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$\tilde{K}_3'$ – A new triangulation of R^n

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Abstract. This paper introduces a new triangulation of R^n. The triangulation is to be applied for solving system of nonlinear equation through a fixed point technique.

Keywords. Triangulation; grid size; fixed point algorithm.

1. Introduction

Since 1967 when Scarf [6] initiated a new algorithm for the computation of fixed point of a given mapping through his concept of primitive set, there have been many attempts to improve it (Khun [4], Todd [9], Eaves [1] and Eaves and Saigal [3]). Basically these algorithms require triangulation, labelling of vertices of triangulation and a particular starting point called pivot point. Todd [10] developed a new triangulation for $R^n \times [0,1]$ and $S^n \times [0,1]$. Eaves [2] and Eaves and Saigal [3] introduced the triangulations for $S^n \times [1,\infty[$ or $R^n \times [1,\infty[$ which decompose those spaces into finer meshes thereby generating an infinite path which approximates f better as deformation tends to infinity. The efficiency of a fixed point algorithm is sensitive to the triangulation used. Hence, the continuous refinement in the triangulation provides the improved computational technique for the fixed point algorithm to calculate the fixed point of a mapping to a greater accuracy.

2. Notation and definition

As usual R denotes the real line, Z the set of integers, E the set of even integers and O the set of odd integers. All the n-dimensional spaces have coordinates 1 through n, while the $(n+1)$-dimensional spaces have coordinates 0 (zero) through n. Similarly, the n $(n+1)$ rows or columns of the matrix are indexed 1 (one) through n (0 through n). Π and Π_{n+1} denote the group of permutation on $\{1,2,\ldots,n\}$ and $\{0,1,2,\ldots,n\}$ respectively. The unit vectors in R^n and R^{n+1} are denoted by $\{e^1,\ldots,e^n\}$ and $\{f^0,\ldots,f^n\}$ respectively.

The standard n-dimensional closed simplex S^n is the convex hull of f^0 through f^n. A K-dimensional simplex (or K simplex) σ is the relative interior of the convex hull of $(K+1)$ affinely independent points $y^0,\ldots,y^k$ called its vertices. We denote the simplex by $\sigma = (y^0,\ldots,y^k)$. A simplex τ is a face of σ iff its vertices are a subset of the vertices of σ and the closure of $(k-1)$-dimensional face of K-simplex σ is called a facet of σ. Two simplices are said to be adjacent if they have a common facet.

3. Preliminaries

3.1 Basically triangulation can be classified into two forms (details can be found in lecture notes of Todd [9]). One form is regular triangulation and the other triangulation with continuous refinement of grid size. The former consists of K, H, J_1 and K_1 developed by Khan [9], Merrill, Tuckәr and Freudenthal (cf. [9] for details) and the latter consists of K_3 due to Eaves an ...aigal [3] and J_3 due to Todd [9]. In between there are J_1, J_2, K_2 types of triangulation developed by Todd [10] and Eaves [2]. J_3 is the refinement of J_1 and K_3 the refinement of K_2.

3.2 We give below the summary of earlier triangulations.

The triangulation K_1 of R^n. Let $K_1^0 = Z^n = \{y \in R^n | y^i \in Z$ for each $i \in N\}$. If $y^0 \in K_1^0$ and Π is a permutation of N, then $K_1(y^0, \Pi)$ denotes n-simplex $(y^0, \ldots, y^n)$ where $y^i = y^{i-1} + u^{\Pi(i)}$ for each $i \in N$.

Finally, let K_1 be the collection of all such $K_1(y^0, \Pi)$. Whenever we set $K_1(y^0, \Pi) = (y^0, \ldots, y^n)$, we suppose the $y^{i,s}$ ordered as above.

The triangulation J_1 of R^n. Let $J_1^0 = Z^n$. Denoting members of J_1^0 (to be called "central vertices") by J_1^{0c} is the set $\{y \in J_1^0 | y_i$ is odd for each $i \in N\}$. If $y^0 \in J_1^{0c}$, Π is a permutation of N, and $s \in R^n$ is a sign vector (each s_i is ± 1), then denote by $j_1(y^0, \Pi, s)$ the n-simplex $(y^0, \ldots, y^n)$ where $y^i = y^{i-1} + S_{\Pi(i)} u^{\Pi(i)}$ for each $i \in N$. Let J_1 be the collection of all such $j_1(y^0, \Pi, s)$.

The triangulation K_2 of S^n. Let Q denote $(n+1) \times n$ matrix

$$\begin{pmatrix} -1 & 0 & \cdots & 0 \\ +1 & -1 & \cdots & 0 \\ 0 & +1 & \cdots & \vdots \\ 0 & 0 & \vdots & -1 \\ \vdots & \vdots & \cdots & +1 \end{pmatrix}$$

and q^j its jth column for $j \in N$. Let $K_2^0(m) = \{y \in S^n | my_i \in Z\}$ for each $i \in N_0$. If $y^0 \in K_2^0(m)$ and Π is a permutation of N, let $\sigma = (y^0, \ldots, y^n)$ where $y^i = y^{i-1} + m^{-1} q^{\Pi(i)}$ for each $i \in N$. If $\sigma \subseteq S^n$, we write $\sigma = k_2(y^0, \Pi)$ (m is implicit). Finally, let $K_2(m)$ be the collection of all such $k_2(y^0, \Pi)$.

The triangulation J_2 of S^n. Let $J_2^0(m) = K_2^0(m)$ and $J_2^{0c}(m) = \{y \in J_2^0(m) | my_i$ is even, $1 \leqslant i < n$ and my_n is odd$\}$. If $y^0 \in J_2^{0c}(m)$, Π a permutation of N, and $s \in R^n$ a sign vector, then $\sigma = (y^0, \ldots, y^n)$ where $y^i = y^{i-1} + m^{-1} s_{\Pi(i)} q^{\Pi(i)}$ for each $i \in N$. If $\sigma \subseteq S^n$, then we write $\sigma = j_2(y^0, \Pi, s)$. Finally, let $J_2(m)$ be the collection of all such $j_2(y^0, \Pi, s)$.

3.3 We will discuss K_3' and J_3 with continuous refinement of grid size due to Todd [10]. K_3' is a modification of K_3 which triangulates $]0, 1] \times R^n$. Then the $(n+1)$-dimensional simplex σ of K_3' is given by $(y^{-1}, \ldots, y^n)$ and pair (y, Π) $K_3'0 \times \Pi_{n+1}$ with $\Pi(j) = 0$ and $\mu l(y) = 0$ if $\Pi^{-1}(l) > j$ is called admissible where $K_3'0$ is the set of vertices given by $\{y \in R^{n+1} : y_0 = 2^{-k},\ 0 \leqslant k \in Z,\ y_0/y_i \in Z$ for $1 \leqslant i \leqslant n\}$ and μ: $K_3'0 - B^{n+1}$ is defined by $\mu_i(y) = 1 - y_i/y_0 \bmod 2$.

The vertices $\{y^{-1}, y^0, \ldots, y^n\}$ of the $(n+1)$-dimensional simplex of K'_3 are defined as

$$y^{-1} = y$$
$$y^i = y^{i-1} + y_0 f^{\Pi(i)}, \text{ where } 0 \leqslant i < j$$
$$y^j = y^{j-1} - \sum_{i=0} \mu\Pi(i) y_0 f^{\Pi(i)} + y_0 f^0 - \; y_0 f^{\Pi(K)},\; K = j+1$$
$$y^k = y^{k-1} + 2y_0 f^{\Pi(K)}, \quad j < k \leqslant n.$$

We denote σ by $K\ (y, \Pi)$. $\tilde{J}_3$ is the triangulation derived for J_3 by Todd [10] by taking $\{v_1, v_2, \ldots, v_n\}$ as set of unit vectors in R^n and $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{n+1}\}$ as the set of unit vectors in R^{n+1}. Suppose $\bar{y} \in Z^n \times [1]$ have all $\bar{y}_i$'s odd let Π be the permutation of $\{1, 2, \ldots, n+1\}$ with $\Pi(j) = n+1$. Let $\bar{s} \in R^n \times [-1]$ be the sign vector such that for $j < k \leqslant n+1$, $\bar{s}_{\Pi(K)}$ is $-1\ (+1)$ if

$$\bar{y}_{\Pi(i)} \text{ is } 1(3) \bmod 4.$$

Then $\tilde{J}_3(y, \Pi, \bar{s})$ denotes simplex $(\bar{y}^0, \ldots, \bar{y}^{n+1})$ where

$$\bar{y}^0 = \bar{y}$$
$$\bar{y}^i = \bar{y}^{i-1} + \bar{s}_{\Pi(i)} \bar{U}^{\Pi(i)}, \quad 1 \leqslant i \leqslant j$$
$$\bar{y}^j = \bar{y}^{j-1} - \sum_{K=j+1}^{n+1} \bar{s}_{\Pi(K)} \bar{U}^{\Pi(K)} + \bar{s}_{n+1} \bar{U}^{n+1}$$
$$\bar{y}^k = \bar{y}^{k-1} + 2\bar{s}_{\Pi(K)}, \quad j < k \leqslant n+1$$

$\tilde{J}$ is the set of all such $\tilde{J}_3\ (\bar{y}, \Pi, \bar{s})$.

4. Main result

We define new triangulation $\tilde{K}'_3$ as follows:

$\tilde{K}'_3 0$ is the set of vertices given by

$\{\bar{y} \in R^{n+1} : \bar{y}_0 = 2^{-k}, \quad 0 \leqslant k \in Z$

$\bar{y}_i / \bar{y}_0 \in Z, \quad \text{for } 1 \leqslant i \leqslant n\}$ and

$\mu : \tilde{K}'_3 — B^{n+1}$ is defined by

$\mu_i(\bar{y}) = 1 - \bar{y}_i / \bar{y}_0 \bmod 2.$

The $(n+1)$-dimensional simplex $\tilde{K}'_3\ (\bar{y}, \Pi)$ denotes the simplex $(\bar{y}^0, \ldots, \bar{y}^{n+1})$, where

$$\bar{y}^0 = \bar{y}$$
$$\bar{y}^i = \bar{y}^{i+1} + \bar{y}_0 f^{\Pi(i)}, \quad 0 \leqslant i < j$$
$$\bar{y}^j = \bar{y}^{j+1} - \sum \mu_{\Pi(i)} \bar{y}_0 f^{\Pi(i)} + \bar{y}_0 f_0 - \sum \bar{y}_0 f^{\Pi(i)} + \bar{y}_0 f^{n+1}$$
$$\bar{y}^k = \bar{y}^{k+1} + 2\bar{y}_0 f^{\Pi(k)}, \quad j < k \leqslant n+1$$

$\tilde{K}'_3$ is the set of such $\tilde{K}'_3\ (\bar{y}\Pi)$ and Π is the permutation on $\{1, 2, \ldots, n+1\}$.

4.1 Theorem. *$\tilde{K}'_3$ is the triangulation of* $[0,1] \times R^n$.

Proof. To prove $\tilde{K}'_3$ triangulation of $[0,1] \times R^n$ we will show that simplices of $\tilde{K}'_3$ together with all these faces cover $[0,1] \times R^n$ and none of their faces intersect.

As every neighbourhood of point of $]0,1[\times R^n$ meets finite number of simplices, it is sufficient to prove the following two conditions:

(1) For each $x \in]0,1[\times R^n \exists \sigma \in \tilde{K}'_3$ such that

(2) Each $x \in]0,1[\times R^n$ lies in the unique face of some $\sigma \in \tilde{K}'_3$

(1) Let k be such that $2^{-k} \leqslant x_0 < 2^{-k+1}$ and for $0 \leqslant i \leqslant n$, let $\bar{y}_i = \bar{2}^k[2^k, x_i]$ where $[a]$ denotes the largest integer not greater than a. Then $\bar{y} \in \tilde{K}'_3 0$ let

$$z = x - \bar{y} \geqslant 0, \quad \mu = \mu(\bar{y})$$

and

$$\bar{w} = z + z_0 \mu.$$

We again consider two cases:

Case I. The largest component $\bar{w}$ is at most $\bar{y}_0$ order the component of $\bar{w}$ using $\Pi \in \{1,2,\ldots,n+1\}$, we have

$$\bar{y}_0 \geqslant \bar{w}_{\Pi(j)} \geqslant \cdots \bar{w}_{\Pi(n)} \geqslant -\bar{w}_0 \tag{1}$$

when $\Pi(j)=0$ and j is maximal then if $j<n$, $\bar{w}_0 > \bar{w}_l$ for $\Pi^{-1}(l) > jZ_1 + Z_0\ \mu l < Z_0$ since $Z_0 \geqslant 0$ we have $\mu l = 0 \Rightarrow (\bar{y}, \Pi)$ is admissible. Denoting the terms in (1) by

$$\alpha_0 = \bar{y}_0$$

$$\alpha_0, \ldots, \alpha_{n+1} = -\bar{w}_0$$

define

$$\beta_i = (\alpha_i - \alpha_{i+1})/\bar{y}_0, \text{ for } 0 \leqslant i < j$$

and

$$\beta_k = (\alpha_{k-1} - \alpha_k)\, 2\bar{y}_0 \text{ for } j \leqslant k \leqslant n+1.$$

Then

$$\beta \geqslant 0 \quad \text{and} \quad \sum_{i=0}^{n+1} \beta_i = 1$$

and

$$x = \sum_{l=0}^{n+1} \beta l \bar{y}^l$$

where

$\{\bar{y}^0, \ldots, \bar{y}^{n+1}\}$ are given by (1) thus $x \in \bar{\sigma} = K(\bar{y}, \Pi) \in \tilde{K}'_3$.

Case II. $\bar{w}_i > y_0$ for $i \in j \leqslant \{1,2,\ldots,n+1\}$. Since $z_i \leqslant y$ for $1 \leqslant i \leqslant n$, we must have $\mu_i = 1$ for $i \in j$ increase the components of corresponding to $i \in j$ by $\bar{y}_0$ and again let $\mu = \mu(\bar{y})$, $z = x - \bar{y}$ and $w = z + z_0\mu$. Here μ_i and $\bar{w}_i$ remain unchanged for $i \notin j$ and for $i \in j$ μ_i becomes 0 and $\bar{w}_i$ is reduced by x_0 implying that x_0 lies between 0 and $-\bar{w}_0$. Thus the component of $\bar{w}$ can be ordered using $\Pi = \{1,2,3,\ldots,n+1\}$ we have that (1) holds with $\Pi(j)=0$ and j is maximal. If $\Pi^{-1}(l) > j$ then $\mu l = 0$ because $l \in j$ or for $z_0 \geqslant 0$ we have $\mu l = 0$ as in case I. Thus $(\bar{y}, \Pi)$ is admissible. Again, as in case I,

β is obtained by showing that $x \in \bar{\sigma}$ for some $\bar{\sigma} = K(\bar{y}, \Pi) \in \tilde{K}'_3$. Thus x is in a face of some simplex $\tilde{K}'_3$.
(II) By condition (i) we can assume that $x \in \bar{\sigma}$ for $\sigma = (\bar{y}^0, \ldots, \bar{y}^{n+1}) = K(\bar{y}, \Pi)$ let

$$x = \sum_{l=0}^{n+1} \beta l \bar{y}^l \text{ with } \beta \geqslant 0 \text{ and } \sum_{l=0}^{n+1} \beta l = 1$$

then x lies in the face of whose vertices are those $\bar{y}^l$ with $l \in L = \{l: \beta l > 0\}$. This face is called carrier of x in $\bar{\sigma}$. It is sufficient to show that $\{\bar{y}^l : l \in L\}$ is uniquely determined from x independent of $\bar{y}$ and Π. Define

$$\alpha = \alpha(x) \text{ by } \alpha_i = \bar{y}_0 \sum_{l=0}^{n} \beta l \text{ for } 0 \leqslant i \leqslant j$$

and

$$\alpha_k = \bar{y}_0 \left(\sum_{l=j}^{k-1} \beta l + \sum_{l=k}^{n+1} \beta l \right) \text{ for } j < k \leqslant n+1$$

$$\Rightarrow \alpha_0 \geqslant \alpha_1 \geqslant \cdots \geqslant \alpha_{n+1} \text{ with } \alpha_{n+1} = -\alpha_j$$

and $\alpha_l > \alpha_{l+1}$ iff $\rho_l = 0$. These α's are in fact the terms of (i) corresponding to x. Again define $x' \in R^{n+1}$ by

$$x'\Pi(l) = \bar{y}\Pi(l) - \mu\Pi(l)\alpha_j + \alpha_l, \quad 0 \leqslant 1 \leqslant n.$$

Then α's are precisely the terms of (1) corresponding to x again it can be easily shown that

$$x' = \sum_{l=1}^{n+1} \beta'_l y^l \Rightarrow \beta' = \beta \text{ and } x' = x.$$

Similarly, we can show that for any $z \in \bar{\sigma}$ we can obtain $\alpha(z)$ and for $l \in L$ iff the position of strict inequality in the element of $\alpha(y^l)$ also corresponds to strict inequality in $\alpha(x)$. We distinguish two cases:

Case 1. When x_0 is not power of $\frac{1}{2}$. We claim that for $l \in L \exists$ unique y^l such that we have $\alpha(y^l)$. let

$$\gamma \in]\bar{y}_0 - x_0, \bar{y}_0]$$

denote $z = z\,(\gamma)$ for γ lies each non-trivial interval between adjacent coordinate of $\alpha(\gamma)$ if $\gamma > x_0 - y_0$, we have for any z

$$Z_i = \begin{cases} x_i \text{ if } x_i/\bar{y}_0 \in E, \\ x_i + x_0 - \bar{y}_0 \text{ if } (x_i + x_0 - \bar{y}_0)/\bar{y}_0 \in 0. \end{cases}$$

To determine $\bar{y}_i$ and $\bar{w}_i$ from x, we apply

$$Z_i = \begin{cases} \bar{y}_i + \bar{y}_0 \text{ for } \bar{w}_i \geqslant \gamma \\ \bar{y}_i \text{ for } \bar{w}_i < \gamma \text{ (this case includes } i = 0). \end{cases}$$

If $\gamma \leqslant x - \bar{y}_0$, then z can be obtained as follows:

$$Z_i = \begin{cases} x_i \text{ if } x_i/\bar{y}_0 \in E, \\ x_i + x_0 - 2\bar{y}_0, \text{ if } (x_i + x_0 - \bar{y}_0)/\bar{y}_0 \in 0. \end{cases}$$

To determine $\bar{y}_i$ and $\bar{w}_i$ from we apply for when

$$Z = \begin{cases} \bar{y}_i + \bar{y}_0 \text{ for } \bar{w}_i \geqslant \gamma \text{ (this case includes } i=0) \\ \bar{y}_i - \bar{y}_0 \text{ for } \bar{w}_i < \gamma. \end{cases}$$

Thus showing that vertices of the carrier of $x \in \bar{\sigma}$ can be obtained from x alone.

Case 2. When x is a power of $\frac{1}{2}$ since each y_0^l is a power of $\frac{1}{2}$ and only two adjacent powers may appear $y_0^l = x \forall l \in L$ showing that all vertices of σ with the same 0th coordinates for facing $\tilde{K}_3'$ thereby proving the result in this case as well.

5. Comparative analysis

We have introduced a new triangulation known as $\tilde{K}_3'$ which is a modification of K_3 and K_3' developed by Eaves and Saigal [8] and Todd [9] respectively. K_3' is purely an extension of K_3' by including the boundary of rectangle used in K_3' on the right hand side. Todd [9] has described various theoretical measures of the efficiency of triangulation for computing fixed point and determine their values for all the triangulations described by him on the basis of their measure. He found that J_1 and K_1 are roughly comparable and both are superior to H. J_1 is less sensitive to different direction than K. In horizontal direction J_3 and K_3' are again comparable, but in the vertical direction J_3 appears to be superior.

Similarly $\tilde{K}_3'$ is comparable with J_3 in horizontal direction and very much equivalent to K_3' as both J_3' and K_3' have been coded in Fortran for the purpose of computation. So $\tilde{K}_3'$ can be coded in a similar way for the computation of non-linear problems.

Besides, this new triangulation is superior because its algorithms will take a few steps to calculate accurately fixed point for any mapping f. Any infinite path generated to the infinite sequence gives distinct and complete simplices of homotopy h defined for the mapping f. In summary.

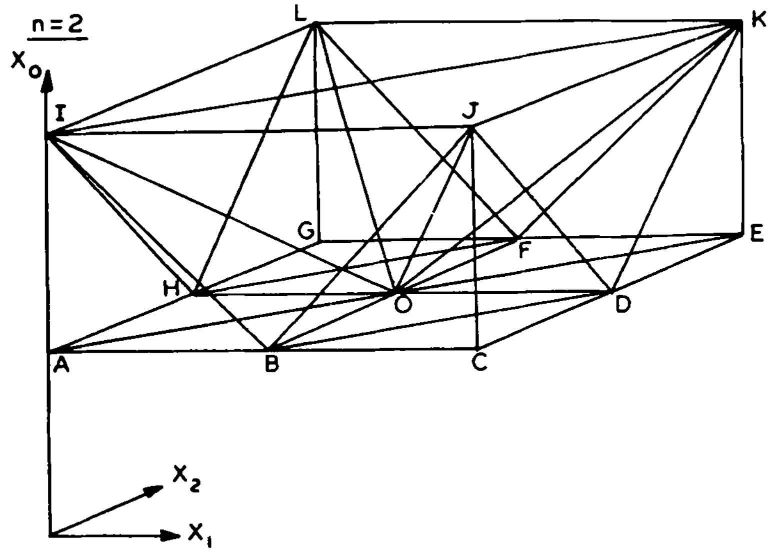
Step 1. Let σ_1 be the unique $(n+1)$-simplex of G with τ_0 as facet where G denotes a triangulation with continuous refinement of grid size. Let y_0^+ with the vertex σ_1 outside $R_{(0)}^n$ be a linear order system showing τ_0 as a complete set $m \leftarrow 1$.

Step 2. Define l: $[0,1] \times R^n \to R^n$ by $l(x) = h(x) - P(x)$ and calculate $l(\bar{y}^+)$. Introduce the column $(l(\bar{y}^+))$ into the current basis of linear order system. Assume that the column $(l(1/(y^+))$ drops from the basis with $\bar{y}'$ a vertex of σ_m.

Step 3. Let τ_m be the complete facet of σ_m opposite $\bar{y}'$. If desired, calculate the fixed point of homotopy h in τ_m and ferminal if it is sufficiently accurate. Let σ_{m+1} be the simple of G sharing the facet τ_m with σ_m let y be the new vertex, set $m \leftarrow m+1$ and return to step 1.

Generally the fixed point of homotopy h in $\bar{\tau}_m$ is calculated only when $\tau_m \subseteq R_{(k)}^n$ for some k.

For $n=2$ following example shows the triangulations of $\tilde{k}_3'$.



This is the triangulation by $\tilde{K}'_3$ of $[\frac{1}{2}, 1] \times [0,1]^2$. A list of the simplices follows. (Note that without the stipulation of admissibility, other simplicies intersecting these (such as AIJK, ABIL, BCJK) would appear).

$:\tilde{K}'_3\,(y,\Pi)$	y	Π	$:\tilde{K}'_3\,(y,\Pi)$	y	Π
(ABOI)	A	(1,2,0)	(AHOI)	A	(2,1,0)
(BODJ)	B	(2,1,0)	(BOIJ)	B	(2,0,1)
(BCDJ)	B	(1,2,0)	(ODJK)	O	(1,0,2)
(ODEK)	O	(1,2,0)	(OFLK)	O	(2,0,1)
(OFEK)	O	(2,1,0)	(HOIL)	H	(1,0,2)
(HOFL)	H	(1,2,0)	(OIJK)	O	(0,1,2)
(HGFL)	H	(2,1,0)	(OILK)	O	(0,2,1)

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