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Vector fields and framings on isolated complete intersection singularities

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Abstract. A theorem is proved for deciding as to when the complex orthogonal complement of a vector field on an isolated, complete intersection germ, is a trivial vector bundle.

Keywords. Vector field; singularity link.

1. Introduction

In this note the following question is studied: Given (V, P), a n-dimensional, isolated, complete intersection germ in some complex space $\mathbb{C}^{n+k}$, and a (holomorphic or continuous) vector field Z on V, non-singular away from P, when is the bundle Z^*, normal to the $\mathbb{C}$-distribution on V-$\{P\}$ spanned by Z, a trivial vector bundle?. I prove.

Theorem. *The bundle Z^* is topologically trivial over $\mathbb{C}$ if and only if the index of Z in V is a multiple of $(n-1)!$.*

Since V is singular at P, one has to make sense of the concept of the index of a vector field in V[3, 10]. In §2, I translate the question above into a general problem in differential topology, which is solved in §3. It is worth noting that the theorem above arises a basic question whose answer is not known, not even in the simplest case when V is non-singular at P: If Z^* is not trivial, how many sections does this bundle admit?.

2.

Let $f_1, \ldots, f_k$ be holomorphic functions on (an open set in) $\mathbb{C}^{n+k}$ defining the germ of V at P, so that the map

$$f = (f_1, \ldots, f_k): (\mathbb{C}^{n+k}, P) \to (\mathbb{C}^k, 0)$$

has an isolated critical point at P. The gradient vector fields $\nabla f_1, \ldots, \nabla f_k$ are linearly independent everywhere on a punctured neighbourhood of P in $\mathbb{C}^{n+k}$ and they define

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a trivialization of the bundle normal to the fibers of f, away from P. Hence, Z and $\nabla f_1,\ldots,\nabla f_k$ define (up to homotopy) a continuous map

$$J(Z)=(Z,\nabla f_1,\ldots,\nabla f_k)\colon M\to W$$

where M is the *link* of P in V, as defined in [7], and W is the Stiefel manifold of complex, orthonormal $(k+1)$-frames in $\mathbb{C}^{n+k}$; W is $(2n-2)$-connected, $\pi_{2n-1}(W)\cong\mathbb{Z}$, see [2], so that $J(Z)$ has a well defined degree in $\mathbb{Z}$. By definition [3, 10], the *index of* Z in V *is the degree of* $J(Z)$, denoted $\mathrm{Ind}(Z,V)$.

Let us now choose $\varepsilon>0$ sufficiently small [7], so that every sphere in $\mathbb{C}^{n+k}$ centered at P and of radius less or equal to ε intersects $V=f^{-1}(0)$ transversely, and let $\delta>0$ be sufficiently small [7] with respect to ε, so that every fiber $V_t=f^{-1}(t)$ with $|t|<\delta$ intersects the sphere S_ε transversely. We let $V'=V_{t'}\cap D_\varepsilon$, where D_ε is the closed ε-disc around P and $0<|t'|<\delta$ is fixed. Then V' is the *Milnor fibre* of P; it is a parallelizable manifold with the homotopy type of a wedge of n-spheres, [6, 7]. Moreover, the boundary of V' is isotopic to $M=V\cap S_\varepsilon$, the link of P, and the vector field Z can be regarded as a continuous, nowhere-zero, vector field Z' on a neighbourhood of the boundary $\partial V'\subset V'$. As such, Z has a well defined index in V': the total number of zeros, counted with their local indices [8], of a continuous extension of Z' to all of V'; by classical homotopy theory (see [11]), this number $\mathrm{Ind}(Z',V')$ does not depend on the choice of the extension of Z', and as noted in [3] it equals $\mathrm{Ind}(Z,V)$, because the gradient vector fields $\nabla f_1,\ldots,\nabla f_k$ are everywhere linearly independent over V', and they are normal to V'.

Since V is a cone over M, by [7], it follows that the bundle Z^* over V-$\{P\}$ is trivial if and only if it is trivial restricted to M. Also $Z^*|M$ is isotopic to the bundle $(Z')^*$, normal to Z' in $V'|\partial V'$. Thus we have established the following lemma.

Lemma 1. *The theorem above is equivalent to claiming that* $(Z')^*$ *is trivial if and only if* $\mathrm{Ind}(Z',V')$ *is a multiple of* $(n-1)!$.

This lemma will follow at once from the results in §3 below, thus proving the theorem stated in the introduction.

3.

We now let X be an arbitrary $2n$-dimensional, compact, almost complex manifold with non-empty boundary M and with trivial tangent bundle TX, and we let Z be a nowhere-zero, continuous vector field on a neighbourhood of M in X. Let $\mathrm{Ind}(Z;X)$ denote the total index of Z in X; as explained before, this means $\mathrm{Ind}(Z,X)$ is the total sum of the zeros of a continuous extension of Z to X, with finite singularities and counted with their local indices [8].

Lemma 2. *If* $\mathrm{Ind}(Z,X)\equiv 0$ *mod* $(n-1)!$, *then* Z *can be completed to a continuous* $\mathbb{C}$*-trivialization of* $TX|M$.

This lemma can be proved directly using obstruction theory. Instead we give a simpler, geometric proof that uses the Hirsh-Poenarú immersion theorem and it has the advantage of proving the lemma above by providing an explicit trivialization of

$TX|M$ with Z as one of its sections, up to homotopy. (Moreover; the Chern classes of X relative to this trivialization are all zero except the top one, which equals $\text{Ind}(Z, X)$. This allows us to think of this number as being a Chern class [5]). To prove lemma 2 we note that, by the theorem of Poincaré-Hopf for closed manifolds [8], there exist around M vector fields of every index in X. Furthermore, by the theorem of Hirsh–Poenarú [9], there exists an immersion

$$\mathscr{I}: X \to \mathbb{R}^{2n}.$$

Thus, if τ is a vector field around M, then we may define a map $\psi(\tau)$: $M \to S^{2n-1}$ by

$$\psi(\tau)(x) = \frac{\mathscr{I}_*((x))}{|\mathscr{I}_*((x))|},$$

where $\mathscr{I}_*$ is the derivative of $\mathscr{I}$, and the degree of $\psi(\tau)$ equals the index of τ in X, because τ can always be extended to all of X minus one point x° around which the immersion can be assumed to be an embedding. It then follows that two vector fields τ and τ° around M are homotopic if and only if they have the same index in X. Now, let Z be as in lemma 2 and let $\text{Ind}(Z, X) = t\cdot(n-1)!$ for an appropriate $t \in \mathbb{Z}$. We choose Z° to be a vector field around M of index t in X, and we define a map

$$\phi(Z^\circ): M \to U(n)$$

by considering $\psi(Z^\circ)$ as above and composing it with the generator of $\pi_{2n-1}(U(n))$, which is isomorphic to $\mathbb{Z}$, see [1]. If F° denotes a trivialization of TX, then we twist F° over M as in [4] using $\phi(Z^\circ)$. We claim that the new trivialization $F(Z)$ of $TX|M$ that we obtain has Z as one of its n-sections, up to homotopy. To see this we observe first that $\psi(Z^\circ)$ has degree t, by the previous discussion. Also, there is a fibration $U(n-1) \hookrightarrow U(n) \to S^{2n-1}$, and an associated long exact homotopy sequence,

$$(*) \qquad \cdots \to \pi_{2n-1}(U(n)) \xrightarrow{p_*} \pi_{2n-1}(S^{2n-1}) \to \pi_{2n-2}(U(n-1)) \to \pi_{2n-2}(U(n)) \to \cdots$$

By Bott's calculations [1], p_* is multiplication by $(n-1)!$. Thus, each section of $F(Z)$ has index $t\cdot(n-1)!$ in X, and the result follows.

Lemma 3. *Let the manifold X have the homotopy type of a wedge of n-spheres.* If Z is a vector field around M, non-singular and such that its complex orthogonal complement Z^* is $\mathbb{C}$-trivial, then $\text{Ind}(Z, X) \equiv 0 \bmod (n-1)!$.

Proof. This is based on a lemma of [5] which states that if F is a complex trivialization of $TX|M$, then F extends to all of X minus one point if n is odd, or to X minus an embedded torus $T = S^n x D^n$ if n is even. Suppose first that n is odd and let β be a trivialization of Z^*. Then Z and β define a trivialization F of $TX|M$. By the result of [5] stated above, F extends to all of X minus a point x°, thus the index of Z in X equals the degree of a complex framing F'' on a small sphere S_σ around x°, i.e. the index of one of the sections that defines F''. The result then follows from the exact sequence (*). Now suppose n is even, let β be as above and let F be the trivialization of $TX|M$ defined by Z and β. Then F extends to $X - T$ and the result follows from the exact sequence (*) together with a result in [5; p. 304], stating that given a complex framing F^* on ∂T, F^* can be obtained (up to homotopy) by twisting in a neighbourhood of a point a framing F_0 on ∂T that has degree 0 in T.

References

[1] Bott R, The stable homotopy of the classical groups, *Proc. Nat. Acad. Sci.* **13** (1957) 933–935

[2] Fuks D B and Rokhlin V A, *Beginner's course in topology*, (Springer Verlag: Universitext) (1984)

[3] Gomez-Mont X, Seade J A and Verjovsky A, The index of a holomorphic flow with an isolated singularity. To appear in *Math. Ann.*

[4] Kervaire M and Milnor J, Groups of homotopy spheres I, *Ann. Math.* **77** (1963) 504–537

[5] Li B and Seade J A, Framings on algebraic knots, *Q. J. Math.* Oxford (2) **38** (1987) 297–306

[6] Looijenga E J N, Isolated singular points on complete intersections, *London Math. Soc. Lect. Note Ser.* **77** (1984)

[7] Milnor J, Singular points of complex hypersurfaces, *Ann. Math. Stud.* # 61 (Princeton: Univ. Press) (1968)

[8] Milnor J, *Topology from the differentiable viewpoint* (Virginia: Univ. Press), 5 printing (1978)

[9] Poenarú V, Sur la theorie des immersions, *Topology* **1** (1962) 81–100

[10] Seade J A, The index of a vector field on a complex surface with singularities, *Contemp. Math.* **58**, part III (1987) (ed.) Verjovsky A, 225–232 A.M.S

[11] Steenrod N E, Topology of fiber bundles *Princeton Math. Ser.* (N.J.: Princeton Univ. Press) (1951)