorff–Young inequality gives the estimate

$$\|S_1^\alpha(w)\|_\infty \leqslant C\|(1-H(\lambda))_+^\alpha\|_{\mathscr{S}_1}.$$

As $(1-H(\lambda))_+^\alpha=\sum_{N=0}^\infty(1-(2N+n)|\lambda|)_+^\alpha P_N(\lambda)$, it is easy to see that $\|(1-H(\lambda))_+^\alpha\|_{\mathscr{S}_1}\leqslant C$ and hence $\|S_1^\alpha(w)\|_\infty\leqslant C$. Thus it is enough to prove the estimate $|S_1^{2m}(w)|\leqslant c|w|^{-2m}$ when $|w|>1$.

Let us set $f=S_1^\alpha$ so that $\hat{f}(\lambda)$ is given by $\hat{f}(\lambda)=\sum_{N=0}^\infty R_N(\lambda,f)P_N(\lambda)$. Here $R_N(\lambda,f)=(1-(2N+n)|\lambda|)_+^\alpha=\psi((2N+n)|\lambda|)$, say. Our aim is to calculate $R_N(\lambda,(is-r^2)^m f)$ and then apply the Hausdorff–Young inequality getting

$$\|(is-r^2)^m f\|_\infty \leqslant C\sum_{N=0}^{\infty}\int|R_N(\lambda,(is-r^2)^m f)|\frac{(N+n-1)!}{N!}|\lambda|^n\,d\lambda. \quad (19)$$

If we can show that the right hand side of (19) is bounded then we are done.

Recall the definition of $R_N(\lambda,f)$. We have

$$R_N(\lambda,f)=c_n\frac{N!}{(N+n-1)!}\int_0^\infty \tilde{f}(\lambda,r)L_N^{n-1}(2|\lambda|r^2)\exp(-|\lambda|r^2)r^{2n-1}\,dr.$$

Since $(isf)^\sim(\lambda,r)=(d/d\lambda)\,\tilde{f}(\lambda,r)$ we have the formula

$$R_N(\lambda,isf)=\frac{d}{d\lambda}R_N(\lambda,f)-c_n\frac{N!}{(N+n-1)!}\int_0^\infty\tilde{f}(\lambda,r)$$
$$\times\frac{d}{d\lambda}\{L_N^{n-1}(2|\lambda|r^2)\exp(-|\lambda|r^2)\}r^{2n-1}\,dr.$$

Now we make use of the following formula satisfied by the Laguerre polynomials (see [7]).

$$r\frac{d}{dr}L_N^{n-1}(r)=NL_N^{n-1}(r)-(N+n-1)L_{N-1}^{n-1}(r).$$

Using this formula, a simple calculation shows that for $\lambda\neq0$ we have

$$R_N(\lambda,isf)=\frac{d}{d\lambda}R_N(\lambda,f)-N|\lambda|^{-1}\{R_N(\lambda,f)-R_{N-1}(\lambda,f)\}+R_N(\lambda,r^2f).$$

So, if we let $\sigma=(2N+n)|\lambda|$ and $\psi_1(\sigma)=R_N(\lambda,(is-r^2)f)$ then we have proved the following formula

$$\psi_1(\sigma)=|\lambda|^{-1}\{\sigma\psi'(\sigma)-N\psi(\sigma)+N\psi(\sigma-2|\lambda|)\}.$$

Let us now recall that $\psi(\sigma)=(1-\sigma)_+^\alpha$ satisfies the following two properties: i) ψ is supported in $(2N+n)|\lambda|\leqslant1$; ii) $\int|\psi((2N+n)|\lambda|)|\,|\lambda|^n\,d\lambda\leqslant C(2N+n)^{-n-1}$.

From the above equation for $\psi_1(\sigma)$ it is clear that ψ_1 also satisfies the property (i). We claim that ψ_1 satisfies property (ii) also. To see this we have

$$\psi_1((2N+n)|\lambda|) = \tfrac{1}{2}n|\lambda|^{-1}\frac{\partial\psi}{\partial N} + 2N|\lambda|^{-1} \times \left\{\frac{\partial\psi}{\partial N} - \psi((2N+n)|\lambda|) - \psi((2(N-1)+n)|\lambda|)\right\}.$$

Using Taylor expansion we can write

$$\psi((2(N-1)+n)|\lambda|) - \psi((2N+n)|\lambda|) + \frac{\partial\psi}{\partial N} = 4|\lambda|^2\int_{N-1}^{N}(t+1-N)\psi''((2t+n)|\lambda|)\,\mathrm{d}t.$$

Since $(2N+n)|\lambda| \leqslant 1$ we get

$$\int 2N|\lambda|^{-1}\left|\psi((2(N-1)+n)|\lambda|) - \psi((2N+n)|\lambda|) + \frac{\partial\psi}{\partial N}\right||\lambda|^n\,\mathrm{d}\lambda \leqslant C\int_{N-1}^{N}\mathrm{d}t\int(1-(2t+n)|\lambda|)_+^{\alpha-2}|\lambda|^n\,\mathrm{d}\lambda \leqslant C(2N+n)^{-n-1}.$$

We also have

$$\int|\lambda|^{-1}\left|\frac{\partial\psi}{\partial N}\right||\lambda|^n\,\mathrm{d}\lambda \leqslant C(2N+n)^{-n-1}.$$

Hence we have proved that

$$\int|\psi_1((2N+n)|\lambda|)||\lambda|^n\,\mathrm{d}\lambda \leqslant C(2N+n)^{-n-1}.$$

Now an iteration of the process shows that $R_N(\lambda,(is-r^2)^k f) = \psi_k$ satisfies the inequality

$$\int|\psi_k((2N+n)|\lambda|)|\quad|\lambda|^n\,\mathrm{d}\lambda \leqslant C_k(2N+n)^{-n-1}$$

provided $(1-\lambda)_+^{\alpha-2k}$ is integrable. Hence when $k=m$ we get

$$\int|R_N(\lambda(is-r^2)^m f)||\lambda|^n\,\mathrm{d}\lambda \leqslant C(2N+n)^{-n-1}$$

and consequently

$$\sum_{N=0}^{\infty}\int|R_N(\lambda,(is-r^2)^m f)|\frac{(N+n-1)!}{N!}|\lambda|^n\,\mathrm{d}\lambda \leqslant C\sum_{N=0}^{\infty}(2N+n)^{-2} \leqslant C.$$

Thus we have proved

$$\|(is-r^2)^m f\|_\infty \leqslant C$$

and hence this completes the proof of Theorem 3.1.

4. Boundedness of the Riesz means

In this section we will prove our main theorem. By rescaling we can assume that $R=1$. So we are considering the operator $T=(1-L)_+^\alpha$. We are going to prove that when $\alpha>\alpha(p)$

$$\|Tf\|_p \leqslant C\|f\|_p, \quad f\in L^p(H^n). \tag{20}$$

To that end we take a partition of unity $\sum_{j=-\infty}^{\infty}\phi(2^j s)=1$ where $\phi\in C_0^\infty(\frac{1}{2},2)$. Let us write $\phi_j^\alpha(s)=(1-s)^\alpha\phi(2^j(1-s))$ and define

$$T_j=\int_0^1 \phi_j^\alpha(s)\,\mathrm{d}E(s). \tag{21}$$

Then $T=\sum_{j=0}^{\infty}T_j$ and (20) will follow once we prove the following proposition:

PROPOSITION 4.1

Assume that $\alpha>\alpha(p)$ and $f\in L^p(H^n)$. Then $\exists\varepsilon>0$ such that

$$\|T_j f\|_p \leqslant C2^{-j\varepsilon}\|f\|_p. \tag{22}$$

We claim that Proposition 4.1 is a consequence of the following estimates:

$$\|T_j f\|_2 \leqslant C2^{-\alpha j}2^{-j((1/p)-(1/2))}\|f\|_p. \tag{23}$$

This estimate will be proved in the next proposition. To complete the proof of (22) we follow Sogge [6] and therefore will be sketchy in the proof.

We first show that whenever B is a ball of radius 2^j then

$$\|T_j f\|_{L^p(B)} \leqslant C2^{-j(\alpha-\alpha(p))}\|f\|_p. \tag{24}$$

This is immediate since

$$\|T_j f\|_{L^p(B)} \leqslant C2^{jQ((1/p)-(1/2))}\|T_j f\|_2$$

in view of the estimate (23). We next show that the kernel of T_j has 'thickness' 2^j. More precisely it is enough to show that for each $\nu>0\,\exists$ an $\varepsilon>0$ for which one has the uniform estimates

$$\int_{|w|>2^{j(1+\nu)}}|S_j^\alpha(w)|\,\mathrm{d}w \leqslant C2^{-j\varepsilon}$$

where S_j^α is the kernel of T_j. To prove this estimate we make use of the kernel estimate

$$|S_R^{2m+1}(w)| \leqslant CR^{Q/2}(1 + R^{1/2}|w|)^{-2m}. \tag{25}$$

Recall that $T_j = \int_0^1 \phi_j(s)\,\mathrm{d}E(s)$, $\phi_j(s) = (1-s)^\alpha \cdot \phi(2^j(1-s))$.

Proceeding as in [6] one obtains the following equation for the kernel S_j^α

$$S_j^\alpha(w) = c_m \int t^{2m+1} S_t^{2m+1}(w)\partial_t^{2m+2}\phi_j^\alpha(t)\,\mathrm{d}t.$$

Since ϕ_j^α is supported in $2^{-j-1} \leqslant (1-t) \leqslant 2^{-j+1}$ we have the bound

$$|\partial_t^{2m+2}\phi_j^\alpha(t)| \leqslant C2^{j(2m+2)}.$$

If we use the estimate (25) what we get is

$$|S_j^\alpha(w)| \leqslant C2^{j(2m+2)}|w|^{-2m}.$$

From the estimate it follows that

$$\int_{|w|>2^{j(1+\nu)}} |S_j^\alpha(w)|\,\mathrm{d}w \leqslant 2^{j(2m+2)} \int_{2^{j(1+\nu)}}^\infty t^{-2m+Q-1}\,\mathrm{d}t$$
$$\leqslant C2^{j(2m+2)}2^{j(1+\nu)(-2m+Q)}.$$

Choosing m large so that $(2m+2)\nu > (Q+1)(\nu+1)$ we can arrange it so that

$$\int_{|w|>2^{j(1+\nu)}} |S_j^\alpha(w)|\,\mathrm{d}w \leqslant C2^{-j\varepsilon}. \tag{26}$$

Now using (24) and (26) one can prove (22). A detailed proof of this fact can be found in [8]. We omit the details here and proceed to the proof of the following:

PROPOSITION 4.2.

Assume that $1 \leqslant p \leqslant 2$ *and* $f \in L^p(H^n)$. *Then we have*

$$\| T_j f \|_2 \leqslant C2^{-\alpha j}2^{-j((1/p)-(1/2))}\| f \|_p. \tag{27}$$

Since $\| T_j f \|_2 \leqslant C2^{-\alpha j}\| f \|_2$ it is enough to prove the inequality when $p=1$. The proof of the inequality is simple when $p=1$. In view of the Plancherel theorem we have

$$\| T_j f \|_2^2 = \int \|(T_j f)^\wedge(\lambda)\|_{\mathrm{HS}}^2 |\lambda|^n\,\mathrm{d}\lambda.$$

Since $T_j = \varphi_j^\alpha(L)$ we have the equation

$$(T_j f)^\wedge(\lambda) = \sum_{N=0}^\infty \varphi_j^\alpha(\nu|\lambda|)P_N(\lambda))\hat{f}(\lambda)$$

where $\nu = (2N+n)$. Due to the orthogonality of the projections $P_N(\lambda)$ we have

$$\| T_j f \|_2^2 = \sum_{N=0}^\infty \int \| \varphi_j^\alpha(\nu|\lambda|)P_N(\lambda)\hat{f}(\lambda)\|_{\mathrm{HS}}^2 |\lambda|^n\,\mathrm{d}\lambda.$$

If we define the function $\varphi_{Nj}(t,|z|)=\varphi_{Nj}(t,r)$ by the equation

$$\varphi_{Nj}(t,r)=\int \exp(-i\lambda t)\varphi_j^{\alpha}(v|\lambda|)L_N^{n-1}(2|\lambda|r^2)\exp(-|\lambda|r^2)|\lambda|^n\,\mathrm{d}\lambda$$

then it is easy to see that $\hat{\phi}_{Nj}(\lambda)=\varphi_j^{\alpha}(v|\lambda|)P_N(\lambda)$ so that

$$\|T_jf\|_2^2=\sum_{N=0}^{\infty}\|\varphi_{Nj}*f\|_2^2. \tag{28}$$

By applying Young's inequality we obtain

$$\|T_jf\|_2^2\leqslant c\|f\|_1^2\sum_{N=0}^{\infty}\|\varphi_{Nj}\|_2^2.$$

Now it is an easy matter to calculate $\|\varphi_{Nj}\|_2$. Recalling the definition of φ_{Nj} we have

$$\begin{aligned}\|\varphi_{Nj}\|_2^2&=\int\int_0^{\infty}|\varphi_j^{\alpha}(v|\lambda|)|^2|\lambda|^{2n}|(L_N^{n-1}(2|\lambda|r^2)\exp(-\lambda r^2)|^2r^{2n-1}\,\mathrm{d}r\,\mathrm{d}\lambda\\&=\int\int_0^{\infty}|\varphi_j^{\alpha}(v|\lambda|)|^2|\lambda|^n(L_N^{n-1}(r^2))^2\exp(-\lambda r^2)r^{2n-1}\,\mathrm{d}r\,\mathrm{d}\lambda.\end{aligned}$$

Since $\varphi_j^{\alpha}(s)=(1-s)^{\alpha}\varphi(2^j(1-s))$, on the support of φ_j^{α} we have $2^{-j-1}\leqslant(1-s)\leqslant 2^{-j+1}$ and hence $(1-v|\lambda|)^{\alpha}\leqslant c2^{-j\alpha}$. We also have

$$\int_0^{\infty}(L_N^{n-1}(r^2))^2\exp(-r^2)r^{2n-1}\,\mathrm{d}r\leqslant cv^{n-1}$$

which is a basic property of the Laguerre functions. Therefore, we have

$$\begin{aligned}\|\varphi_{Nj}\|_2^2&\leqslant cv^{n-1}2^{-2j\alpha}\int|\varphi(2^j(1-v|\lambda|)|^2|\lambda|^n\,\mathrm{d}\lambda\\&\leqslant c2^{-2j\alpha}2^{-j}v^{-2}.\end{aligned} \tag{29}$$

Hence we have proved

$$\|T_jf\|_2^2\leqslant c\|f\|_1^2 2^{-2\alpha j}2^{-j}\sum_{N=0}^{\infty}(2N+n)^{-2}$$

or

$$\|T_jf\|_2\leqslant c2^{-\alpha j}2^{-j/2}\|f\|_1. \tag{30}$$

This completes the proof of Proposition 4.2.

Note: After this paper was written the author came to know that D. Muller has obtained the same result by using almost similar methods in the paper 'On Riesz means of eigenfunction expansions for the Kohn-Laplacian' (preprint).

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