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A class of totally geodesic foliations of Lie groups

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Abstract. This paper is devoted to classifying the foliations of G with leaves of the form gKh^{-1} where G is a compact, connected and simply connected Lie group and K is a connected closed subgroup of G such that G/K is a rank-1 Riemannian symmetric space. In the case when $G/K = S^n$, the homotopy type of space of such foliations is also given.

Keywords. Foliations; rank-1 Riemannian symmetric space; cutlocus.

1. Introduction

The study of fibrations of spheres by great spheres is a very interesting problem in geometry and it is very important in the theory of Blaschke manifolds. In [1], Gluck and Warner have studied the great circle fibrations of the three spheres. In that paper they have proved very interesting results. When we look at the problem group theoretically, we see that all the results of [1] go through, for foliations of G with leaves of the form gKh^{-1} where G is a compact, connected and simply connected Lie group and K is a connected closed subgroup of G such that G/K is a rank-1 Riemannian symmetric space (see [2]), except perhaps the theorem 3 for the pair (G, K) such that $G/K = \mathbb{CP}^n$, $\mathbb{HP}^n$ etc. Thus this paper is a group theoretic generalisation of [1].

Throughout we will be working with foliations of G with leaves of the form gKh^{-1} where g, $h \in G$. We call such foliations *admissible*.

Let us fix some notations which we will be using throughout:

G-Compact, connected, simply connected Lie group.
K-connected and closed subgroup of G such that G/K is a rank-1 Riemannian symmetric space.
$\mathbf{g}$-Lie algebra of G.
$\mathbf{k}$-Lie algebra of K.
$Gr(\mathbf{g}, \mathbf{k}) = \{Ad_g(\mathbf{k}) : g \in G\}$.
$\tilde{G}r(\mathbf{g}, \mathbf{k})$-universal cover of $Gr(\mathbf{g}, \mathbf{k})$.
$\mathscr{F}$-admissible foliation of G.

Unless we specify, we will always be working with continuous foliations.

Let $\mathscr{L} := \{gKh^{-1} : g, h \in G\}$. $\mathscr{L}$ is the set of all possible leaves of the foliations $\mathscr{F}$ of G. The group $G \times G$ acts on $\mathscr{L}$ as follows:

$$G \times G \times \mathscr{L} \to \mathscr{L}$$

given by

$$((g,h),L)\mapsto gLh^{-1}.$$

This action of $G\times G$ on $\mathscr{L}$ is transitive. Let us denote the isotropy group of the action at the point K by I_K. Then we have the following

Lemma 1. $I_K=\{(g,h)\in N(K)\times N(K): gh^{-1}\in K\}$

Proof.

$$gKh^{-1}=K\Rightarrow gh^{-1}\in K.$$

Hence

$$gKg^{-1}=gKh^{-1}hg^{-1}=Khg^{-1}=K$$

$\Rightarrow g\in N(K)$, and hence $h\in N(K)$. Therefore,

$$gKh^{-1}=K\Rightarrow g,h\in N(K).$$

Conversely, let $g,h\in N(K)$ be such that $gh^{-1}\in K$.

$$gKh^{-1}=gKg^{-1}gh^{-1}=Kgh^{-1}=K.$$

Hence $(g,h)\in I_K$. Thus

$$I_K=\{(g,h)\in N(K)\times N(K): gh^{-1}\in K\}.$$

Q.E.D.

From the above lemma it follows that

$$\frac{G\times G}{I_K}\simeq\mathscr{L}.$$

We topologise $\mathscr{L}$ using the bijection $G\times G/I_K\simeq\mathscr{L}$. From now on we shall denote by $G\times G/I_K$ for $\mathscr{L}$.

Let $\mathscr{F}$ be an admissible foliation of G. We define an equivalence relation $\sim$ on G as follows. For x, $y\in G$, define $x\sim y$ iff x, $y\in L$ for some leaf L in $\mathscr{F}$. This is an equivalence relation on G. Let us denote by $M_{\mathscr{F}}=G$, the quotient space of the equivalence relation. Then it can easily be seen that $G\to M_{\mathscr{F}}$ is a fibre bundle with fibres gKh^{-1}. If we write down the homotopy long exact sequence of this fibration, we see that $M_{\mathscr{F}}$ is simply connected.

Furthermore, we have a continuous inclusion i: $M_{\mathscr{F}}\hookrightarrow G\times G/I_K$ which is a homeomorphism on to its image. We will identify $M_{\mathscr{F}}$ with its image. Thus given an admissible foliation $\mathscr{F}$ of G, we get a simply connected topological submanifold $M_{\mathscr{F}}\subseteq G\times G/I_K$. Observe that I_K is a closed subgroup of $G\times G$ such that $K\times K\subseteq I_K\subseteq N(K)\times N(K)$. The fact that G/K is a rank-1 Riemannian symmetric space implies that $(G/K)\times(G/K)\to G\times G/I_K$ is a finite sheeted simply connected covering. Since $M_{\mathscr{F}}$ is simply connected, we can lift $M_{\mathscr{F}}$ to a simply connected submanifold of $G/K\times G/K$. We note that there is no uniqueness in the lifting and any two liftings will differ by an element of $\pi_1((G\times G)/I_K)=(I_K/K\times K)$. We will fix one lifting and continue to denote the lifted submanifold also by $M_{\mathscr{F}}$.

Remark. If we take $G=SU(2)$ and $K=S^1$, then the lifting of $M_{\mathscr{F}}$ corresponds to choosing an orientation of the great circle fibration of S^3. Because of this reason, after fixing one lifting we say that our foliation is *oriented*.

From now on we will be working with a fixed oriented admissible foliation only. After this much of preliminaries, presently we prove two lemmas from which our main theorem of this section is almost immediate.

Lemma 2. *For* g, $h \in G$,

$$gKh^{-1} \cap K \neq \emptyset \Leftrightarrow d(\bar{g}, \bar{e}) = d(\bar{h}, \bar{e}),$$

where d *denotes the Riemannian distance on* G/K *and* $\bar{g}$ *denotes the image of the left coset* gK *in* G/K.

Proof.

$$gKh^{-1} \cap K \neq \emptyset \Leftrightarrow gK \cap Kh \neq \emptyset.$$

This is true iff $\bar{g} \in K\bar{h}$ where $K\bar{h}$ is the K-orbit of $\bar{h}$ in G/K, the K action being the induced action of G on G/K. G (and hence K) acts on G/K via isometries. Furthermore K fixes $\bar{e}$. This means that $d(\bar{g}, \bar{e}) = d(\bar{h}, \bar{e})$.

Conversely, let $d(\bar{g}, \bar{e}) = d(\bar{h}, \bar{e}) = r$ (say). Let $S(\bar{e}, r) \subseteq T_{\bar{e}}(G/K)$ be the geodesic sphere of radius r. Let γ_g be the geodesic from $\bar{e}$ to $\bar{g}$ (respectively, γ_h be the geodesic from $\bar{e}$ to $\bar{h}$). Both $\gamma_g'(0)$ and $\gamma_h'(0) \in S(\bar{e}, r)$. Since G/K is a rank-1 Riemannian symmetric space, there exists a $k \in K$ such that $k\gamma_g'(0) = \gamma_h'(0)$ (see [2]). This means that $\bar{g}$, $\bar{h}$ are in the same K-orbit in G/K.

Q.E.D.

So, given $M_{\mathscr{F}} \subseteq (G/K) \times (G/K)$, we know from the above lemma that for any two distinct points of $M_{\mathscr{F}} (\bar{g}_1, \bar{h}_1)$ and $(\bar{g}_2, \bar{h}_2)$ of $M_{\mathscr{F}}$, $d(\bar{g}_1, \bar{g}_2) \neq d(\bar{h}_1, \bar{h}_2)$. But what is more interesting is

Lemma 3. *Either* $d(\bar{g}_1, \bar{g}_2) > d(\bar{h}_1, \bar{h}_2)$ *or* $d(\bar{g}_1, \bar{g}_2) < d(\bar{h}_1, \bar{h}_2)$ *for any two distinct points* $(\bar{g}_1, \bar{h}_1)$ *and* $(\bar{g}_2, \bar{h}_2)$ *in* $M_{\mathscr{F}}$.

Proof. The proof follows easily from the fact that $M_{\mathscr{F}}$ is path connected.

Given an oriented admissible foliation $\mathscr{F}$ we have a submanifold $M_{\mathscr{F}} \subseteq (G/K) \times (G/K)$. We can ask if the converse is true or not. The answer to this question is our main theorem in this section which we prove now.

Theorem 1. *A submanifold* $M \subseteq G/K \times G/K$ *corresponds to an oriented admissible foliation* $\mathscr{F}$ *of* G *iff* $M = Graph(f)$, *for some strictly distance decreasing map* f *from either of the* G/K *factor to the other.*

Proof. Let $M = M_{\mathscr{F}}$ for some oriented admissible foliation $\mathscr{F}$ of G. Then from the above lemma we know that for any two distinct points $(\bar{g}_1, \bar{h}_1)$ and $(\bar{g}_2, \bar{h}_2)$ in $M_{\mathscr{F}}$ either

$$d(\bar{g}_1, \bar{g}_2) > d(\bar{h}_1, \bar{h}_2)$$

or

$$d(\bar{g}_1, \bar{g}_2) < d(\bar{h}_1, \bar{h}_2).$$

Hence we may (and we will) assume that $d(\bar{g}_1, \bar{g}_2) > d(\bar{h}_1, \bar{h}_2)$. This implies that $d(\bar{g}_1, \bar{g}_2) > 0$ always. Hence the map

$$\Phi\colon M_{\mathscr{F}} \to \frac{G}{K}$$

defined by $\Phi(\bar{g}, \bar{h}) = \bar{g}$ is continuous and injective. This implies that Φ is an open map and hence a homeomorphism of $M_{\mathscr{F}}$ on to (G/K).

Now we define another map

$$f\colon \frac{G}{K} \to \frac{G}{K}$$

by $f(\bar{g}) = \bar{h}$, where $(\bar{g}, \bar{h}) \in M_{\mathscr{F}}$. Then by definition f is strictly distance decreasing and $M_{\mathscr{F}} = \text{graph}(f)$.

Conversely, let $M = \text{graph}(f)$. We will show that $M = M_{\mathscr{F}}$ for some oriented admissible foliation $\mathscr{F}$ of G. Let us denote by $\mathscr{F}$ the collection of all submanifolds of the form gKy^{-1} where $y \in f(\bar{g})$ is any element in the coset $f(\bar{g})$. i.e., $\mathscr{F} = \{gKy^{-1}: g \in G$ and $y \in f(\bar{g})\}$. We will show that $\mathscr{F}$ is an oriented foliation of G. Since f is strictly distance decreasing, $g_1Ky_1^{-1} \cap g_2Ky_2^{-1} = \varnothing$ for $\bar{g}_1 \neq \bar{g}_2$ in G/K and $y_i \in f(\bar{g}_i)$. First we shall show that G is a continuous union of submanifolds L where $L \in \mathscr{F}$. Since $G \to G/K$ is a principal K-bundle, local sections always exist. We will use this fact to prove our claim.

Consider $(\bar{g}, f(\bar{g})) \in \text{Graph}(f)$. Let V be an open neighbourhood of $f(\bar{g})$ such that G has a local section S_2 over V. Let U be an open neighbourhood of $\bar{g}$ such that G has a local section S_1 over U. We may assume that U is contained in $f^{-1}(V)$ which is an open neighbourhood of $\bar{g}$. Hence $f(U) \subseteq V$. Now we define a map $\Phi_U\colon U \times K \to G$ by $\Phi_U(\bar{g}, k) = S_1(\bar{g})kS_2(f(\bar{g}))^{-1}$. The map Φ_U is continuous by the definition. Clearly Φ_U is injective.

This shows that Φ_U is open. Since G/K is compact, we can choose finitely many of those U's which cover whole of G. Hence the subset of G which is the finite union of the images of Φ_U must be open. i.e. $\cup \Phi_U(U \times K) \subseteq G$ is open. Since M is compact and each gKh^{-1} is a closed submanifold of G, this subset $\cup \Phi_U(U \times K)$ must be compact and hence closed. Thus $M = \text{graph}(f)$ defines an oriented admissible foliation of G. Q.E.D.

Remark. Any constant map $f\colon G/K \to G/K$ corresponds to a coset foliation, and conversely.

2. Differentiability of the foliation

So far we have not said anything about the differentiability of the foliation. In this section we shall see how the differentiability of the foliation is related to that of the map.

If we start with a strictly distance decreasing differentiable map f such that $\|df\| < 1$, we can conclude that the foliation has to be differentiable. We can ask if the converse is true or not. Our main theorem in this section answers this question afirmatively.

Theorem 2. *An oriented foliation $M_{\mathscr{F}} = graph(f)$ is differentiable iff f is differentiable and $\|df\| < 1$.*

Given $M_{\mathscr{F}}$ we get a continuous integrable distribution of the tangent bundle TG of G. Consider $(\bar{g}, \bar{h}) \in M_{\mathscr{F}}$. Then gKh^{-1} is a leaf for the foliation $\mathscr{F}$ of G. For $p \in gKh^{-1}$, the tangent space at p tangential to the foliation is $L_{gkh^{-1}*}(Ad_h(\mathbf{k}))$.

For a foliation $M_{\mathscr{F}}$ the leaf gKh^{-1}, passing through the identity e of G, is of the form gKg^{-1}. For there exists a $k \in K$ such that $gkh^{-1} = e$. This shows that $h = gk$ and hence $gKh^{-1} = gKg^{-1}$. The tangent space tangential to this leaf at any point gkg^{-1} is $Ad_g(\mathbf{k})$.

Let us denote by $Gr(\mathbf{g}, \mathbf{k})$ the set of all n-dimensional subspaces of $\mathbf{g}$ which are of the form $Ad_g(\mathbf{k})$ where $n = \dim K$. i.e. $Gr(\mathbf{g}, \mathbf{k}) := \{Ad_g(\mathbf{k}) \colon g \in G\}$. G acts on $Gr(\mathbf{g}, \mathbf{k})$ via Ad action and this action is transitive. The isotropy group of the action at the point $\mathbf{k}$ is $N(K)$. We identify $Gr(\mathbf{g}, \mathbf{k})$ with $G/N(K)$. As we had mentioned earlier, since G/K is a rank-1 Riemannian symmetric space, $(G/K) \to [G/N(K)]$ is finite sheeted simply connected covering. Let us denote by $\tilde{G}r(\mathbf{g}, \mathbf{k})$ the universal cover of $Gr(\mathbf{g}, \mathbf{k})$. $\tilde{G}r(\mathbf{g}, \mathbf{k})$ can be identified with G/K. Because we are working with oriented foliations which we explained earlier, we have to take the space $\tilde{G}r(\mathbf{g}, \mathbf{k})$ instead of $Gr(\mathbf{g}, \mathbf{k})$.

Let us denote by E the set of all possible tangent planes tangential to all admissible foliations $\mathscr{F}$ of G. Observe that E is a subbundle of the Grassmannian bundle of all n-dimensional subspaces of the tangent space at each point of G. We have the canonical projection from $\Pi_1 \colon E \to G$ and a natural map from $G \times Gr(\mathbf{g}, \mathbf{k}) \to E$ defined as

$$(g, Ad_h(\mathbf{k})) \mapsto L_{g*}(Ad_h(\mathbf{k})).$$

Let us denote by $\tilde{E}$ the universal cover of E. Then we see that this natural map lifts to a map from $G \times \tilde{G}r(\mathbf{g}, \mathbf{k}) \to \tilde{E}$. Further $\tilde{E}$ is equivalent to $G \times \tilde{G}r(\mathbf{g}, \mathbf{k})$ over G. If we recall that, for $x \in gKh^{-1}$ where $gKh^{-1} \in \mathscr{F}$, $T_x(gKh^{-1}) = L_{x*}(Ad_h(\mathbf{k}))$ the following map

$$\pi_2 \colon E \to \frac{G \times G}{I_K}$$

defined by

$$\pi_2(L_{g*}(Ad_h(\mathbf{k})) = (\overline{gh}, \bar{h})$$

is natural. This map π_2 can be lifted to map

$$\Pi_2 \colon E \to \frac{G}{K} \times \frac{G}{K}.$$

Again we remark that, we are lifting this map, because we are working with oriented foliations. Hence we get a map $\tilde{\Pi}_2 \colon \tilde{E} \to G/K \times G/K$ which is a K-bundle.

Following the notations of [1], we define, for $(\overline{g_0}, \overline{h_0}) \in (G/K) \times (G/K)$

$$\Delta(\overline{g_0}, \overline{h_0}) = \{(\bar{g}, \bar{h}) \in \frac{G}{K} \times \frac{G}{K} \colon d(\bar{g}, \overline{g_0}) = d(\bar{h}, \overline{h_0})\}.$$

$\Delta(\overline{g_0}, \overline{h_0})$ is called the *diagonal* centred at the point $(\overline{g_0}, \overline{h_0})$. Then from lemma 2, we see that if $(\bar{g}, \bar{h}) \in \Delta(\overline{g_0}, \overline{h_0})$, the submanifolds gKh^{-1} and $g_0Kh_0^{-1}$ intersect nontrivially. Hence if $(\overline{g_0}, \overline{h_0}) \in M_{\mathscr{F}}$ then $\Delta(\overline{g_0}, \overline{h_0})$ intersects $M_{\mathscr{F}}$ only at $(\overline{g_0}, \overline{h_0})$ and nowhereelse.

Remark. Given $(u, v) \in T_{(\overline{g_0}, \overline{h_0})}((G/K) \times (G/K))$, (u, v) is tangential to $\Delta(\overline{g_0}, \overline{h_0})$, iff $\| u \| = \| v \|$. This fact it very easy to verify. We will not prove this.

We call the directions tangential to the fibres of the map Π_2 as horizontal directions and the ones tangentials to the fibres of the map Π_1 as vertical. Since Π_2 is a projection in the second coordinate, $\Pi_1^{-1}(g) = g \times \tilde{G}r(\mathbf{g}, \mathbf{k})$ is mapped diffeomorphically on to its image, which is a diagonal centred at any point in the image of $\Pi_1^{-1}(g)$ under Π_2, as can be seen easily.

Now we start proving the theorem.

Proof of Theorem 2. Let $\mathscr{F}$ be an oriented admissible foliation of G. From the construction of the bundle $\tilde{E}$, we have a section $S_{\mathscr{F}}$ of $\tilde{E}$ over G such that at each point $x \in G$, $S_{\mathscr{F}}(x)$ is the tangent plane tangential to the oriented admissible foliation $\mathscr{F}$ of G at that point, i.e. if $x \in gKh^{-1}$ for $gKh^{-1} \in \mathscr{F}$, then $s_{\mathscr{F}}(x) = T_x(gKh^{-1})$.

From the definition of $\tilde{\Pi}_2 : \tilde{E} \to (G/K) \times (G/K)$, we see that

$$\tilde{\Pi}_2^{-1}(M_{\mathscr{F}}) = s_{\mathscr{F}}(G).$$

Let us now assume that f is differentiable and that $\|df\| < 1$. Then for any $(u, v) \in T_{(\bar{g}, f(\bar{g}))}(M_{\mathscr{F}})$, since $\|df\| < 1$, $\|u\| > \|v\|$. Hence (u, v) cannot be tangential to $\Delta(\bar{g}, f(\bar{g}))$. This means that $M_{\mathscr{F}}$ is nowhere tangential to diagonals is $(G/K) \times (G/K)$. Hence $\Pi_2^{-1}(M_{\mathscr{F}}) = S_{\mathscr{F}}(G)$ is a differentiable submanifold of $\tilde{E}$ which is nowhere tangentials to vertical directions. This shows that $\Pi_1 | s_{\mathscr{F}}(G) : S_{\mathscr{F}}(G) \to G$ is diffeomorphism. Therefore the section $S_{\mathscr{F}}$ is differentiable and hence the foliation is differentiable.

Conversely, let us start with a differentiable foliation $\mathscr{F}$. This implies that $M_{\mathscr{F}} = \mathrm{Graph}(f)$, for some strictly distance decreasing map $f : (G/K) \to (G/K)$, is a differentiable submanifold of $(G/K) \times (G/K)$. In general from the differentiability of the submanifold we cannot say the map is differentiable. Because the map is strictly distance decreasing, we can conclude that it is differentiable.

Now, we shall show that $\|df\| < 1$. f is strictly distance decreasing implies that $\|df\| \leqslant 1$. Since $S_{\mathscr{F}}$ is a differentiable section, $S_{\mathscr{F}}(G) = \tilde{\Pi}_2^{-1}(M_{\mathscr{F}})$ cannot be tangential to any vertical direction. But one can easily see that $S_{\mathscr{F}}(G) = \tilde{\Pi}_2^{-1}(M_{\mathscr{F}})$ is tangential to verticals iff $M_{\mathscr{F}}$ is tangential to diagonals $(G/K) \times (G/K)$. This coupled with the fact that $M_{\mathscr{F}}$ is tangentials to some diagonal direction iff $\|df\| = 1$ at some point implies that $\|df\| < 1$. This finishes the proof. Q.E.D.

3. Topology of the space of foliations

In this section we shall describe the homotopy type of the space of foliations $\mathscr{F}$ of G, when $G/K = S^n$.

Let

$$\mathscr{D}\left(\frac{G}{K}\right) = \{f : G/K \to G/K : f \text{ is strictly distance decreasing}\}.$$

We observe that we have a one-one map from $\mathscr{D}(G/K)$ on to the space of all admissible foliations $\mathscr{F}$ of G with fixed orientation. Hence we study the space $\mathscr{D}(G/K)$ with compact open topology to understand the space of all admissible oriented foliations.

Theorem 3. *$\mathscr{D}(G/K)$ is path connected and deformation retracts to the space of constant maps when $G/K = S^n$.*

Remark. The proof uses Borsuk-Ulam theorem as in [1]. We do not have an analogous result for other rank-1 Riemannian symmetric spaces. Even if we do not have Borsuk-Ulam type of result, it is enough if we can prove that the image of G/K for any $f \in \mathscr{D}(G/K)$, is contained in a ball of radius less than half the diameter of G/K. For, given any $f \in \mathscr{D}(G/K)$ the cut-locus of any point $f(x)$ is not in the image of f. This tells us that the image has diameter less than $\pi/2$ where $\operatorname{diam}(G/K) = \pi/2$. Since two geodesics starting from a point diverge up to a distance $\pi/4$, we can use the radial contraction along geodesics starting from the centre of the ball, provided we prove that the image lies inside a ball of radius less than $\pi/4$.

Proof. First we shall prove that $\mathscr{D}(G/K)$ is path connected when $G/K = S^n$. Let $f \in \mathscr{D}(G/K)$. Since f is strictly distance decreasing, $f(G/K)$ cannot be the whole of G/K. Then by Borsuk-Ulam theorem there exists a pair of antipodal points (x_+, x_-) in S^n such that $f(x_+) = f(x_-) = y$. This shows that $\operatorname{Im}(f)$ lies inside a ball of radius less than $\pi/2$ where the metric on S^n is normalized such that $\operatorname{diam} S^n = \pi$. Then we use radial contraction along geodesics issuing from y, to homotope f through distance decreasing maps to the constant map y, the centre of the hemisphere containing the image of f. This proves that $\mathscr{D}(G/K)$ is path connected when $G/K = S^n$.

Claim. There is a continuous map $c: \mathscr{D}(G/K) \to G/K$ such that $\operatorname{Im}(f)$ for each $f \in \mathscr{D}(G/K)$ is contained in the open hemisphere centred at $c(f)$.

The proof follows along the same lines as that of S^2 for which a proof is given in Lemma 9.3 of [1]. Hence we omit the proof here. Q.E.D.

Once we have the theorem the following corollary is immediate.

COROLLARY 1.

The space of all admissible foliations has exactly $\# I_K/(K \times K)$ path components.

Thus the space of all admissible oriented foliations has the homotopy type of disjoint union of $\# I_K/(K \times K)$ copies of G/K.

Theorem 4. *Let $(G/K) = S^n$. Then every oriented admissible K-foliation contains two parallel leaves.*

Proof. In Theorem C of [1] it is proved that any great circle fibration of S^3 must contain some orthogonal pair of circles. We remark that orthogonal circles are parallel and conversely. Hence we refer to Theorem C of [1], as the proof follows along the same lines as that of S^2. Q.E.D.

Concluding remarks

It will be interesting to know how far we can go in proving results of a similar kind in the higher rank cases. If we start with a strictly distance decreasing map then we do get an admissible oriented foliation $\mathscr{F}$ of G. But we do not have a complete answer at present.

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References

[1] Gluck H and Warner F, Great circle fibrations of the three sphere, *Duke Math. J.*, **50** (1983) 107–131
[2] Helgason S, *Differential geometry, Lie groups and symmetric spaces* (New York: Academic Press) (1978)