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Some uncertainty inequalities

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Abstract. We prove an uncertainty inequality for the Fourier transform on the Heisenberg group analogous to the classical uncertainty inequality for the Euclidean Fourier transform. Inequalities of similar form are obtained for the Hermite and Laguerre expansions.

Keywords. Heisenberg group; Fourier transform; Hermite expansion; Laguerre expansion.

1. Introduction

The classical uncertainty inequality for the Fourier transform on $\mathbb{R}^n$ states that

$$\left(\int |x|^2 |f(x)|^2 \,\mathrm{d}x\right)\left(\int |\xi|^2 |\hat{f}(\xi)|^2 \,\mathrm{d}\xi\right) \geqslant \frac{n^2}{4} \tag{1}$$

for all normalized L^2 functions f. Here the Fourier transform of the function f is defined by

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int \exp(-ix\cdot\xi) f(x)\,\mathrm{d}x. \tag{2}$$

The above inequality can be written in the form

$$\| |x| f \|_2 \, \| (-\Delta)^{1/2} f \|_2 \geqslant \frac{n}{2} \tag{3}$$

where $(-\Delta)^{1/2}$ is defined by the equation

$$((-\Delta)^{1/2} f)\hat{}(\xi) = |\xi| \hat{f}(\xi). \tag{4}$$

The purpose of this note is to prove similar inequalities when $-\Delta$ is replaced by some other differential operator.

First we consider the Hermite operator $H = -\Delta + |x|^2$ on $\mathbb{R}^n$. For normalized L^2 functions on $\mathbb{R}^n$ we prove that

$$\| |x| f \|_2 \, \| H^{1/2} f \|_2 \gtrless \frac{\sqrt{n}}{2}. \tag{5}$$

Here $H^{1/2} f$ is defined by

$$H^{1/2} f(x) = \sum (2N + n)^{1/2} P_N f(x) \tag{6}$$

wherever f has the Hermite expansion

$$f(x)=\sum P_N f(x).$$

For example when $n=1$ we get the inequality

$$\left(\int t^2|f(t)|^2\,dt\right)^{1/2}\left(\sum(2k+1)|\hat{f}(k)|^2\right)^{1/2}\geqslant\tfrac{1}{2} \tag{7}$$

where $\hat{f}(k)$ are the Hermite coefficients of the function f.

Next we consider the operator L,

$$L=-\Delta_x-\Delta_y+\tfrac{1}{4}(|x|^2+|y|^2)-1\sum_{j=1}^{n}\left(x_j\frac{\partial}{\partial y_j}-y_j\frac{\partial}{\partial x_j}\right) \tag{8}$$

on $\mathbb{C}^n$. This operator is connected with special Hermite expansions. For this operator we prove the inequality

$$\|\,|z|f\,\|_2\,\|L^{1/2}f\|_2\geqslant\frac{\sqrt{n}}{2} \tag{9}$$

for all normalized L^2 functions f on $\mathbb{C}^n$. If we consider only radial functions the above inequality becomes an uncertainty inequality for the Laguerre expansions. For example, when $n=1$ we have

$$\left(\int_0^\infty t^2|f(t)|^2\,dt\right)^{1/2}\left(\sum_{k=0}^{\infty}(2k+1)|\hat{f}(k)|^2\right)^{1/2}\geqslant\frac{\sqrt{n}}{2} \tag{10}$$

where $\hat{f}(k)$ are defined by

$$\hat{f}(k)=\int_0^\infty f(t)L_k(\tfrac{1}{2}t)\exp(-\tfrac{1}{4}t)\,dt \tag{11}$$

where $L_k(t)$ are the usual Laguerre polynomials.

Finally we consider the sublaplacian $\mathscr{L}$ on the Heisenberg group H^n. For normalized functions f on $L^2(H^n)$ we prove that

$$\left(\int\int|z|^2|f(z,t)|^2\,dz\,dt\right)^{1/2}\left(\int\int|\mathscr{L}^{1/2}f(z,t)|^2\,dz\,dt\right)^{1/2}\geqslant\sqrt{n}\left(\frac{\pi}{2}\right)^{(n+1)/2} \tag{12}$$

The above inequality can also be written in the form

$$\left(\int\int|z|^2|f(z,t)|^2\,dz\,dt\right)^{1/2}\left(\int|\lambda|^n\|\hat{f}(\lambda)H(\lambda)^{1/2}\|_{HS}^2\,d\lambda\right)^{1/2}\geqslant\sqrt{n}\left(\frac{\pi}{2}\right)^{(n+1)/2} \tag{13}$$

where $\hat{f}(\lambda)$ is the Fourier transform of f and $H(\lambda)$ are certain Hermite operators to be defined later.

All the three inequalities are easy to prove. A very simple proof of the inequality (1) can be found in Folland [1]. It so happens that we can simply adapt the same proof to all the three situations which we are interested in.

2. Uncertainty inequalities for Hermite and Laguerre expansions

The n dimensional Hermite functions are denoted by $\Phi_\alpha(x)$. They are eigen functions of the Hermite operator $H = (-\Delta + |x|^2)$.

$$(-\Delta + |x|^2)\Phi_\alpha(x) = (2|\alpha| + n)\Phi_\alpha(x). \tag{14}$$

Let P_N denote the projection of $L^2(\mathbb{R}^n)$ onto the eigenspace spanned by $\{\Phi_\alpha : |\alpha| = N\}$. Then we can write

$$H^{1/2} f = \sum (2N + n)^{1/2} P_N f. \tag{15}$$

We also need to consider the operators A_j, A_j^+ which are defined by

$$A_j = \frac{\partial}{\partial \xi_j} + \xi_j, \quad A_j^+ = -\frac{\partial}{\partial \xi_j} + \xi_j. \tag{16}$$

These operators act on the Hermite functions in the following way:

$$\begin{aligned} A_j \Phi_\alpha &= (2\alpha_j)^{1/2} \Phi_{\alpha - e_j} \\ A_j^+ \Phi_\alpha &= (2\alpha_j + 1)^{1/2} \Phi_{\alpha + e_j} \end{aligned} \tag{17}$$

where $e_j = (0, 0, \ldots 1, \ldots 0)$ with 1 in the jth place. Then it is clear that the operators $H^{-1/2} A_j$ and $H^{-1/2} A_j^+$ are bounded on $L^2(\mathbb{R}^n)$.

Let $a \in \mathbb{R}^n$ and $f \in L^2(\mathbb{R}^n)$ be such that $\int |f(x)|^2 \, dx = 1$. Then we will prove the following theorem.

Theorem 2.1.

$$\left(\int |x - a|^2 |f(x)|^2 \, dx \right) \left(\sum_{N=0}^{\infty} (2N + n) \| P_N f \|_2^2 \right) \geq \frac{n}{4}.$$

Proof. The proof of this theorem is elementary. We start with the identity

$$\frac{\partial}{\partial x_j} \{(x_j - a_j) f\} = (x_j - a_j) \frac{\partial}{\partial x_j} f + f.$$

Then we have

$$\int f(x) \bar{f}(x) \, dx = \int \frac{\partial}{\partial x_j} \{(x_j - a_j) f\} \bar{f}(x) \, dx - \int (x_j - a_j) \bar{f}(x) \frac{\partial f}{\partial x_j} \, dx.$$

Integration by parts gives

$$\int |f(x)|^2\,dx = -\int (x_j-a_j)f(x)\frac{\partial \bar{f}}{\partial x_j}\,dx - \int (x_j-a_j)\bar{f}(x)\frac{\partial f}{\partial x_j}\,dx.$$

Now we write

$$(x_j-a_j)f\frac{\partial \bar{f}}{\partial x_j} = (x_j-a_j)fA_j\bar{f} - (x_j-a_j)x_j|f|^2$$

$$(x_j-a_j)\bar{f}\frac{\partial f}{\partial x_j} = -(x_j-a_j)\bar{f}A_j^+ f + (x_j-a_j)x_j|f|^2.$$

Using this it follows that

$$\int |f(x)|^2\,dx = -\int (x_j-a_j)fA_j\bar{f} + \int (x_j-a_j)\bar{f}A_j^+ f.$$

Applying Cauchy–Schwarz inequality we get

$$\int |f(x)|^2\,dx \leqslant \left(\int |x_j-a_j|^2|f|^2\,dx\right)^{1/2}(\|A_jf\|_2 + \|A_j^+ f\|_2).$$

Since $A_jH^{-1/2}$ and $A_j^+H^{-1/2}$ are bounded on L^2 we get

$$\int |f(x)|^2\,dx \leqslant 2\left(\int |x_j-a_j|^2|f(x)|^2\,dx\right)^{1/2}\|H^{1/2}f\|_2.$$

Squaring and summing over j gives

$$\left(\int |x-a|^2|f(x)|^2\,dx\right)\left(\sum(2N+n)\|P_Nf\|_2^2\right) \geqslant \frac{n}{4}.$$

Let us now take up the case of the operator L. The operator L can be written as

$$L = -\tfrac{1}{2}\sum_{j=1}^{n}(Z_j\bar{Z}_j + \bar{Z}_jZ_j)$$

where Z_j and $\bar{Z}_j$ are the vector fields defined by

$$Z_j = \partial_j + \tfrac{1}{4}\bar{z}_j, \quad \bar{Z}_j = \bar{\partial}_j - \tfrac{1}{4}z_j$$

where

$$\partial_j = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad \bar{\partial}_j = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

We also have to recall some properties of the Weyl transform. For the Weyl transform we refer to Mauceri [3] and the references thereof.

The Weyl transform which we denote by W takes functions on $\mathbb{C}^n$ into bounded operators on $L^2(\mathbb{R}^n)$. There is a Plancherel formula for the Weyl transform:

$$\|f\|_2^2 = (2\pi)^{-n}\|W(f)\|_{HS}^2.$$

We also have the identities

$$W(Z_j f) = iW(f)A_j^+$$
$$W(\bar{Z}_j f) = iW(f)A_j,$$

so that $W(L^{-1/2}Z_j f) = W(f)A_j^+ H^{-1/2}$ and $W(L^{-1/2}\bar{Z}_j f) = W(f)A_j H^{-1/2}$. The operators $Z_j L^{-1/2}$ and $\bar{Z}_j L^{-1/2}$ are easily seen to be bounded on $L^2(\mathbb{C}^n)$.

Let $a \in \mathbb{C}^n$ and $f \in L^2(\mathbb{C}^n)$ be normalized. Then we can prove the following inequality.

Theorem 2.2

$$(\int |z-a|^2 |f(z)|^2 \, dz)(\int |L^{1/2} f(z)|^2 \, dz) \geqslant \frac{n}{4}$$

Proof. The proof of this is very similar to the proof of the previous theorem. We start with the equation.

$$\partial_j\{(z_j - a_j)f\} = f + (z_j - a_j)\partial_j f.$$

Proceeding as before we obtain

$$\int |f(z)|^2 \, dz = -\int (z_j - a_j) f \overline{(\bar{\partial}_j f)} \, dz - \int (z_j - a_j) \bar{f} (\partial_j f) \, dz,$$

where we have used the fact that $(\partial_j \bar{f}) = \overline{(\bar{\partial}_j f)}$. Since $\partial_j = Z_j - \frac{1}{4}\bar{z}_j$, $\bar{\partial}_j = \bar{Z}_j + \frac{1}{4} z_j$ an easy calculation gives us

$$\int |f(z)|^2 \, dz = -\int (z_j - a_j) f \overline{(\bar{Z}_j f)} \, dz - \int (z_j - a_j) \bar{f} (Z_j f) \, dz.$$

Applying Cauchy-Schwarz inequality and recalling that the operators $Z_j L^{-1/2}$ and $\bar{Z}_j L^{-1/2}$ are bounded on $L^2(\mathbb{C}^n)$ we get

$$\int |f(z)|^2 \, dz \leqslant 2\left(\int |z_j - a_j|^2 |f(z)|^2 \, dz\right)^{1/2} \left(\int |L^{1/2} f(z)|^2 \, dz\right)^{1/2}.$$

Summing over j proves the theorem.

To deduce the uncertainty inequality for the Laguerre expansions we proceed as follows. When f is a radial function the Weyl transform reduces to the Laguerre transform:

$$W(f) = \sum_{N=0}^{\infty} R_N(f) P_N$$

where $R_N(f)$ are given by

$$R_N(f) = \frac{N!}{(N+n-1)!} \int_0^{\infty} f(r) L_N^{n-1}(\tfrac{1}{2}r^2) \exp(-\tfrac{1}{4}r^2) r^{2n-1} \, dr.$$

Since $W(L^{1/2} f) = W(f) H^{1/2}$ we have

$$W(L^{1/2} f) = \sum_{N=0}^{\infty} (2N+n)^{1/2} R_N(f) P_N$$

which gives

$$\| W(L^{1/2}f)\|_{HS}^2 = \sum_{N=0}^{\infty} (2N+n)\frac{(N+n-1)!}{N!}|R_N(f)|^2.$$

In view of the Plancherel formula we get

$$\| L^{1/2}f \|_2^2 = (2\pi)^{-n}\sum (2N+n)\frac{(N+n-1)!}{N!}|R_N(f)|^2.$$

Thus we have the following result.

Theorem 2.3. *For f in $L^2(\mathbb{R}_+, r^{2n-1}\,\mathrm{d}r)$ with norm 1 we have the inequality*

$$\left(\int_0^{\infty} r^2|f(r)|^2 r^{2n-1}\,\mathrm{d}r\right)\left(\sum (2N+n)\frac{(N+n-1)!}{N!}|R_N(f)|^2\right) \geqslant (2\pi)^n\frac{n}{4}.$$

If we take $n=1$ and define $f(r) = g(\sqrt{r})$ then we have

$$R_N(g) = \int_0^{\infty} f(r)L_N(\tfrac{1}{2}r)\exp(-\tfrac{1}{4}r)\,\mathrm{d}r = \hat{f}(N)$$

and what we have is the following uncertainty inequality for the Laguerre expansions (of type 0).

$$\left(\int_0^{\infty} r^2|f(r)|^2\,\mathrm{d}r\right)\left(\sum (2k+1)|\hat{f}(k)|^2\right) \geqslant \frac{\pi}{2}.$$

3. Uncertainty inequality for the Heisenberg group

On the n-dimensional Heisenberg group H^n we consider the following left invariant vector fields

$$Z_j = \partial_j + i\bar{z}_j\partial_t, \quad \bar{Z}_j = \bar{\partial}_j - iz_j\partial_t, \quad T = \partial_t. \tag{18}$$

They generate the Lie algebra of H^n and the sublaplacian $\mathscr{L}$ is defined by

$$\mathscr{L} = -\frac{1}{2}\sum_{j=1}^{n} (Z_j\bar{Z}_j + \bar{Z}_jZ_j). \tag{19}$$

All the infinite dimensional irreducible unitary representations of H^n are parametrized by $\lambda \in \mathbb{R}$, $\lambda \neq 0$ and they all can be realized on $L^2(\mathbb{R}^n)$. For all the facts about the representations π_λ of H^n and for other results which we use on H^n we refer to Geller [2] (see also Mauceri [4]).

The Fourier transform of a function f on H^n which we denote by $\hat{f}(\lambda)$ is a bounded linear operator on $L^2(\mathbb{R}^n)$ for each $\lambda \neq 0$. The Fourier transform satisfies the following Plancherel's theorem:

$$\| f \|_2^2 = \frac{2^{n-1}}{\pi^{n+1}}\int |\lambda|^n \,\| \hat{f}(\lambda)\|_{HS}^2\,\mathrm{d}\lambda. \tag{20}$$

Each representation π_λ determines a Lie algebra representation $d\pi_\lambda$. The operators $d\pi_\lambda(Z_j)$ and $d\pi_\lambda(\bar{Z}_j)$ are closable operators. Let $W_j(\lambda)$ and $\bar{W}_j(\lambda)$ denote their closures. Then they have a very simple description as follows let Φ_α^λ be the scaled Hermite functions defined by

$$\Phi_\alpha^\lambda(x) = (2|\lambda|^{1/2})^{n/2}\Phi_\alpha(2|\lambda|^{1/2}x). \tag{21}$$

Then Geller [2] has proved that for each $\lambda > 0$

$$W_j(\lambda)\Phi_\alpha^\lambda(x) = (2|\lambda|(\alpha_j+1))^{1/2}\Phi_{\alpha+e_j}^\lambda(x) \tag{22}$$

$$\bar{W}_j(\lambda)\Phi_\alpha^\lambda(x) = (2|\lambda|\alpha_j)^{1/2}\Phi_{\alpha+e_j}^\lambda(x). \tag{23}$$

For $\lambda < 0$, $W_j(\lambda) = \bar{W}_j(-\lambda)$, $\bar{W}_j(\lambda) = W_j(-\lambda)$. Let $P_N(\lambda)$ denote the projection of $L^2(\mathbb{R}^n)$ onto the eigenspace spanned by $\{\Phi_\alpha^\lambda : |\alpha| = N\}$ and let $H(\lambda) = \sum_{N=0}^\infty (2N+n)|\lambda|\, P_N(\lambda)$. Then it is clear that the operators $H(\lambda)^{-1/2}W_j(\lambda)$ and $H(\lambda)^{-1/2}\bar{W}_j(\lambda)$ are bounded on $L^2(\mathbb{R}^n)$ for each $\lambda \neq 0$. We also have the identities

$$\begin{aligned} (Z_jf)\hat{}(\lambda) &= i\hat{f}(\lambda)W_j(\lambda) \\ (\bar{Z}_jf)\hat{}(\lambda) &= i\hat{f}(\lambda)\bar{W}_j(\lambda) \\ (Tf)\hat{}(\lambda) &= i\lambda\hat{f}(\lambda). \end{aligned} \tag{24}$$

Finally, when f is a zonal function i.e. a function of the form $f(|z|, t)$ then the Fourier transform $\hat{f}(\lambda)$ is explicitly given by the formula

$$\hat{f}(\lambda) = \sum_{N=0}^\infty R_N(\lambda, f)P_N(\lambda) \tag{25}$$

where $R_N(\lambda, f)$ are defined by

$$R_N(\lambda, f) = \frac{N!}{(N+n-1)!}\int_0^\infty \tilde{f}(r,\lambda)L_N^{n-1}(2|\lambda|r^2)\exp-(|\lambda|r^2)r^{2n-1}\,dr. \tag{26}$$

In the above $\tilde{f}(r, \lambda)$ is the Fourier transform of f in the t variable and $L_N^{n-1}(r)$ are the Laguerre polynomials of type $(n-1)$.

Now for normalized L^2 functions f on H^n we can prove the following uncertainty inequality

$$\left(\iint t^2|f(z,t)|^2\,dz\,dt\right)^{1/2}\left(\int |\lambda|^n \|\lambda\hat{f}(\lambda)\|_{HS}^2\,d\lambda\right)^{1/2} \geq \left(\frac{\pi}{2}\right)^{(n+1)/2} \tag{27}$$

The proof of this is elementary. First one proves that

$$\|f\|_2^2 \leq 2\,\|tf\|_2\,\|\partial_t f\|_2 \tag{28}$$

and then uses the Plancherel formula

$$\|\partial_t f\|_2^2 = \frac{2^{n-1}}{\pi^{n+1}}\int |\lambda|^n\,\|\lambda\hat{f}(\lambda)\|_{HS}^2\,d\lambda \tag{29}$$

where we have used the relation $(\partial_t f)\hat{}(\lambda) = i\lambda\hat{f}(\lambda)$.

From inequality (27) it is quite clear that if $f(z, t)$ has compact support in the t

variable then $\hat{f}(\lambda)$ cannot have compact support and vice versa. Actually much more is true. In [5] Price and Sitaram have proved that if both the sets $\{t: f(z,t) \neq 0\}$ and $\{\lambda: f(\lambda) \neq 0\}$ have finite measure then $f \equiv 0$. But things are quite different when we consider f as a function of z.

Let f be zonal function viz. a function of the form $f(z,t) = \varphi(t)g(z)$ where g is radial. Then the Fourier transform of f is given by

$$\hat{f}(\lambda) = \tilde{\varphi}(\lambda) \sum_{N=0}^{\infty} R_N(\lambda, g) P_N(\lambda) \tag{30}$$

where

$$R_N(\lambda, g) = \frac{N!}{(N+n-1)!} \int_0^{\infty} g(r) L_N^{n-1}(2|\lambda| r^2) \exp(-|\lambda| r^2) r^{2n-1}\, dr.$$

If g is compactly supported then by choosing $\tilde{\varphi}$ to have compact support we can make $\hat{f}(\lambda)$ also to have compact support. Nevertheless the following uncertainty inequality is valid.

Theorem 3.1. *For a normalized L^2 function f on H^n one has*

$$\left(\iint |z-a|^2 |f(z,t)|^2\, dz\, dt\right)^{1/2} \left(\int |\lambda|^n \| \hat{f}(\lambda)(H(\lambda))^{1/2} \|_{HS}^2\, d\lambda\right)^{1/2}$$
$$\geqslant \sqrt{n}\left(\frac{\pi}{2}\right)^{(n+1)/2}.$$

Proof. Let $\tilde{f}(z,\lambda)$ denote the Fourier transform of f in the t variable. Then we have

$$\int |\tilde{f}(z,\lambda)|^2\, dz = \int \partial_j((z_j - a_j)\tilde{f})\bar{\tilde{f}}\, dz - \int (z_j - a_j)\bar{\tilde{f}}\partial_j \tilde{f}\, dz.$$

Integrating by parts in the first integral we get

$$\int |\tilde{f}(z,\lambda)|^2\, dz = -\int (z_j - a_j)\tilde{f}(\overline{\bar{\partial}_j \tilde{f}})\, dz - \int (z_j - a_j)\bar{\tilde{f}}(\partial_j \tilde{f})\, dz.$$

Define $Z_j(\lambda)$ and $\bar{Z}_j(\lambda)$ by the equations

$$Z_j(\lambda) = \partial_j - \lambda \bar{z}_j, \quad \bar{Z}_j(\lambda) = \bar{\partial}_j + \lambda z_j. \tag{31}$$

Then a simple calculation shows that

$$\int |\tilde{f}(z,\lambda)|^2\, dz = -\int (z_j - a_j)\tilde{f}(\bar{Z}_j(\lambda)\tilde{f})\, dz - \int (z_j - a_j)\bar{\tilde{f}}(Z_j(\lambda)\tilde{f})\, dz.$$

But

$$Z_j(\lambda)\tilde{f}(z,\lambda) = (Z_j f)^{\sim}(z,\lambda)$$
$$\bar{Z}_j(\lambda)\tilde{f}(z,\lambda) = (\bar{Z}_j f)^{\sim}(z,\lambda)$$

and therefore,

$$\int |\tilde{f}(z,\lambda)|^2\, dz = -\int (z_j - a_j)\bar{\tilde{f}}(Z_j f)^{\sim}\, dz - \int (z_j - a_j)\tilde{f}(\bar{Z}_j f)^{\sim}\, dz.$$

Integrating with respect to λ, applying Cauchy–Schwarz and Plancherel theorem for the Fourier transform in the t variable we obtain

$$\|f\|_2^2 \leqslant \|(z_j - a_j)f\|_2 \{\|Z_j f\|_2 + \|\bar{Z}_j f\|_2\}. \tag{32}$$

In view of the Plancherel formula for the Heisenberg group

$$\|Z_j f\|_2^2 = \frac{2^{n-1}}{\pi^{n+1}} \int |\lambda|^n \|\hat{f}(\lambda) W_j(\lambda)\|_{\mathrm{HS}}^2 \,\mathrm{d}\lambda.$$

Using the fact that $H(\lambda)^{-1/2} W_j(\lambda)$ is bounded on $L^2(\mathbb{R}^n)$ we get

$$\|Z_j f\|_2 \leqslant \frac{2^{(n-1)/2}}{\pi^{(n+1)/2}} \left(\int |\lambda|^n \|\hat{f}(\lambda)(H(\lambda))^{1/2}\|_{\mathrm{HS}}^2 \,\mathrm{d}\lambda \right)^{1/2}. \tag{33}$$

Similarly

$$\|\bar{Z}_j f\|_2 \leqslant \frac{2^{(n-1)/2}}{\pi^{(n+1)/2}} \left(\int |\lambda|^n \|\hat{f}(\lambda) H(\lambda)^{1/2}\|_{\mathrm{HS}}^2 \,\mathrm{d}\lambda \right)^{1/2}. \tag{34}$$

Hence

$$\|f\|_2^2 \leqslant \left(\frac{2}{\pi}\right)^{(n+1)/2} \|(z_j - a_j)f\|_2 \left(\int |\lambda|^n \|\hat{f}(\lambda) H(\lambda)^{1/2}\|_{\mathrm{HS}}^2 \,\mathrm{d}\lambda \right)^{1/2} \tag{35}$$

Summing over j proves the theorem.

If we use the fact that $(\mathscr{L}^{1/2} f)\hat{}(\lambda) = \hat{f}(\lambda) H(\lambda)^{1/2}$ then we get the inequality

$$\| |z - a| f \|_2 \|\mathscr{L}^{1/2} f\|_2 \geqslant \sqrt{n} \left(\frac{\pi}{2}\right)^{(n+1)/2} \tag{36}$$

as advertised.

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