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Decomposition of the de Rham complex

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Abstract. We give a reformulation of Deligne and Illusie's characteristic p proof of the degeneration of the "Hodge-to-de Rham" spectral sequence, which replaces patching in the derived category by an explicit quasi-isomorphism.

Keywords. de Rham complex; Hodge decomposition; liftings of varieties.

Deligne and Illusie [1] showed that if X is a smooth variety of dimension d over a perfect field k of characteristic p, such that $d < p$, and X has a lifting to a scheme flat (and hence smooth) over $W_2(k)$, the ring of Witt vectors of length 2 over k, then we have an isomorphism in the derived category of bounded complexes of coherent $\mathcal{O}_X$ modules

$$F_*\Omega^{\cdot}_{X/k} \cong \oplus_{i=0}^{d}\Omega^{i}_{X/k}[-i].$$

This easily implies the degeneration of the "Hodge-to-de Rham" spectral sequence

$$E_1^{p,q} = H^q(X,\Omega^p_{X/k}) \Rightarrow H^{p+q}_{DR}(X/k),$$

as well as the vanishing theorem of Kodaira-Akizuki-Nakano

$$H^q(X,\Omega^p_{X/k}\otimes\mathscr{L}^{-1}) = 0 \quad \forall p+q<d$$

for an ample invertible sheaf $\mathscr{L}$ on X. This implies both statements for smooth proper varieties in characteristic 0 as well.

Our aim here is to give a quick proof of this result, using the following facts:

(i) If X is smooth over k, and B^i_X, Z^i_X are the sheaves of locally exact and closed i-forms, respectively, with their natural $\mathcal{O}_X$ module structures (as $\mathcal{O}_X$-submodules of $F_*\Omega^i_{X/k}$), and

$$0 \to B^i_X \to Z^i_X \xrightarrow{C} \Omega^i_{X/k} \to 0$$

(where C is the Cartier operator) has extension class

$$\xi_i \in \mathrm{Ext}^1_X(\Omega^i_{X/k}, B^i_X)$$

then ξ_1 is the obstruction to lifting the *pair* (X, F) to a pair $(X^{(2)}, F^{(2)})$, where $X^{(2)}$

is flat over $W_2(k)$, and $F^{(2)}: X^{(2)} \to X^{(2)}$ is a morphism, covering the Frobenius automorphism of $W_2(k)$, and restricting to the absolute Frobenius morphism of X. A proof of this is given in the appendix to [2].

(ii) If we apply the functor $\mathrm{Hom}_X(\Omega^1_{X/k}, --)$ to the exact sheaf sequence (d = exterior derivative)

$$0 \to \mathcal{O}_X \to F_*\mathcal{O}_X \xrightarrow{d} B^1_X \to 0,$$

to obtain the exact sequence

$$\cdots \to \mathrm{Ext}^1_X(\Omega^1_{X/k}, \mathcal{O}_X) \to \mathrm{Ext}^1_X(\Omega^1_{X/k}, F_*\mathcal{O}_X) \xrightarrow{\alpha} \mathrm{Ext}^1_X(\Omega^1_{X/k}, B^1_{X/k})$$

$$\xrightarrow{\partial} \mathrm{Ext}^2_X(\Omega^1_{X/k}, \mathcal{O}_X) \to \cdots$$

then $\partial(\xi_1) \in \mathrm{Ext}^2_X(\Omega^1_{X/k}, \mathcal{O}_X)$ is the obstruction to lifting X to a scheme $X^{(2)}$ flat over $W_2(k)$. This follows easily from the argument in the appendix to [2] (used in proving (i)).

Granting these statements, we see that if X is a smooth variety over k which has a flat lifting to a scheme over $W_2(k)$, then

$$\partial(\xi_1) = 0,$$

so that we have:

Lemma 1. *If X lifts to a scheme flat over $W_2(k)$, then*

$$\xi_1 = \alpha(\zeta)$$

for some $\zeta \in \mathrm{Ext}^1_X(\Omega^1_{X/k}, F_*\mathcal{O}_X)$. Hence we have a pushout diagram of exact sequences

$$\begin{array}{ccccccccc} 0 \to & F_*\mathcal{O}_X & \to & \mathscr{V} & \to & \Omega^1_{X/k} & \to 0 \\ & d\downarrow & & \downarrow & & \| & \\ 0 \to & B^1_X & \to & Z^1_X & \to & \Omega^1_{X/k} & \to 0 \end{array}$$

for some locally free $\mathcal{O}_X$-module $\mathscr{V}$.

The top row may be regarded as a quasi-isomorphism (which we denote ψ_1) between the 2-term complex (which we denote $\mathscr{C}_1$)

$$F_*\mathcal{O}_X \to \mathscr{V}$$

and

$$\Omega^1_{X/k}[-1].$$

The pushout diagram induces a map of complexes

$$\varphi_1: \mathscr{C}_1 \to F_*\Omega^{\cdot}_{X/k},$$

which induces an isomorphism on first cohomology sheaves. This is done as follows: if $\beta: \mathscr{V} \to Z^1_X$ is the vertical map in the pushout diagram, and $\gamma: \mathscr{V} \to F_*\Omega^1_{X/k}$ is obtained from β by composing with the inclusion $Z^1_X \subset F_*\Omega^1_{X/k}$, then we have a diagram whose vertical maps give a map of complexes ($d = \dim X$)

$$\begin{array}{ccccccccccc} 0 \to & F_*\mathcal{O}_X & \to & \mathcal{V} & \to & 0 & \to \cdots \to & 0 & \to & 0 \\ & \| & & \gamma\downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 \to & F_*\mathcal{O}_X & \to & F_*\Omega^1_{X/k} & \to & F_*\Omega^2_{X/k} & \to \cdots \to & F_*\Omega^d_{X/k} & \to & 0. \end{array}$$

The complex $\mathscr{C}_i^{\otimes i}$ is clearly quasi-isomorphic to the complex $(\Omega^1_{X/k})^{\otimes i}[-i]$; the multiplicative structure on the de Rham complex yields a map "$\varphi_i^{\otimes i}$" of complexes

$$C_1^{\otimes i} \to F_*\Omega^{\cdot}_{X/k},$$

which yields the exterior product map

$$\Lambda^i : (\Omega^1_{X/k})^{\otimes i} \to \Omega^i_{X/k}$$

on i-th cohomology sheaves. If $i < p$, the characteristic, then as in [1], we see that the map Λ^i is a *split surjection*; hence we can find an injective $\mathcal{O}_X$-linear map

$$h_i : \Omega^i_{X/k} \to (\Omega^1_{X/k})^{\otimes i}$$

which splits the surjection Λ^i. Let $\mathscr{C}_i$ be the complex of locally free $\mathcal{O}_X$-modules obtained from the pullback diagram with exact rows

$$\begin{array}{ccccccccc} 0 \to & (\mathscr{C}_i)^0 & \cdots & (\mathscr{C}_i)^{i-1} & \to & (\mathscr{C}_i)^i & \to & \Omega^i_{X/k} & \to 0 \\ & \| & & \| & & \downarrow & & h_i\downarrow & \\ 0 \to & (\mathscr{C}_1^{\otimes i})^0 & \cdots & (\mathscr{C}_1^{\otimes i})^{i-1} & \to & (\mathscr{C}_1^{\otimes i})^i & \to & (\Omega^1_{X/k})^{\otimes i} & \to 0 \end{array}$$

Thus, there is a quasi-isomorphism

$$\psi_i : \mathscr{C}_i \to \Omega^i_{X/k}[-i],$$

and "$\varphi_1^{\otimes i}$" induces a map of complexes

$$\varphi_i : \mathscr{C}_i \to F_*\Omega^{\cdot}_{X/k}$$

which yields an isomorphism on the ith cohomology sheaves.

Hence, if $d < p$, the map

$$\sum_{i=0}^{d} \varphi_i : \oplus_{i=0}^{d} \mathscr{C}_i \to F_*\Omega^{\cdot}_{X/k}$$

is a quasi-isomorphism (define $\mathscr{C}_0$ to be $\mathcal{O}_X$ in degree 0, and φ_0 to be the map induced by the inclusion $\mathcal{O}_X \to F_*\mathcal{O}_X$). Since there is a quasi-isomorphism

$$\oplus_{i=0}^{d} \psi_i : \oplus_{i=0}^{d} \mathscr{C}_i \to \oplus_{i=0}^{d} \Omega^i_{X/k}[-i],$$

we deduce that there is an isomorphism in the derived category

$$F_*\Omega_{X/k} \to \oplus_{i=0}^{d} \Omega^i_{X/k}[-i].$$

Remark. The argument may be made more elementary, by giving a direct proof of the lemma, which does not appeal to the results in the Appendix of [2] (however, the proof given there is along the same lines as the argument given below; it is also the main point of the argument in [1], apart from the patching argument).

If A is a smooth k-algebra of finite type, and B is a flat $W_2(k)$-algebra lifting A i.e. satisfying $B \otimes_{W_2(k)} k \cong A$, then for any $W_2(k)$-algebra C, any k-algebra homomorphism $f: A \to C \otimes_{W_2(k)} k$ "lifts" to a $W_2(k)$-algebra homomorphism $B \to C$. Hence, if X is a smooth k-variety which lifts to a flat $W_2(k)$-scheme Y, then for each affine open set $U \subset Y$, the absolute Frobenius morphism of $V = X \cap U$ lifts to a $W_2(k)$-semi-linear morphism $U \to U$ (take $B = H^0(\mathcal{O}_U)$, $A = H^0(\mathcal{O}_V)$, and $C = H^0(F_* \mathcal{O}_U)$).

If $U = \operatorname{Spec} B$, $V = \operatorname{Spec} A$, then such a lifting of Frobenius corresponds to the choice of a function $\varphi: B \to A$ such that, identifying A with pB, (p=characteristic of k), we have formulae

$$\varphi(a+b) = \varphi(a) + \varphi(b) - \sum_{j=1}^{p-1} \left(\binom{p}{j} \Big/ p \right) \bar{a}^j \bar{b}^{p-j},$$

$$\varphi(ab) = \bar{a}^p \varphi(b) + \bar{b}^p \varphi(a),$$

for all $a, b \in B$, where $\bar{a}$, $\bar{b}$ denote the images of a, b in A. The lifting of Frobenius is given by the map $B \to B$, $a \mapsto a^p + p\varphi(a)$. (Note that

$$\binom{p}{j} \Big/ p$$

is an integer.)

Given such a lifting of Frobenius, we have a derivation $A \to Z_A^1$ (where Z_A^1 is the A-module corresponding to $Z^1_{\operatorname{Spec} A}$), given by $a \mapsto a^{p-1} da + d\varphi(\tilde{a})$, where $\tilde{a} \in B$ lifts a. This is well defined, and induces an A-linear map $\Omega^1_{A/k} \to Z^1_A$ which splits the surjection given by the Cartier operator.

Hence if $\{U_i = \operatorname{Spec} B_i\}$ is an affine open cover of $X^{(2)}$, $\{V_i = \operatorname{Spec} A_i\}$ the corresponding open cover of X, and $\varphi_i: B_i \to B_i$ are lifts of Frobenius, then on $V_i \cap V_j = \operatorname{Spec} A_{ij}$, the derivation $a \mapsto d\varphi_i(\tilde{a}) - d\varphi_j(\tilde{a})$ yields an A_{ij}-linear map $c_{ij}: \Omega^1_{A_{ij}/k} \to B^1_{A_{ij}}$, such that $\{c_{ij}\}$ gives a Čech 1-cocycle representing the extension class

$$\xi_1 \in \operatorname{Ext}^1_X (\Omega^1_{X/k}, B^1_X).$$

But then $a \mapsto \varphi_i(\tilde{a}) - \varphi_j(\tilde{a})$ is a derivation $A_{ij} \to F_* A_{ij}$, giving a map $b_{ij}: \Omega^1_{A_{ij}} \to F_* A_{ij}$, and $\{b_{ij}\}$ is a 1-cocyle representing a class

$$\zeta \in \operatorname{Ext}^1_X (\Omega^1_{X/k}, F_* \mathcal{O}_X)$$

satisfying $\alpha(\zeta) = \xi_1$. The existence of such a class ζ is precisely the assertion of the lemma.

References

[1] Deligne P and Illusie L, Rélevements modulo p^2 et décomposition du complexe de de Rham, *Invent. Math.* **89** (1987) 247–270

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