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Compact operators on a Banach space into the space of almost periodic functions

M V DESHPANDE* and M N PAREKH

Department of Mathematics, Indian Institute of Technology, Bombay 400076, India

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Abstract. Let S be a topological semigroup and AP(S) the space of continuous complex almost periodic functions on S. We obtain characterizations of compact and weakly compact operators from a Banach space X into AP(S). For this we use the almost periodic compactification of S obtained through uniform spaces. For a bounded linear operator T from X into AP(S), let T_s be the translate of T by s in S defined by $T_s(x) = (Tx)_s$. We define topologies on the space of bounded linear operators from X into AP(S) and obtain the necessary and sufficient conditions for an operator T to be compact or weakly compact in terms of the uniform continuity of the map $s \to T_s$. If S is a Hausdorff topological semigroup, we also obtain characterizations of compact and weakly compact multipliers on AP(S) in terms of the uniform continuity of the map $s \to \mu_s$, where μ_s denotes the unique vector measure corresponding to the operator T_s.

Keywords. Compact operators; almost periodic functions.

1. Introduction

Let S be a compact Hausdorff topological space and $C(S)$, the space of bounded continuous complex functions on S with uniform topology. Characterizations of compact and weakly compact linear operators from a Banach space X into $C(S)$ are given by Dunford and Schwartz [5, theorem VI.7.3, 7.7]. We consider the case in which S is further a topological semigroup with a uniform structure. We obtain the necessary and sufficient conditions for operators from X into $C(S)$ to be compact or weakly compact. Let $(S, \mathscr{U})$ be a compact Hausdorff topological semigroup, where the uniformity $\mathscr{U}$ is left translation invariant. (That is, for every entourage U in $\mathscr{U}$, $(s,t) \in U$ implies $(us, ut) \in U$ for all $u \in S$). Let H be a dense subsemigroup of $(S, \mathscr{U})$. For Banach spaces X and Y let BL(X, Y) be the space of bounded linear operators from X into Y. When $X = Y$ we write BL(X) for BL(X, Y). For $T \in \mathrm{BL}(X, C(S))$ and $s \in S$, define $T_s: X \to C(S)$ by $T_s(x) = (Tx)_s$, where $(Tx)_s$ is the right translate of Tx. As usual X^* denotes the topological dual of X.

In §3, it is proved that an operator $T: X \to C(S)$ is compact if and only if the map $s \to T_s$ from H into BL(X, $C(S)$) is uniformly continuous in the norm topology of BL(X, $C(S)$). We define a topology τ on BL(X, $C(S)$) and characterize weakly compact operators T in terms of the uniform continuity of the maps $s \to T_s$ from H into

*To whom all correspondence should be addressed

$\mathrm{BL}(X, C(S))$ with respect to the topology τ. In these results we consider the maps $s \to T_s$ for s in the dense subsemigroup H of S because this is convenient for its applications considered in §3.

For a topological semigroup S, let $\mathrm{AP}(S)$ denote the space of continuous complex almost periodic functions on S. In §3, we characterize compact and weakly compact operators from a Banach space X into $\mathrm{AP}(S)$. For this, we first construct the almost periodic compactification S^a of S by using uniform spaces. The space S^a is a compact Hausdorff topological semigroup and in fact coincides with the almost periodic compactification of S obtained by [4] using operators. It is well-known that the multiplication in S^a is jointly uniformly continuous. But what is essential for the results obtained here is the fact that the uniformity of S^a is left translation invariant. It is proved in [4] and [8] that the space $\mathrm{AP}(S)$ is isometrically isomorphic to $C(S^a)$. But we give an independent proof of this fact (theorem 1.6). Theorems of §3 are then applied to the operators from X into $C(S^a)$ to obtain the desired characterizations.

Section 5 deals with the operators in $\mathrm{BL}(C(S))$ where S is a compact Hausdorff topological semigroup with left translate invariant uniformity. Let $\mathscr{B}$ be the σ-field of Borel sets in S and $\mathrm{rca}(S)$ the space of regular countably additive scalar valued measures defined on $\mathscr{B}$. For a Banach space X, let X^{**} denote the second dual of X and $F(S, \mathscr{B}, X^{**})$, the space of set functions μ on $\mathscr{B}$ into X^{**} such that $\mu(\cdot)x^* \in \mathrm{rca}(S)$ for every $x^* \in X^*$, where $(\mu(\cdot)x^*)(E) = \mu(E)x^*$, $E \in \mathscr{B}$. Let $T \in \mathrm{BL}(C(S), X)$ and μ be the unique set function which represents the operator T [5, theorem VI.7.2]. Then, proposition 4.3 shows that for each $s \in S$ the set function $\mu_s : \mathscr{B} \to X^{**}$ defined by $\mu_s(E) = \mu(Es)$ represents the operator $T_s \in \mathrm{BL}(C(S), X)$ where $T_s(f) = T(f_s)$, $f \in C(S)$. In theorem 4.4 it is proved that a multiplier T in $\mathrm{BL}(C(S))$ is compact if and only if the map $s \to \mu_s$ from a dense subsemigroup H of S into $F(S, \mathscr{B}, C(S)^{**})$ is uniformly continuous in the topology on $F(S, \mathscr{B}, C(S)^{**})$ defined in terms of semivariation. We also characterize weakly compact multipliers in $\mathrm{BL}(C(S))$ in terms of the uniform continuity of the map $s \to \mu_s$ from H into $(F(S, \mathscr{B}, C(S)^{**}, \tau')$ (theorem 4.5). In this case the set function μ_s is in fact a vector measure defined on $\mathscr{B}$ and takes values in $C(S)$. Using these results and the almost periodic compactification of S, in theorems 4.6 and 4.7 we obtain the necessary and sufficient conditions for multipliers in $\mathrm{BL}(\mathrm{AP}(S))$ to be compact or weakly compact. If μ is a translation invariant vector measure then $\mu_s - \mu_t = 0$, where $\mu_s(E) = \mu(Es)$. Our results show that for vector measures corresponding to compact multipliers, $\mu_s - \mu_t$ is 'small' if s, t are 'near' to one another. This is thus a consequence of the assumption that the uniform structure on S is left translation invariant.

2. Preliminaries

We begin this section with the following definitions from [1].

DEFINITION 1.1

A set S is called a topological semigroup if S is a semigroup with identity e and if S has a Hausdorff topology such that the multiplication on S is separately continuous. That is, for each $t \in S$ the maps $s \to st$ and $s \to ts$ are continuous functions.

DEFINITION 1.2

For f in $C(S)$ and $s \in S$, define the right translate f_s of f by $f_s(t)=f(ts)$, $t \in S$ and the orbit of f, $O_R(f)=\{f_s : s \in S\}$.

DEFINITION 1.3

A function f in $C(S)$ is right almost periodic if $O_R(f)$ is relatively compact in the norm topology of $C(S)$.

Left almost periodic functions are similarly defined. The set of right almost periodic functions coincides with the set of left almost periodic functions [7, p. 167]. We denote this common set by AP(S) and call its functions almost periodic. For the proof of the following lemma, we refer [6, theorem 4.2].

Lemma 1.4. *The space AP(S) is translation invariant norm closed linear subspace of C(S) and hence a Banach space.*

To obtain results mentioned in the introduction we construct the almost periodic compactification of a topological semigroup as follows: Construction of S^a: Let S be a topological semigroup. Denote by $\tilde{S}$ the quotient structure of S determined by the equivalence relation, s is related to t if and only if $f_s = f_t$ for all f in AP(S). Then $\tilde{S}$ is a semigroup with the multiplication $\tilde{s} \cdot \tilde{t} = \widetilde{st}$. For $\varepsilon > 0$ and f in AP(S) define

$$U(\varepsilon, f) = \{(\tilde{s}, \tilde{t}) \in \tilde{S} \times \tilde{S} : \| f_s - f_t \| < \varepsilon \text{ for all } s \in \tilde{s}, t \in \tilde{t}\}.$$

Then the family of sets of the form $U(\varepsilon, f)$ forms a subbase for a uniformity, say $\tilde{\mathscr{U}}$, on $\tilde{S}$. We note that $\tilde{\mathscr{U}}$ is left translation invariant, since $\| f_{us} - f_{ut} \| \leqslant \| f_s - f_t \|$ for all $u \in S$. The proof of the following theorem is not difficult and hence it is omitted.

Theorem 1.5. *The space $(\tilde{S}, \tilde{\mathscr{U}})$ is a totally bounded Hausdorff topological semigroup and its completion, say S^a, is a compact Hausdorff topological semigroup on which the multiplication is jointly uniformly continuous.*

Theorem 1.6. *Let S be a topological semigroup. The homomorphism $\rho : S \to (\tilde{S}, \tilde{\mathscr{U}}) \subset S^a$ defined by $\rho(s) = \tilde{s}$ is continuous with $\rho(S)$ dense in S^a. The induced map $\tilde{\rho} : C(S^a) \to C(S)$ given by $\tilde{\rho}(\hat{f}) = \hat{f} \circ \rho$ is an isometric algebra isomorphism of $C(S^a)$ onto AP(S).*

Proof. By [1, theorem 1.7(iii)] for each f in AP(S) the map $s \to f_s$ is norm continuous from S into AP(S). Therefore, it follows that ρ is continuous. Also, since $\rho(S) = (\tilde{S}, \tilde{\mathscr{U}})$, a dense subspace of S^a, $\tilde{\rho}$ is one-one and an isometry. Moreover $\tilde{\rho}$ preserves the ordinary multiplication of functions, so AP(S) is an algebra and $\tilde{\rho}$ is an algebra isomorphism. Therefore it remains to prove that $\tilde{\rho}$ maps $C(S^a)$ onto AP(S). By [1, theorem 1.8 (ii)′], $\tilde{\rho}\ [C(S^a)] \subset$ AP(S). To prove the reverse inclusion, let $f \in$ AP(S). Define $\tilde{f} : (\tilde{S}, \tilde{\mathscr{U}}) \to \mathbb{C}$ by $\tilde{f}(\tilde{s}) = f(s)$. It is then easy to see that $\tilde{f}$ is uniformly continuous and hence extends to a uniformly continuous function, say $\hat{f}$, on S^a. The extended function $\hat{f}$ satisfies $\tilde{\rho}(\hat{f})(s) = \hat{f} \circ \rho(s) = \hat{f}(\tilde{s}) = \tilde{f}(\tilde{s}) = f(s)$. That is, $\tilde{\rho}(\hat{f}) = f$. Thus $\tilde{\rho}[C(S^a)] =$ AP(S).

Remark 1. The space obtained above coincides with the almost periodic compactification of S in [4] and [8]. This follows from [4, theorem 6.3] and the above theorem.

Remark 2. When S has sufficiently many semicharacters to separate the points of S then $\tilde{s} = \{s\}$ for each s and hence $\tilde{S}$ defined above coincides with S.

DEFINITION 1.7

Let S be a topological semigroup and X, a Banach space. For an operator $T: X \to C(S)$ and $s \in S$, define $T_s: X \to C(S)$ by $T_s(x) = (Tx)_s$, the right translate of Tx.

DEFINITION 1.8

Let $\mathscr{B}$ denote the field of Borel sets in a topological space S. If μ is a function on $\mathscr{B}$ with values in a Banach space, then the semi-variation of μ over $E \in \mathscr{B}$ is defined as

$$\|\mu\|(E) = \sup \left\| \sum_{i=1}^{n} \alpha_i \mu(E_i) \right\|,$$

where the supremum is taken over all finite collections of disjoint Borel sets in E and all finite set of scalars $\alpha_1, \ldots, \alpha_n$ with $|\alpha_i| \leqslant 1$.

The following theorems are from [5]:

Theorem 1.9. *Let X be a Banach space and X^*, the topological dual of X. If J is the natural embedding of X into X^{**}, the JX is X^*-dense in X^{**}.*

Theorem 1.10. *Let S be a compact Hausdorff space and let T be a bounded linear operator from a Banach space X into $C(S)$. Then there exists a mapping $\tau: S \to X^*$ which is continuous with the X-topology in X^* such that*

(1) $Tx(s) = \tau(s)(x), \quad x \in X, \quad s \in S$

(2) $\|T\| = \sup_{s \in S} \|\tau(s)\|$.

Conversely, if such a map τ is given, then the operator T defined in (1) is a bounded linear operator from X into $C(S)$ with norm given by (2). The operator T is weakly compact if and only if τ is continuous with the X^{**} topology in X^*. The operator T is compact if and only if τ is continuous with the norm topology in X^*.

Theorem 1.11. *Let S be a compact Hausdorff space and let T be a weakly compact operator from $C(S)$ to a Banach space X. Then there exists a vector measure μ defined on the Borel sets in S and having values in X such that*

(a) $x^*\mu$ is in rca(S), $\quad x^* \in X^*$.

(b) $Tf = \int_S f(s)\mu(ds), \quad f \in C(S)$.

(c) $\|T\| = \|\mu\|(S)$

(d) $T^*x^* = x^*\mu$.

Conversely if μ is a vector measure on the Borel sets in S to the Banach space X which satisfies (a) then the operator T defined by (b) is a weakly compact operator from $C(S)$ to X whose norm is given by (c) and whose adjoint is given by (d).

Theorem 1.12. *An operator $T: C(S) \to X$ is compact if and only if the vector measure*

$\mu:\mathscr{B}\to X$ corresponding to it as in the above theorem takes its values in a compact subset of X.

Throughout the paper all topological spaces are assumed to be Hausdorff. Hence we shall not mention this in further discussion.

3. Operators in BL$(X, C(S))$

Let S be a compact uniform topological semigroup such that the uniformity is left translation invariant. Let H be a dense subsemigroup of S. We denote the relative uniformity on H by $\mathscr{U}$. In this section we obtain characterizations of compact and weakly compact operators from a Banach space X into $C(S)$. We crucially use theorem 1.10 and the fact that the uniformity on S is left translation invariant.

The following theorem gives the necessary and sufficient condition for an operator T in BL$(X, C(S))$ to be compact.

Theorem 2.1. *A bounded linear operator $T: X\to C(S)$ is compact if and only if the map $\theta:(H,\mathscr{U})\to$ BL$(X, C(S))$ defined by $\theta(s)=T_s$ is uniformly continuous in the norm topology of* BL$(X, C(S))$.

Proof. From theorem 1.10 and the fact that $(H,\mathscr{U})$ is dense in S we see that T is compact if and only if τ/H: $(H,\mathscr{U})\to(X^*,\|\cdot\|)$ defined by $\tau(s)(x)=Tx(s)$ is uniformly continuous. To prove the theorem first assume that T is compact. Let $\varepsilon>0$. Then by the uniform continuity of τ, there exists a basic entourage U in $\mathscr{U}$ such that $(s,t)\in U$ implies $\|\tau(s)-\tau(t)\|<\varepsilon/2$. Since the uniformity $\mathscr{U}$ is left translation invariant, $(s,t)\in U$ implies $(us,ut)\in U$ for all $u\in S$. Hence whenever

$$(s,t)\in U,\ \|\tau(us)-\tau(ut)\|<\varepsilon,\quad \text{for all } u\in S. \tag{1}$$

But as

$$\begin{aligned}\|\tau_s-\tau_t\| &= \sup_{u\in S}\|\tau(us)-\tau(ut)\|\\ &= \sup_{u\in S}\sup_{\|x\|\leq 1}|\tau(us)(x)-\tau(ut)(x)|\\ &= \sup_{u\in S}\sup_{\|x\|\leq 1}|Tx(us)-Tx(ut)|\\ &= \sup_{\|x\|\leq 1}\sup_{u\in S}|(Tx)_s(u)-(Tx)_t(u)|\\ &= \sup_{\|x\|\leq 1}\|T_s(x)-T_t(x)\|\\ &= \|T_s-T_t\|\\ &= \|\theta(s)-\theta(t)\|,\end{aligned}$$

the uniform continuity of θ follows from (1). By reversing the chain of equalities and arguments used above we see that if θ is uniformly continuous in the norm topology of BL$(X, C(S))$ then $\tau:(H,\mathscr{U})\to(X^*,\|\cdot\|)$ is uniformly continuous. This proves the compactness of T.

Remark 1. If we define the left translate ${}_sT$ of T in BL$(X, C(S))$ by ${}_sT(x) = {}_s(Tx)$, the left translate of Tx, then the result similar to the above theorem can be obtained. Combining the two we get the following result: If T in BL$(X, C(S))$ is compact then the map $s \to T_s$ is norm continuous if and only if the map $s \to {}_sT$ is norm continuous.

Remark 2. Let X, Y be Banach spaces on which a group G acts. Then the notion of uniformly almost periodic multiplier from X into Y is defined by [3]. If $Y = S(G)$, a symmetric segal algebra on G then a characterization of a compact multipliers is obtained in [9, theorem 3]. If X and $C(S)$ are as in the above theorem we define uniformly almost periodic operators from X into $C(S)$ in a manner analogous to that of [3]. (We say that $T \in$ BL$(X, C(S))$ is uniformly almost periodic if the set $\{T_s : s \in S\}$ is relatively compact in the uniform operator topology on BL$(X, C(S))$. Theorem 2.1 then shows that if a bounded linear operator $T: X \to C(S)$ is compact then it is uniformly almost periodic in this sense.

We now characterize weakly compact operators from X into $C(S)$. For Banach spaces X, Y we first define a topology on BL(X, Y) as follows: For $\varepsilon > 0$ and $x^{**} \in X^{**}$, define a neighbourhood of zero,

$$N(\varepsilon, x^{**}) = \{T \in \mathrm{BL}(X, Y) : \| T^{**}(x^{**}) \| < \varepsilon\}.$$

The family of such sets forms a subbase for a neighbourhood system of zero in BL(X, Y). We denote by τ the topology generated in this manner. One can easily see that τ is a locally convex Hausdorff topology and lies between the uniform and the strong operator topology of BL(X, Y).

Before proving the next theorem, we note the following: Let A be a dense subspace of a uniform space X and (Y, w), a Banach space Y with its weak topology. Then every uniformly continuous function from A into (Y, w) has a continuous extension to X. If (Y, w) is complete there is nothing to prove. If it is not, let $\bar{Y}$ denote its completion. The result then follows from the fact that $(\bar{Y})^* = (Y, w)^*$.

Theorem 2.2. *An operator T in* BL$(X, C(S))$ *is weakly compact if and only if the map θ defined in Theorem* 2.1 *is uniformly continuous with respect to the topology τ on* BL$(X, C(S))$.

Proof. Since $(H, \mathscr{U})$ is dense in S, it follows from theorem 1.10 and the discussion above that T is weakly compact if and only if τ/H: $(H, \mathscr{U}) \to (X^*, w)$ is uniformly continuous. Now assume that T is weakly compact. To prove the continuity of θ, let $N(\varepsilon, x^{**})$ be any arbitrary subbasic neighbourhood of zero in τ. Then by the uniform continuity of τ, there exists a subbasic entourage U in $\mathscr{U}$ such that $(s, t) \in U$ implies $|x^{**}(\tau(s) - \tau(t))| < \varepsilon/8$. If J denotes the natural embedding of X into X^{**} then, by theorem 1.9, there exists a net $\{x_\alpha\}$ in X such that $J(x_\alpha)$ converges to x^{**} in the weak star topology of X^{**}. Therefore there exists $\alpha_0 = \alpha_0(s, t)$ such that

$$|J(x_\alpha)(\tau(s) - \tau(t)) - x^{**}(\tau(s) - \tau(t))| < \varepsilon/8 \text{ for all } \alpha \geqslant \alpha_0.$$

Thus if $(s, t) \in U$ and $\alpha \geqslant \alpha_0$,

$$|J(x_\alpha)(\tau(s) - \tau(t))| \leqslant |J(x_\alpha)(\tau(s) - \tau(t)) - x^{**}(\tau(s) - \tau(t))| + |x^{**}(\tau(s) - \tau(t))|$$
$$< \varepsilon/8 + \varepsilon/8 = \varepsilon/4 \qquad (1)$$

Now as $\mathscr{U}$ is translation invariant $(s,t)\in U$ implies $(us,ut)\in U$ for all $u\in S$. Hence from (1) whenever $(s,t)\in U$ and $\alpha\geqslant\alpha_0$ we have $|J(x_\alpha)(\tau(us)-\tau(ut))|<\varepsilon/4$, for all u in S. But

$$\begin{aligned}\sup_{u\in S}|J(x_\alpha)(\tau(us)-\tau(ut))| &= \sup_{u\in S}|\tau(us)(x_\alpha)-\tau(ut)(x_\alpha)|\\ &= \sup_{u\in S}|Tx_\alpha(us)-Tx_\alpha(ut)|\\ &= \sup_{u\in S}|(Tx_\alpha)_s(u)-(Tx_\alpha)_t(u)|\\ &= \sup_{u\in S}|T_sx_\alpha(u)-\ T_tx_\alpha(u)|\\ &= \|T_s(x_\alpha)-T_t(x_\alpha)\|.\end{aligned}$$

Thus for all $\alpha\geqslant\alpha_0$,

$$(s,t)\in U\Rightarrow\|T_s(x_\alpha)-T_t(x_\alpha)\|\leqslant\varepsilon/4. \tag{2}$$

Now let $f^*\in C(S)^*$ be such that $\|f^*\|\leqslant 1$. Again since $J(x_\alpha)$ converges to x^{**} in the weak star topology, there exists $\beta_0=\beta_0(f^*,s,t)$ such that

$$|J(x_\alpha)(T_s^*f^*-T_t^*f^*)-x^{**}(T_s^*f^*-T_t^*f^*)|<\varepsilon/4 \text{ for all } \alpha\geqslant\beta_0.$$

Let $\gamma_0\geqslant\alpha_0,\ \beta_0$. Then for $(s,t)\in U$ and $\alpha\geqslant\gamma_0$ we have from (2)

$$\begin{aligned}|((T_s)^{**}-(T_t)^{**})(x^{**})(f^*)| &= |x^{**}(T_s^*f^*-T_t^*f^*|\\ &\leqslant |x^{**}(T_s^*f^*-T_t^*f^*)-J(x_\alpha)(T_s^*f^*-T_t^*f^*)|\\ &\quad+|J(x_\alpha)(T_s^*f^*-T_t^*f^*)|\\ &<\varepsilon/4+|T_s^*f^*(x_\alpha)-T_t^*f^*(x_\alpha)|\\ &<\varepsilon/4+|f^*(T_s(x_\alpha)-T_t(x_\alpha))|\\ &\leqslant\varepsilon/4+\|f^*\|\,\|T_s(x_\alpha)-T_t(x_\alpha)\|\\ &\leqslant\varepsilon/4+\varepsilon/4=\varepsilon/2.\end{aligned}$$

Thus if $(s,t)\in U$,

$$\|T_s^{**}x^{**}-T_t^{**}x^{**}\|=\sup_{\|f^*\|\leqslant 1}|T_s^{**}x^{**}(f^*)-T_t^{**}x^{**}(f^*)||<\varepsilon.$$

Equivalently, $T_s-T_t\in N(\varepsilon,x^{**})$. This proves that the map θ is uniformly continuous with respect to the topology τ on $BL(X,C(S))$. Conversely, suppose that θ is uniformly continuous in τ. To prove T is weakly compact, it is enough to show that $\tau:(H,\mathscr{U})\to X^*$ is uniformly continuous in the weak topology of X^*. Let $V(\varepsilon,x^{**})$ be any arbitrary weak neighbourhood of zero in X^*. Since θ is uniformly continuous, for the neighbourhood of zero in τ determined by ε and x^{**} there exists U in $\mathscr{U}$ such that

$$(s,t)\in U\Rightarrow|(T_s^{**}-T_t^{**})(x^{**})(f^*)|<\varepsilon\quad\text{for all }\|f^*\|\leqslant 1.$$

Define $m_e: C(S)\to\mathbb{C}$ by $m_e(f)=f(e)$ where e is the identity of S. Then $m_e\in C(S)^*$ and $\|m_e\|\leqslant 1$. Therefore

$$|(T_s^{**}-T_t^{**})(x^{**})(m_e)|<\varepsilon,\quad\text{whenever }(s,t)\in U. \tag{3}$$

Now, let $\{x_\alpha\}$ be a net in X such that $J(x_\alpha)$ converges to x^{**} in the weak-star topology. Then

$$\begin{aligned}
|(T_s^{**} - T_t^{**})(x^{**})(m_e)| &= |x^{**}(T_s^* m_e - T_t^* m_e)| \\
&= \lim_\alpha |J(x_\alpha)(T_s^* m_e - T_t^* m_e)| \\
&= \lim_\alpha |T_s^* m_e(x_\alpha) - T_t^* m_e(x_\alpha)| \\
&= \lim_\alpha |m_e(T_s(x_\alpha)) - m_e(T_t(x_\alpha))| \\
&= \lim_\alpha |Tx_\alpha(s) - Tx_\alpha(t)| \\
&= \lim_\alpha |(\tau(s) - \tau(t))(x_\alpha)| \\
&= \lim_\alpha |J(x_\alpha)(\tau(s) - \tau(t))| \\
&= |x^{**}(\tau(s) - \tau(t))|.
\end{aligned}$$

Thus from (3), $(s,t)\in U$ implies $|x^{**}(\tau(s) - \tau(t))| < \varepsilon$. Equivalently, $\tau(s) - \tau(t) \in V(\varepsilon, x^{**})$, which proves that the map τ is uniformly continuous in the weak topology of X^*. This completes the proof.

4. Operators in BL($X, AP(S)$)

Let S be a topological semigroup and X, a Banach space. As mentioned in the introduction, here we apply results of §3 to the operators from X in $C(S^a)$, where S^a is the almost periodic compactification of S. (theorem 1.5). We note that the space $(\tilde{S}, \tilde{\mathscr{U}})$ constructed in §2 is dense in the compact Hausdorff topological semigroup S^a. Also the uniformity $\tilde{\mathscr{U}}$ is left translation invariant. This meets the requirement of results in §3.

For the simplicity of notation throughout this section we write $(S, \mathscr{U})$ for the space $(\tilde{S}, \tilde{\mathscr{U}})$ and denote the element $\tilde{s}$ of $\tilde{S}$ by s itself. As mentioned in Remark 2 of §3, $\tilde{s} = \{s\}$ when S has sufficiently many semicharacters to separate the points of S. Indeed this is the case when S is a locally compact abelian group. Let $\tilde{\rho}$ be the isometric isomorphism of $C(S^a)$ onto AP(S) (theorem 1.6). For proving the next result we first observe here the following:

If $f \in \mathrm{AP}(S)$ and $\hat{f}$ denotes the continuous extension of f to S^a then $\tilde{\rho}(\hat{f}) = f$. Further for any $s \in S$, $(\hat{f_s}) = (\hat{f})_s$ and $\|(\hat{f})_s\| = \|f_s\|$. Also if $K \in \mathrm{BL}(X, \mathrm{AP}(S))$ then $T = \tilde{\rho}^{-1} \circ K \in \mathrm{BL}(X, C(S^a))$ and $T_s = (\tilde{\rho}^{-1} \circ K)_s = \tilde{\rho}^{-1} \circ K_s$. The following theorem characterizes compact operators from X into AP(S).

Theorem 3.1. *A bounded linear operator* $K: X \to \mathrm{AP}(S)$ *is compact if and only if the map* $\psi: (S, \mathscr{U}) \to \mathrm{BL}(X, \mathrm{AP}(S))$ *defined by* $\psi(s) = K_s$ *is uniformly continuous in the norm topology of* BL(X, AP(S)).

Proof. It is easy to see that $K: X \to \mathrm{AP}(S)$ is compact if and only if the operator $T = \tilde{\rho}^{-1} \circ K: X \to C(S^a)$ is compact. But by theorem 2.1, this is equivalent to the fact that the map $\theta: (S, \mathscr{U}) \to \mathrm{BL}(X, C(S^a))$ defined by $\theta(s) = T_s$ is uniformly continuous in the norm topology. Now from the above observations and the fact that $(S, \mathscr{U})$ is dense in S^a, we have, for s, t in S,

$$\begin{aligned}\|T_s - T_t\| &= \sup_{\|x\| \leq 1} \|T_s(x) - T_t(x)\| \\ &= \sup_{\|x\| \leq 1} \sup_{u \in S} |(\tilde{\rho}^{-1} \circ K_s)(x)(u) - (\tilde{\rho}^{-1} \circ K_t)(x)(u)| \\ &= \sup_{\|x\| \leq 1} \sup_{u \in S} |(\widehat{Kx})(us) - (\widehat{Kx})(ut)| \\ &= \sup_{\|x\| \leq 1} \sup_{u \in S} |K_s x(u) - K_t x(u)| \\ &= \|K_s - K_t\| \\ &= \|\psi(s) - \psi(t)\|.\end{aligned}$$

From this the assertion of the theorem follows easily.

We now obtain the characterization of a weakly compact operator in $\mathrm{BL}(X, \mathrm{AP}(S))$ in terms of the topology on $\mathrm{BL}(X, \mathrm{AP}(S))$ defined in §3.

Theorem 3.2. *A bounded linear operator $K: X \to \mathrm{AP}(S)$ is weakly compact if and only if the map ψ defined in theorem* 3.1 *is uniformly continuous in the topology τ on* $\mathrm{BL}(X, \mathrm{AP}(S))$.

Proof. As in the above theorem we identify the space $\mathrm{AP}(S)$ with $C(S^a)$. Then $K: X \to \mathrm{AP}(S)$ is weakly compact if and only if $T = \tilde{\rho}^{-1} \circ K: X \to C(S^a)$ is weakly compact. But by theorem 2.2, this is equivalent to the uniform continuity of the map $\theta: (S, \mathscr{U}) \to (\mathrm{BL}(X, C(S^a)), \tau)$ given by $\theta(s) = T_s$. Let $N(\varepsilon, x^{**})$ be any arbitrary subbasic neighbourhood of zero in $(\mathrm{BL}(X, \mathrm{AP}(S)), \tau)$. Then by the uniform continuity of θ, for the neighbourhood of zero in $(\mathrm{BL}(X, C(S^a)), \tau)$ determined by ε and x^{**}, there exists U in $\mathscr{U}$ such that $(s, t) \in U$ implies $\|(T_s)^{**}x^{**} - (T_t)^{**}x^{**}\| < \varepsilon$. But from the observations made before theorem 3.1 one can easily see that $\|(T_s)^{**}x^{**} - (T_t)^{**}x^{**}\| = \|K_s^{**}x^{**} - K_t^{**}x^{**}\|$. This shows that θ is uniformly continuous if and only if the map $s \to K_s$ is uniformly continuous with respect to topology τ. This proves the theorem.

5. Compact and weakly compact multipliers in BL(AP(S))

Let S, $\mathscr{U}$ and H be as in §3 and X a Banach space. In this section we first obtain characterizations of compact and weakly compact multipliers in $\mathrm{BL}(C(S))$. When S is not compact, these results are then applied to the operators on $C(S^a)$, where S^a is the almost periodic compactification of S. As in §4, we identify the space $\mathrm{AP}(S)$ with $C(S^a)$ to obtain the desired characterizations.

Let $\mathscr{B}$ denotes the σ-field of Borel sets in S and rca(S), the space of regular countably additive scalar valued measures defined on $\mathscr{B}$.

Notation: For a set function $\mu: \mathscr{B} \to X^{**}$ and $x^* \in X^*$ we write $\mu(\cdot)x^*$ for the scalar valued set function defined by $(\mu(\cdot)x^*)(E) = \mu(E)x^*$, $E \in \mathscr{B}$. Let $F(S, \mathscr{B}, X^{**}) = \{\mu: \mathscr{B} \to X^{**}: \mu(\cdot)x^* \in \mathrm{rca}(S) \text{ for every } x^* \in X^*\}$.

DEFINITION 4.1

For a set function $\mu:\mathscr{B}\to X$ and $s\in S$ define $\mu_s:\mathscr{B}\to X$ by $\mu_s(E)=\mu(Es)$, $E\in\mathscr{B}$.

DEFINITION 4.2

An operator $T: C(S)\to X$ is a multiplier if $(Tf)_s = T(f_s)$, $f\in C(S)$, $s\in S$.

For a linear operator $T: C(S)\to X$ and $s\in S$ we define $T_s: C(S)\to X$ by $T_s(f)=T(f_s)$, $f\in C(S)$. This may be compared with the definition of T_s given in §3. When $X=C(S)$ and T is a multiplier the two meanings for T_s coincide.

PROPOSITION 4.3

Let $T\in BL(C(S),X)$ and μ be the unique set function which represents T [5, theorem VI.7.2]. Then for any $s\in S$, $T_s\in \mathrm{BL}(C(S),X)$ and the set function μ_s represents T_s.

Proof. It is easy to see that if $T\in \mathrm{BL}(C(S),X)$, $T_s\in \mathrm{BL}(C(S),X)$ for any $s\in S$. Let $E\in\mathscr{B}$ and $\phi_E\in C(S)^{**}$ be defined by $\phi_E(\lambda)=\lambda(E)$, $\lambda\in \mathrm{rca}(S)$. Then by [5, theorem VI.7.2], the set function $\mu:\mathscr{B}\to X^{**}$ defined by $\mu(E)=T^{**}(\phi_E)$ represents the operator T and $T^*x^*=\mu(\cdot)x^*\in\mathrm{rca}(S)$ for every $x^*\in X^*$. Further $\|T\|=\|\mu\|(S)$, the semivariation of μ. Now, for $x^*\in X^*$ and $f\in C(S)$, $(T^*x^*)_s(f)=T^*x^*(f_s)=x^*(T(f_s))=x^*(T_s(f))=(T_s)^*x^*(f)$. Thus $(T^*x^*)_s=(T_s)^*x^*$ for every $x^*\in X^*$. But then $\mu_s(E)x^*=\mu(Es)x^*=T^{**}(\phi_{Es})(x^*)=\phi_{Es}(T^*x^*)=T^*x^*(Es)=(T^*x^*)_s(E)=T_s^*x^*(E)=\phi_E(T_s^*x^*)=T_s^{**}(\phi_E)\cdot(x^*)$. Hence $\mu_s(E)=T_s^{**}(\phi_E)$ and for every $x^*\in X^*$, $\mu_s(\cdot)x^*=T_s^*x^*\in\mathrm{rca}(S)$. As in [5, theorem VI. 7.2] one can then prove that $\|T_s\|=\|\mu_s\|(S)$. This shows that the set function μ_s represents the operator T_s.

To obtain the results of this section we define topologies on $F(S,\mathscr{B},X^{**})$ as follows:

(1) τ_V-topology on $F(S,\mathscr{B},X^{**})$: For $\mu\in F(S,\mathscr{B},X^{**})$ define $p(\mu)=\|\mu\|(S)$, where $\|\mu\|(S)$ denotes the semivariation of μ. Let $\varepsilon>0$ and $V(p,\varepsilon)=\{\mu: p(\mu)<\varepsilon\}$. Then the family of sets of this type forms a subbase at zero for a locally convex topology on $F(S,\mathscr{B},X^{**})$. We denote this topology by τ_V.

(2) τ'-topology on $F(S,\mathscr{B},X^{**})$: For $\varepsilon>0$ and $f^{**}\in C(S)^{**}$ define

$$V(\varepsilon,f^{**})=\left\{\mu\in F(S,\mathscr{B},X^{**}): \sup_{\|x^*\|\leq 1} |f^{**}(\mu(\cdot)x^*)|<\varepsilon\right\}.$$

The family of sets of this type forms a subbase at zero for a topology on $F(S,\mathscr{B},X^{**})$. We denote this topology by τ'.

We use results of §3 to characterize compact and weakly compact multipliers in $\mathrm{BL}(C(S))$ in terms of the topologies defined in (1) and (2). It may be noted that the uniformity $\mathscr{U}$ on S is assumed to be left translation invariant.

Theorem 4.4. *A multiplier T in $BL(C(S))$ is compact if and only if the map $\phi: H\to(F(S,\mathscr{B},C(S)^{**},\tau_V)$ defined by $\phi(s)=\mu_s$ is uniformly continuous, where μ_s is the unique set function which represents T_s. In this case in fact the set function μ_s is a vector measure on $\mathscr{B}$ with values in $C(S)$ and its range is relatively compact.*

Proof. Since T is a multiplier, $T_s(f)=T(f_s)=(Tf)_s$ for every $s\in S$ and $f\in C(S)$. By theorem 2.1 T is compact if and only if the map $s\to T_s$ is uniformly norm continuous from H into $\mathrm{BL}(C(S))$. But since the set function μ_s represents the operator T_s, we have $\|T_s - T_t\| = \|\mu_s - \mu_t\|$ (S). This shows that the map $s \to T_s$ is uniformly norm continuous if and only if $\phi: s \to \mu_s$ is uniformly continuous with respect to the topology τ_V. The fact that μ_s is a vector measure on $\mathscr{B}$ with values in $C(S)$ and its range is relatively compact is a consequence of theorem 1.12.

Theorem 4.5. *A multiplier T in* $\mathrm{BL}(C(S))$ *is weakly compact if and only if the map defined in theorem 4.4 is uniformly continuous with respect to the topology τ' on $F(S, \mathscr{B}, C(S)^{**})$ and in this case the set function μ_s is a vector measure on $\mathscr{B}$ with values in $C(S)$.*

Proof. By theorem 2.2, T is weakly compact if and only if the map $s \to T_s$ is uniformly continuous from H into $\mathrm{BL}(C(S))$ with respect to the topology τ (defined in §2) on $\mathrm{BL}(C(S))$, where $T_s(f) = (Tf)_s$. But since T is a multiplier, $T_s(f) = (Tf)_s = T(f_s)$. Also if μ_s is the set function which represents the operator T_s, then for every $f^* \in C(S)^*$, $T_s^* f^* = \mu_s(\cdot) f^* \in \mathrm{rca}(S)$. Let $N(\varepsilon, f^{**})$ and $V(\varepsilon, f^{**})$ be the neighbourhoods of zero in $(\mathrm{BL}(C(S)), \tau)$ and $(F(S, \mathscr{B}, C(S)^{**}), \tau')$ respectively. Then

$$\begin{aligned}\|(T_s)^{**} f^{**} - (T_t)^{**} f^{**}\| &= \sup_{\|f^*\| \leq 1} |f^{**}(T_s^* f^* - T_t^* f^*)| \\ &= \sup_{\|f^*\| \leq 1} |f^{**}(\mu_s(\cdot) f^* - \mu_t(\cdot) f^*)|.\end{aligned}$$

This shows that $T_s - T_t \in N(\varepsilon, f^{**})$ if and only if $\mu_s - \mu_t \in V(\varepsilon, f^{**})$. Hence the map $s \to T_s$ is uniformly τ-continuous if and only if $\phi: s \to \mu_s$ is uniformly τ'-continuous. By theorem 1.11 it follows that μ_s is a vector measure on $\mathscr{B}$ with values in $C(S)$. This proves the theorem.

Let S be a topological semigroup. To characterize compact and weakly compact multipliers on $\mathrm{AP}(S)$ we now apply results obtained above to the operators on $C(S^a)$, where S^a denotes the almost periodic compactification of S. Let $\mathscr{B}$ denote the σ-field of Borel sets of S^a and let the space $(S, \mathscr{U})$ be as in §4. Since the space $\mathrm{AP}(S)$ is isometrically isomorphic to $C(S^a)$, the first and the second dual of $\mathrm{AP}(S)$ are identified with those of $C(S^a)$ up to an isometric isomorphism.

The following theorems give the characterizations of compact and weakly compact multipliers on $\mathrm{AP}(S)$.

Theorem 4.6. *A multiplier K in* $\mathrm{BL}(\mathrm{AP}(S))$ *is compact if and only if the map $s \to \mu_s$ from $(S, \mathscr{U})$ into $(F(S^a, \mathscr{B}, \mathrm{AP}(S)^{**}), \tau_V)$ is uniformly continuous, where μ_s is the unique set function which represents the operator on $C(S^a)$ corresponding to K.*

Proof. It is easy to see that a multiplier K in $\mathrm{BL}(\mathrm{AP}(S))$ is compact if and only if the corresponding operator on $C(S^a)$ is compact. But since $(S, \mathscr{U})$ is dense in S^a, by theorem 4.4, this is equivalent to the uniform continuity of the map $s \to \mu_s$ from $(S, \mathscr{U})$ into $F(S^a, \mathscr{B}, C(S^a)^{**})$ with respect to the topology τ_V, where the set function μ_s represents the operator on $C(S^a)$ corresponding to K. The theorem now follows from the fact that the space $C(S^a)^{**}$ is isometrically isomorphic to $\mathrm{AP}(S)^{**}$.

Theorem 4.7. *A multiplier K in* $\mathrm{BL}(\mathrm{AP}(S))$ *is weakly compact if and only if the map defined in theorem 4.6 is uniformly continuous with respect to the topology τ' on $F(S^a, \mathscr{B}, \mathrm{AP}(S)^{**})$.*

Proof. As in theorem 4.6 we identify the space AP(S) with $C(S^a)$. The required assertion then follows by applying theorem 4.5 and using the arguments similar to those in the proof of the above theorem.

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