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Criterion for smoothness of Schubert varieties in Sl(*n*)/*B*

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Abstract. Let $G = Sl(n)$ and B, the Borel subgroup of G consisting of upper triangular matrices. Let $w \in S_n$ and $X(w) = \overline{BwB}(\text{mod } B)$, the associated Schubert variety in G/B. In this paper, we give a geometric criterion for the smoothness of $X(w)$. This criterion admits a neat combinatorial description in terms of the permutation w.

Keywords. Schubert variety; smoothness, geometric criterion.

1. Introduction

In this note, we give a criterion for a Schubert variety in the flag variety $Sl(n)/B$ to be smooth.

In the series Geometry of G/P I-V, (cf. [9], [4], [7], [8], [5]), a standard monomial theory is developed for semi-simple algebraic groups. This theory consists in the construction of an explicit basis for $H^0(X, L)$, X being a Schubert variety in G/B and L being an ample line bundle on G/B. This theory has led to many important geometric and representation-theoretic consequences, one of them being the determination of the singular loci of Schubert varieties (cf [6]). In [5], using standard monomial theory, we determine explicitly the ideal defining a Schubert variety X under $X \hookrightarrow G/B$. Using this we compute in [6] the rank of the Jacobian matrix and thus obtain a description of the singularities of X. This description takes a particularly simple form for $G = Sl(n)$, which we describe below.

$$\text{Let } G = Sl(n), \quad B = \{\text{upper triangular matrices in } G\},$$

$T = \{\text{diagonal matrices in } G\}$, $W = N(T)/T$ (note that $W = S_n$). For $1 \leqslant i < j \leqslant n$, let s_{ij} denote the element in W which corresponds to the transposition of i and j. For $w \in W$, let e_w denote the point wB in G/B and $X(w)$ the Schubert variety in G/B. For $\tau \leqslant w$, let $T(w, \tau)$ denote the Zariski tangent space to $X(w)$ at e_τ. Then we have (cf [6]),

$$\dim T(w, \tau) = \#\{s_{ij} \mid w \geqslant \tau s_{ij}\}.$$

In this note, we give a criterion for smoothness of $X(w)$, in terms of the permutation representing w. The main result of this note is the following theorem.

Theorem 1. *Let* $w=(a_1\ldots a_n)$. *Then* $X(w)$ *is singular* $\Leftrightarrow$ *there exist* i, j, k, l, *with* $1\leqslant i<j<k<l\leqslant n$, *such that either*

$$(*)\qquad \begin{cases} 1.\ a_k<a_l<a_i<a_j \\ \textit{or} \\ 2.\ a_l<a_j<a_k<a_i. \end{cases}$$

Theorem 1 is proved in two steps. We first establish the equivalence of smoothness and equidimensionality of some canonical projections (cf. Theorem 2.1). We then show (cf. Theorem 2.2), the equivalence of the failure of equidimensionality and condition (*) of Theorem 1 above.

The sections are organised as follows: In §1, we gather some preliminaries and we prove the main theorem in §2. In §3, we state a conjecture on the singular components of $X(w)$.

1. Preliminaries

Let G, B, T, W be as in the introduction. Let R be the root system of G relative to T. Let R^+ be the set of positive roots of R relative to B and let S be the system of simple roots in R^+. We shall index the simple roots as $\alpha_1,\ldots,\alpha_{n-1}$ (cf [2]). For $w\in W$, we shall denote $X(w)_r$ (resp. $X(w)_\ell$) the Schubert variety in G/B (resp. $B\backslash G$). When we write $X(w)$, it is to be understood that any statement on $X(w)$ holds for both $X(w)_r$ and $X(w)_\ell$.

PROPOSITION 1.1

(cf. [1]) $\dim X(w)_\ell=\dim X(W)_r=\{\alpha\in R^+\,|\,w^{-1}(\alpha)<0\}$.

PROPOSITION 1.2

(cf. [3]) Let $w\in W$. a. w has a reduced expression $w=u'_{n-1}\cdots u'_1$, where u'_i is either Id or a left-end segment of $s_i\cdots s_{n-1}$ (here s_i denotes the reflection with respect to α_i). b. w has a reduced expression $w=v'_1\cdots v'_{n-1}$, where v'_i is either Id or a right-end segment of $s_{n-1}\ldots s_i$.

PROPOSITION 1.3

(cf [6]) *For* $\tau\leqslant w$, *let* $T(w,\tau)$ *denote the Zariski tangent space to* $X(w)$ *at* e_τ. *Then*

$$\dim T(w,\tau)=\#\{\alpha\in R^+\,|\,w\geqslant\tau s_\alpha\}.$$

Notation 1.4. For $w\in W$, let $R_w=\{\alpha\in R^+\,|\,w\geqslant s_\alpha\}$, $S_w=\{\alpha\in R_w|\alpha$ does not involve $\alpha_1\}$ $N_w=\{\alpha\in R_w|\alpha$ involves $\alpha_1\}$.

Remark 1.5. Let $w=(a_1\cdots a_n)$. Also, let $w=w's_1\cdots s_{m-1}$, where $w'=u'_{n-1}\cdots u'_2$ (as given by Proposition 1.2(a)). This in particular implies that $a_m=1$. Let $a_1=t$. Then, $\#N_w=\min\{t,m\}-1$ (here, one uses the fact that $w\geqslant$the transposition s_{ij} (where $i<j)\Leftrightarrow a_k\geqslant j$ for some $k\leqslant i$ and $a_\ell\leqslant i$ for some $l\geqslant j$).

Lemma 1.6. *Let* $w=w'_1s_1\cdots s_{m-1}$ *(as given by Propositional* 2(a)). *Then* $\#\{R_w-R_{w'}\}\geqslant m-1$.

Proof. Fix j, $2 \leqslant j \leqslant m-1$. Let $i \leqslant j$ be such that $s_j, s_j s_{j-1} s_j, \ldots, s_j s_{j-1} \ldots s_{i+1} \ldots s_{j-1} s_j$ are $\leqslant w'$, but $s_j s_{j-1} \cdots s_{i+1} s_i s_{i+1} \cdots s_j$ is not $\leqslant w'$ (note that such an i exists, since $s_1 \nleqslant w'$). The facts that $s_j s_{j-1} \ldots s_i \leqslant w'$, $s_{i-1} s_i \ldots s_j \leqslant s_1 s_2 \ldots s_{m-1}$, and $w'_1 s_1 \ldots s_{m-1}$ is a reduced expression for w, imply that $s_j s_{j-1} \ldots s_i s_{i-1} s_i \ldots s_j \leqslant w$. Thus the root $\alpha = \varepsilon_{i-1} - \varepsilon_{j+1} \in R_w - R_{w'}$. From this we obtain $\#\{R_w - R_{w'}\} \geqslant m-1$.

Lemma 1.7. *Let* $w = s_{t-1} \ldots s_1 w''$ (*cf. Proposition* 1.2(b)). Then $\#\{R_w - R_{w''}\} \geqslant t-1$. (Proof is similar to that of Lemma 1.6).

2. Main theorem

For $w \in W$, let $X(w)_r$ (resp. $X(w)_\ell$) be the Schubert variety in G/B (resp. $B \backslash G$) associated to w. Let Q_i, $1 \leqslant i \leqslant n-1$ be the parabolic subgroups defined inductively as

$$Q_1 = P_1$$

$$\begin{aligned} Q_2 &= Q_1 \cap P_2 \\ Q_3 &= Q_2 \cap P_3 \\ \vdots \quad & \quad \vdots \\ Q_{i+1} &= Q_i \cap P_{i+1}. \end{aligned}$$

For $0 \leqslant i \leqslant n-1$, let $\pi_{i,r}$ (resp. $\pi_{i,\ell}$) be the projection $Q_i/B \to Q_i/Q_{i+1}$ (resp. $B \backslash Q_i \to Q_{i+1} \backslash Q_i$); here, $Q_0 = G$, $Q_n = B$.

Theorem 2.1. *Let $w \in W$. Then, $X(w)$ is smooth $\Leftrightarrow$ there exist $w_0, w_1, \ldots, w_n$, $w_i \in W(Q_i)$, (with $w_0 = w$) such that for $0 \leqslant i \leqslant n-1$,*

$$(\dagger) \qquad \begin{cases} 1.\textit{either } \pi_{i,\ell}|_{X(w_i)_\ell} \textit{ or } \pi_{i,r}|_{X(w_i)_r} \textit{ is equidimensional.} \\ 2. w_{i+1} \textit{ is given by } X(w_i)_\ell \cap B\backslash Q_i \textit{ or } X(w_i)_r \cap Q_i/B \textit{ according} \\ \textit{as } \pi_{i,\ell}|_{X(w_i)_\ell} \textit{ or } \pi_{i,r}|_{X(w_i)_r} \textit{ is equidimensional.} \end{cases}$$

Proof. a. Let $X(w)$ be smooth. We have,

$$\#N_w = \min\{t, m\} - 1 = m - 1, \text{ say}$$

(with notation as in Remark 1.5). Let T_w denote the tangent space to $X(w)$ at e_{Id}. Then the smoothness hypothesis (cf. [6], Theorem 1) implies that

$$(\#R_w =) \dim T_w = \ell(w). \tag{1}$$

Using the reduced expression $w = w' s_1 \ldots s_{m-1}$, where $w' = u'_{n-1} \ldots u'_2$, we have (cf. Lemma 1.6),

$$\#R_w \geqslant \#R_{w'} + m - 1 \geqslant \ell(w') + m - 1 = \ell(w).$$

Hence (1) implies equality everywhere. In particular, we have $\#R_{w'} = \ell(w')$, which implies the smoothness of $X(w')$. Let $w_1 = w'$ and w_i, $2 \leqslant i \leqslant n$, be the elements as in ($\dagger$) (which exist by induction hypothesis applied to w'). We shall now show that $X(w)_\ell \to P_1 \backslash G$ is equidimensional. We have (in view of smoothness), $\#R_w = \ell(w)$. Also

$$\#R_w = \#S_w + \#N_w = \#S_w + m - 1$$

($\#N_w = m-1$, by assumption) and

$$\ell(w) = \ell(w') + m - 1.$$

Hence we obtain $\ell(w') = \#S_w$ which in turn implies that $\#R_{w'} = \#S_w$ (by smoothness of $X(w')$). Thus we get $S_w = R_{w'}$ (note that $S_w \supseteq R_{w'}$). This now implies $w's_i < w'$, $2 \leqslant i \leqslant m-1$ (for, $w's_i > w'$ for some i, $2 \leqslant i \leqslant m-1$ would imply $\ell(w's_i) > \ell(w') = \#S_w \geqslant \#R_{w's_i} \geqslant \ell(w's_i)$ which is a contradiction). From this we obtain using [3], Lemma A.4

$$X(w)_\ell \cap B\backslash P_1 = X(w')_\ell$$

which implies the required equidimensionality.

b. Assume that (†) holds. We shall show that $X(w)$ is smooth. Let us suppose that the map $X(w)_\ell \to P_1\backslash G$ is equidimensional and that $X(w)_\ell \cap B\backslash P_1 = X(w_1)_\ell$. As above if $w = w's_1 \ldots s_{m-1}$, then we obtain (in view of equidimensionality)

$$X(w)_\ell \cap B\backslash P_1 = X(w')_\ell \text{ and } w' = w_1. \tag{2}$$

Now (2) implies $S_w = R_{w'}$ (note that $S_w \supseteq R_{w'}$). Also, by induction, we have, $X(w')$ is smooth (note that $w' = w_1$) and hence $\#R_{w'} = \ell(w')$. Now

$$m - 1 \geqslant \#N_w = \#R_w - \#S_w = \#R_w - \#R_{w'} \geqslant m - 1$$

(cf. Lemma 1.6). Hence equality holds everywhere. In particular, we obtain $\#R_w = \#S_w + m - 1 = \#R_{w'} + m - 1 = \ell(w') + m - 1$ (since $X(w')$ is smooth) $= \ell(w)$. Thus we obtain $\dim T_w = \ell(w)$, which proves that $X(w)$ is smooth. This completes the proof of Theorem 2.1.

Theorem 2.2. *Let $w \in W$. Let $w = (a_1 \ldots a_n)$. Then $X(w)$ is singular $\Leftrightarrow$ there exist i, j, k, l, $1 \leqslant i < j < k < l \leqslant n$ such that either*

$$(*) \qquad \begin{cases} 1.\ a_k < a_\ell < a_i < a_j \\ \text{or} \\ 2.\ a_\ell < a_j < a_k < a_i \end{cases}$$

Proof. We shall show that (*) holds $\Leftrightarrow$ (†) of Theorem 2.1 does not hold, from which the required result would follow (in view of Theorem 2.1).

a. Given that (*) holds, to show that (†) does not hold. If possible, let us assume that (†) holds. Let us suppose that $X(w)_\ell \to P_1\backslash G$ is equidimensional and that

$$\text{(I)} \qquad X(w_1)_\ell = X(w)_\ell \cap B\backslash P_1.$$

Then writing $w = w's_1 \ldots s_{m-1}$, we have $w' = w_1$. Hence in view of (I) we have, $w's_i < w'$, $2 \leqslant i \leqslant m-1$. Let $w' = (1 b_2 b_3 \ldots b_n)$. Then $w = w'(23 \ldots m1 \ldots) = (b_2 b_3 \ldots b_m 1 \ldots b_n)$, where

$$\text{(A)} \qquad b_2 > b_3 > \cdots > b_m$$

(since $w' > w's_i$; note that $w' > w's_i \Leftrightarrow w'(\varepsilon_i - \varepsilon_{i+1}) (= \varepsilon_{b_i} - \varepsilon_{b_{i+1}})$ is > 0). We may assume by induction that (*) does not hold for $(a_2 a_3 \ldots a_n)$. Hence for (*) to hold for $(a_1 a_2 \ldots a_n)$, we must have that $1 \in (a_i, a_j, a_k, a_\ell)$. This implies $a_k = 1$ in (1) (resp. $a_\ell = 1$ in (2)). This implies $a_i < a_j$ in (1) (resp. $a_j < a_k$ in (2)), where a_i, a_j belong to $(b_2,\dots,b_m)$ in (1) (resp. a_j, $a_k \in \{b_2,\dots,b_m\}$ in (2)), which is not possible in view of (A) above (note that for a_t in $\{b_2,\dots,b_m\}$, $a_t = b_{t+1}$).

b. Given that (†) of Theorem 2.1 does not hold, to show (*) holds. Using induction hypothesis, we may assume that (†) of Theorem 2.1 holds for $i \geqslant 2$. Let us then suppose that $\pi_{1,\ell}$ and $\pi_{1,r}$ are not equidimensional. Then considering $w = w's_1 \dots s_{m-1}$ and $w = s_{t-1} \dots s_1 w''$, we have the existence of integers p, q such that

$$w = (t..p..p+1..)\ (\text{resp. } w = (t..a_q a_{q+1}..1..)$$

where $p+1 \leqslant t$ and $a_q < a_{q+1}$. We shall now show that (*) holds. Let $a_q = b$, $a_{q+1} = c$. We divide the proof into the following two cases.

Case 1. 1 comes after $p+1$.
In this case (2) holds with $a_i = t$, $a_j = p$, $a_k = p+1$, $a_\ell = 1$.

Case 2. 1 comes before $p+1$.
We first note that $p+1 \neq b, c$ (since b and c come before 1 while $p+1$ comes after 1). We further divide this case into the following three sub cases.

Sub case 2(*a*). $b > p+1$.
In this case, (1) holds with $a_i = b$, $a_j = c$, $a_k = 1$, $a_\ell = p+1$.

Sub case 2(*b*). $b < p+1$, $t > c$.
In this case (2) holds with $a_i = t$, $a_j = b$, $a_k = c$, $a_\ell = 1$.

Sub case 2(*c*). $b < p+1$, $t < c$.
In this case, (1) holds with $a_i = t$, $a_j = c$, $a_k = 1$, $a_\ell = p+1$. This completes the proof of Theorem 2.2.

Remark 2.3. For $G = S\ell(4)$, $X(w)$, $w = (4231), (3412)$ are the only two singular Schubert varieties. The phenomenon in the general case thus seems to be just a generalization of that in $S\ell(4)$.

3. A conjecture

In this section, we state a conjecture giving the components of the singular locus of $X(w)$. We first start with the following remark on $S\ell(4)$.

Remark 3.1 a. For $w = (3412)$, $\tau = (1324)$ is a singularity (and is in fact the only singularity).
b. For $w = (4231)$, $\tau = (2143)$ is a singularity (and is in fact the only singularity).
A similar phenomenon seems to hold in the general case, as given by the following Lemma.

Lemma 3.1. *Let $X(w)$ be singular. Let $w = (a_1 \dots a_n)$. Let i, j, k, ℓ be such that $1 \leqslant i < j < k < \ell \leqslant n$.*

1. Let $a_k < a_\ell < a_i < a_j$. Let τ be obtained from w by replacing a_i, a_j, a_k, a_ℓ by a_k, a_i, a_ℓ, a_j. Then e_τ is singular on $X(w)$.

2. Let $a_\ell < a_j < a_k < a_i$. Let τ be obtained by replacing a_i, a_j, a_k, a_ℓ by a_j, a_ℓ, a_i, a_k respectively. Then e_τ is singular on $X(w)$.

Proof. Let $T(w,\tau)$ be the tangent space to $X(w)$ at e_τ. Let

$$A = \{s_\alpha | w \geqslant \tau s_\alpha\} \text{ and } B = \left\{(x,y) \middle| \begin{array}{l} 1. x = a_r,\ y = a_s, \text{ for some } r < s \\ 2. x > y \end{array} \right\}.$$

We have, $\dim T(w,\tau) = \#A$ and $\dim X(w) = \#B$. We shall prove the result by showing that $\#A > \#B$. We shall prove the result for (1) (the proof for (2) is similar). For simplicity of notation, let us denote a_i, a_j, a_k, a_ℓ by c, d, a, b respectively, so that we have $a < b < c < d$. We have

$$w = (..c..d..a..b..)$$
$$\tau = (..a..c..b..d..)$$

(note that the entries in w and τ are the same in the rest of the places). We shall now define an injective mapping $f\colon B \to A$. Let $x = a_u$, $y = a_v$. In the discussion below, s_{pq} (where $1 \leqslant p < q \leqslant n$) will denote the transposition $(1..(p-1)q(p+1)..(q-1)p(q+1)..n)$. We define f by distinguishing the following cases.

Case 1. $x, y \notin \{a,b,c,d\}$.
Set $f(x,y) = s_{uv}$ (note that τ itself is $\geqslant \tau s_{uv}$).

Case 2. $x \notin \{a,b,c,d\}$, $y \in \{a,b,c,d\}$.
We divide this case into the following four sub cases.

Case 2(*a*). $u < i$, i.e., x comes before c, so that $w = (..x..c..d..a..b..)$ $\tau = (..x..a..c..b..d..)$. Set

$$f(x,y) = \begin{cases} s_{uj}, & \text{if } y = c \\ s_{u\ell}, & \text{if } y = d \\ s_{ui}, & \text{if } y = a \\ s_{uk}, & \text{if } y = b \end{cases}$$

Case 2(*b*). $i < u < j$.
We have $w = (..c..x..d..a..b..)$, $\tau = (..a..x..c..b..d..)$. Set

$$f(x,y) = \begin{cases} s_{u\ell} & \text{if } y = d \\ s_{uk}, & \text{if } y = b \\ s_{uj}, & \text{if } y = a \end{cases}$$

Case 2(*c*). $j < u < k$.
We have $w = (..c..d..x..a..b..)$ and $\tau = (..a..c..x..b..d..)$. Set

$$f(x,y) = \begin{cases} s_{u\ell}, & \text{if } y = b \\ s_{uk}, & \text{if } y = a \end{cases}$$

Case 2(*d*). $k < u < l$.
We have $w = (..c..d..a..x..b..)$, $\tau = (..a..c..b..x..d..)$. In this case $y = b$ is the only possibility. We set $f(x,y) = s_{u\ell}$.

Case 3. $x\in\{a,b,c,d\}$, $y\notin\{a,b,c,d\}$.
We divide this case into the following sub cases.

Case 3(*a*): $i<v<j$.
We set $f(x,y)=s_{iv}$.

Case 3(*b*): $j<v<k$.
We have $w=(..c..d..y..a..b..)$, $\tau=(..a..c..y..b..d..)$. Set

$$f(x,y)=\begin{cases} s_{iv}, & \text{if } x=c \\ s_{jv}, & \text{if } x=d \end{cases}$$

Case 3(*c*): $k<v<\ell$.
We have $w=(..c..d..a..y..b..)$, $\tau=(..a..c..b..y..d..)$. Set

$$f(x,y)=\begin{cases} s_{iv}, & \text{if } x=a \\ s_{jv}, & \text{if } x=c \\ s_{kv}, & \text{if } x=d \end{cases}$$

Case 3(*d*): $\ell<v$.
We have $w=(..c..d..a..b..y..)$, $\tau=(..a..c..b..d..y..)$. Set

$$f(x,y)=\begin{cases} s_{iv}, & \text{if } x=a \\ s_{jv}, & \text{if } x=c \\ s_{kv}, & \text{if } x=b \\ s_{\ell v}, & \text{if } x=d \end{cases}$$

Case 4: $x,y\in\{a,b,c,d\}$. Set

$$f(x,y)=\begin{cases} s_{ij}, & \text{if } (x,y)=(c,a) \\ s_{jk}, & \text{if } (x,y)=(c,b) \\ s_{j\ell}, & \text{if } (x,y)=(d,a) \\ s_{k\ell}, & \text{if } (x,y)=(d,b) \end{cases}$$

Thus we obtain an injective mapping $f\colon B\hookrightarrow A$. Now we have at least one element in A which $\notin f(B)$, namely s_{ik}. Hence we obtain $\#A \gneq \#B$. From this it follows that e_τ is singular on $X(w)$. This completes the proof of Lemma 3.1.

We conclude this note by stating a conjecture on the singular components of $X(w)$.

Conjecture. Let $w=(a_1\ldots a_n)$. Let Z be the set of all $\tau'\leqslant w$ such that either (1) or (2) below holds.

1. There exist i,j,k,ℓ, $1\leqslant i<j<k<\ell\leqslant n$ such that

 (a) $a_k<a_\ell<a_i<a_j$

 (b) if $\tau'=(b_1\ldots b_n)$, then there exist i',j',k',ℓ', $1\leqslant i'<j'<k'<\ell'\leqslant n$ such that $b_{i'}=a_k$, $b_{j'}=a_i$, $b_{k'}=a_\ell$, $b_{\ell'}=a_j$

 (c) if τ(resp. w') is the element obtained from w(resp. τ') by replacing a_i, a_j, a_k, a_ℓ respectively by a_k, a_i, a_ℓ, a_j (resp. $b_{i'}$, $b_{j'}$, $b_{k'}$, $b_{\ell'}$ respectively by $b_{j'}, b_{\ell'}$, $b_{i'}$, $b_{k'}$), then $\tau'\geqslant\tau$ and $w'\leqslant w$.

2. There exist i,j,k,ℓ, $1\leqslant i<j<k<\ell\leqslant n$, such that

 (a) $a_\ell<a_j<a_k<a_i$

 (b) if $\tau'=(b_1\ldots b_n)$, then there exist i',j',k',ℓ', $1\leqslant i'<j'<k'<\ell'\leqslant n$ such that $b_{i'}=a_j$, $b_{j'}=a_\ell$, $b_{k'}=a_i$, $b_{\ell'}=a_k$

(c) if τ(resp. w') is the element obtained from w(resp. τ') by replacing a_i, a_j, a_k, a_ℓ respectively by a_j, a_ℓ, a_i, a_k (resp. $b_{i'}$, $b_{j'}$, $b_{k'}$, $b_{\ell'}$ respectively by $b_{k'}$, $b_{i'}$, $b_{\ell'}$, $b_{j'}$), then $\tau' \geqslant \tau$ and $w' \leqslant w$.

Conjecture: Singular locus of $X(w) = \bigcup_{\lambda} X(\lambda)$ where λ runs over the maximal (under the Brühat order) elements of Z.

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