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Extension of certain types of generating relations

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Abstract. This paper gives a new expansion formula which essentially involves a double sum. As a consequence of our main result, eq. (6), various types of generating relations are seen to emerge and relevance with some known results is pointed out briefly. The usefulness of our main result is also indicated by considering its application to a probabilistic method for a lattice path enumeration problem.

Keywords. Generating relations; combinatorial coefficients; Laguerre polynomials; lattice path enumeration.

1. Introduction and the main result

For any real or complex λ, the binomial (or combinatorial) coefficients are defined by

$$\binom{\lambda}{n} = (-1)^n(-\lambda)_n/n!, \quad n\in\{0,1,2,\ldots\}, \tag{1}$$

where $(a)_n$ denotes the usual Pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{if } n=0, \\ a(a+1)\cdots(a+n-1), & \forall n\in\mathcal{N}. \end{cases} \tag{2}$$

In his paper Carlitz [1] derived a generating function for the Laguerre polynomials (see also [8, p. 378])

$$\sum_{n=0}^{\infty} L_n^{(\alpha+\beta n)}(x+ny)t^n = \frac{\exp(-xw)(1+w)^{\alpha+1}}{1-\beta w + yw(1+w)}, \tag{3}$$

where w is implicitly defined in terms of t by

$$w = t\exp(-yw)(1+w)^{\beta+1}. \tag{4}$$

For subsequent extension of Carlitz's work, one may refer to Srivastava [7].

The objective of this paper is to obtain a new generalization of (3) in the light of Gould's important combinatorial relation [4, p. 196, eq. (6.1)] (see also [8, p. 356, eq. (13)]). Our main result is given below:

Theorem. *For arbitrary parameters α, β and γ (independent of n and k), let*

$$S_n(\alpha,\beta,\gamma;z) = \sum_{k=0}^{n} \binom{\alpha+(\beta+1)n}{n-k} z^k/k!(\gamma+(\beta+1)n). \tag{5}$$

Then, for arbitrary x and y,

$$\sum_{n=0}^{\infty} S_n(\alpha, \beta, \gamma; x + yn)t^n$$
$$= (1 + w)^{\alpha} \exp[yw\gamma/(\beta + 1)] \sum_{n,k=0}^{\infty} \binom{\alpha - \gamma}{n} \binom{n + k + \gamma/(\beta + 1)}{n}^{-1} \cdot$$
$$\times (-w/(1 + w))^n [w(x - y\gamma/(\beta + 1))]^k / k! \, (\gamma + (\beta + 1)k), \tag{6}$$

provided that $\mathrm{Re}[\gamma/(\beta + 1)] > 0$, *w is a function of t given by*

$$w = t \exp(yw)(1 + w)^{\beta + 1}, \tag{7}$$

and the parameters assume such values that both sides of (6) *exist.*

2. Proof of (6)

Let us replace t by $t^{\beta+1}$ in an equivalent form of the Carlitz's formula (3) (see Raina [5, p. 592, eq. (2.3)]). Both sides of the resulting equation is multiplied by $t^{\gamma-1}\,\mathrm{d}t$, then we get

$$\sum_{n,k=0}^{\infty} \binom{\alpha + (\beta + 1)(n + k)}{n} [(x + y(n + k))^k / k!] t^{\gamma + (\beta + 1)(n + k) - 1} \, \mathrm{d}t \tag{8}$$
$$= \frac{e^{xw}(1 + w)^{\alpha + 1}}{1 - \beta w - yw(1 + w)} t^{\gamma - 1} \frac{\mathrm{d}t}{\mathrm{d}w} \cdot \mathrm{d}w,$$

where w is now given by

$$w = \exp(yw)[t(1 + w)]^{\beta + 1}. \tag{9}$$

On obtaining $\mathrm{d}t/\mathrm{d}w$ from (9) by logarithmic differentiation, substituting it on the right side of (8), then on invoking (9) and integrating the resulting equation from (8) yields

$$\sum_{n,k=0}^{\infty} \binom{\alpha + (\beta + 1)(n + k)}{n} [(x + y(n + k))^k / k!] \left. \frac{t^{\gamma + (\beta + 1)(n + k)}}{\gamma + (\beta + 1)(n + k)} \right]_0^t \tag{10}$$
$$= (\beta + 1)^{-1} \int_0^w w^{\gamma/(\beta + 1) - 1} (1 + w)^{\alpha - \gamma} \exp[w(x - y\gamma/(\beta + 1))] \, \mathrm{d}w,$$

provided that $\mathrm{Re}[\gamma/(\beta + 1)] > 0$.

By appealing to a special case of an integral given in [2, p. 71, eq. (2.11.4)] to evaluate the integral on the right side, and then making use of a linear transformation [8, p. 33, eq. (19)] for the ${}_2F_1$-function involved, the resulting equation from (10) in accompaniment with (7) and the defining equation (5) is seen to lead to the desired result (6) upon setting t for $t^{\beta+1}$.

Remark 1. The formula (6) can also be expressed in the equivalent form

$$\sum_{n=0}^{\infty} \frac{\gamma}{\gamma + (\beta + 1)n} L_n^{(\alpha + \beta n)}(-x - ny)t^n \tag{11}$$

$$= (1+w)^{\alpha} \exp[ywy/(\beta+1)]$$
$$\Xi_1[\gamma-\alpha, \gamma/(\beta+1), 1; 1+\gamma/(\beta+1); w/(1+w), w(x-y\gamma/(\beta+1))],$$

where w (as before) is given by (7), and the function $\Xi_1[x, y]$ is one of the Humbert's confluent hypergeometric function of two variables [8, p. 59, eq. (43)].

Remark 2. It may be observed that in the limiting case when $|\gamma| \to \infty$, (11) corresponds (formally) to the Carlitz's formula (3), a fact which can be verified.

3. Applications

The main result (6) can be specialized and applied to derive several classes of generating functions. Most of these results have relevance with the results recorded in Srivastava-Manocha monograph [8]. We leave to the interested reader to verify the outcome of such results from (6). We however point out the application of (6) in a certain probabilistic method in finding the solution of a functional equation involving generating functions for lattice path enumeration problem.

As explained in [3], let A_n denote the number of paths in a plane from the origin to the point $(n, \alpha+\beta n)$, where α and β are positive integers, with unit steps in the positive horizontal and vertical directions, which never touch the line $y=\alpha+\beta x$, except at the end point. If p is the probability that the particle starting from the origin moves one unit to the right and with probability $q=(1-p)$ one unit up, then the probability that the particle stops at $(n, \alpha+\beta n)$ is $A_n p^n q^{\alpha+\beta n}$.

If we let

$$A(t) = \sum_{n=0}^{\infty} A_n t^n, \tag{12}$$

then for sufficiently small p, the particle will eventually reach the line $y=\alpha+\beta x$, with probability 1, satisfying

$$q^{\alpha} A(pq^{\beta}) = 1. \tag{13}$$

To solve the functional eq. (13), we use (6) in the following manner:

We set $\gamma=\alpha$, and $x=y=0$, in (6), and then putting $1+w=q^{-1}$, in the resulting simplified form, so that $w=(1-q)/q=p/q$, we then have

$$\sum_{n=0}^{\infty} \binom{\alpha+(\beta+1)n}{n} t^n/(\alpha+(\beta+1)n) = q^{-\alpha}, \tag{14}$$

where

$$t = pq^{\beta}. \tag{15}$$

A comparison of (13) and (14) in view of (12) at once gives the number of paths A_n in the plane by

$$A_n = \frac{1}{\alpha+(\beta+1)n} \binom{\alpha+(\beta+1)n}{n}. \tag{16}$$

In a similar way, other simpler cases as investigated in [3, pp. 50–51] may be treated with the help of our general formula (6). We conclude by remarking that the present work can be extended further on the lines of recent papers of Raina [6] and Srivastava and Raina [9].

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