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On the inverse Laplace transform of the product of a general class of polynomials and the multivariable H-function

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Abstract. In this paper we evaluate the inverse Laplace transform of

$$s^{-\eta}(s^{l_1}+\lambda_1)^{-\sigma}(s^{l_2}+\lambda_2)^{-\rho}$$
$$\times S_n^m[xs^{-u}(s^{l_1}+\lambda_1)^{-v}(s^{l_2}+\lambda_2)^{-w}]S_{n'}^{m'}[ys^{-u'}(s^{l_1}+\lambda_1)^{-v'}(s^{l_2}+\lambda_2)^{-w'}]$$
$$\times H[z_1 s^{-u_1}(s^{l_1}+\lambda_1)^{-v_1}(s^{l_2}+\lambda_2)^{-w_1},\ldots,z_r s^{-u_r}(s^{l_1}+\lambda_1)^{-v_r}(s^{l_2}+\lambda_2)^{-w_r}].$$

Due to the general nature of the multivariable H-function involved herein, the inverse Laplace transform of the product of a large number of special functions involving one or more variables, occurring frequently in the problems of theoretical physics and engineering sciences can be obtained as simple special cases of our main findings. For the sake of illustration, we obtain here the inverse Laplace transform of a product of the Hermite polynomials, the Jacobi polynomials and r different modified Bessel functions of the second kind. A theorem obtained by Srivastava and Singh [7] follows as a special case of our main result.

Keywords. The Laplace transform; the multivariable H-function; a general class of polynomials.

1. Introduction

We define the Laplace transform of the function $f(x)$ as

$$\phi(s) = L\{f(x);s\} = \int_0^\infty \exp(-sx)f(x)\,\mathrm{d}x. \tag{1}$$

The function $f(x)$ is called the inverse Laplace transform of $\phi(s)$ and will be denoted by $L^{-1}\{\phi(s)\}$ in this paper.

Main result

$$L^{-1}\{s^{-\eta}(s^{l_1}+\lambda_1)^{-\sigma}(s^{l_2}+\lambda_2)^{-\rho}$$
$$\times S_n^m[xs^{-u}(s^{l_1}+\lambda_1)^{-v}(s^{l_2}+\lambda_2)^{-w}]S_{n'}^{m'}[ys^{-u'}(s^{l_1}+\lambda_1)^{-v'}(s^{l_2}+\lambda_2)^{-w'}]$$
$$\times H[z_1 s^{-u_1}(s^{l_1}+\lambda_1)^{-v_1}(s^{l_2}+\lambda_2)^{-w_1},\ldots,z_r s^{-u_r}(s^{l_1}+\lambda_1)^{-v_r}(s^{l_2}+\lambda_2)^{-w_r}]\}$$

$$
= t^{\eta + l_1\sigma + l_2\rho - 1} \sum_{k=0}^{[n/m]} \sum_{k'=0}^{[n'/m']} \frac{(-n)_{mk}(-n')_{m'k'}}{k!\,k'!}
\times A_{n,k} A'_{n',k'} (x t^{u+vl_1+wl_2})^k (y t^{u'+v'l_1+w'l_2})^{k'}
$$

$$
\times H^{O,N+2\;:M',N';\ldots;M^{(r)},N^{(r)};1,0;1,0}_{P+2,Q+3:P',Q';\ldots;P^{(r)},Q^{(r)};0,1;0,1} \left[\begin{matrix} z_1 t^{u_1+v_1l_1+w_1l_2} \\ \vdots \\ z_r t^{u_r+v_rl_1+w_rl_2} \\ \lambda_1 t^{l_1} \\ \lambda_2 t^{l_2} \end{matrix} \right.
$$

$$
\begin{matrix}
(1-\sigma-vk-v'k';v_1,\ldots,v_r,1,0),(1-\rho-wk-w'k';w_1,\ldots,w_r,0,1), \\
(1-\sigma-vk-v'k';v_1,\ldots,v_r,0,0),(1-\rho-wk-w'k';w_1,\ldots,w_r,0,0), \\
(a_j;\alpha'_j,\ldots,\alpha_j^{(r)},0,0)_{1,P} \\
(1-\eta-l_1\sigma-l_2\rho-(u+vl_1+wl_2)k-(u'+v'l_1+w'l_2)k';u_1+v_1l_1+w_1l_2,\ldots, \\
---\qquad\qquad\vdots \\
u_r+v_rl_1+w_rl_2,l_1,l_2),(b_j;\beta'_j,\ldots,\beta_j^{(r)},0,0)_{1,Q}\;:
\end{matrix}
$$

$$
\left. \begin{matrix} (c'_j,\gamma'_j)_{1,P'};\ldots;(c_j^{(r)},\gamma_j^{(r)})_{1,P^{(r)}};---;--- \\ (d'_j,\delta'_j)_{1,Q'};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,Q^{(r)}};(0,1);(0,1) \end{matrix} \right] \tag{2}
$$

where the symbols S_n^m and $S_{n'}^{m'}$ occurring in the left hand side of (2) denote product of Srivastava's general class of polynomials [4, p. 1, eq. (1)] and the H-functions occurring in its left hånd side and right hand side stand for the H-function of r and $r+2$ variables respectively [6] expressed with slightly changed notations in this paper [5, p. 251, eq. (C.1)] and the following conditions are satisfied. The quantities $u, v, w, u', v', w', u_i, v_i, w_i$, $i=1,\ldots,r$ are all positive, $\mathrm{Re}(s)>0$,

$$
\mathrm{Re}(\eta+l_1\sigma+l_2\rho)+\min_{1\leq j\leq M^{(i)}}\left[\mathrm{Re}(u_i+v_il_1+w_il_2)\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0,
$$

$$
\sum_{j=1}^{P}\alpha_j^{(i)}+\sum_{j=1}^{P(i)}\gamma_j^{(i)}-\sum_{j=1}^{Q}\beta_j^{(i)}-\sum_{j=1}^{Q(i)}\delta_j^{(i)}-(u_i+v_il_1+w_il_2)<0,\quad i=1,\ldots,r;
$$

$$
U_i=-\sum_{j=N+1}^{P}\alpha_j^{(i)}+\sum_{j=1}^{N(i)}\gamma_j^{(i)}-\sum_{j=N^{(i)}+1}^{P(i)}\gamma_j^{(i)}-\sum_{j=1}^{Q}\beta_j^{(i)}+\sum_{j=1}^{M(i)}\delta_j^{(i)}-\sum_{j=M^{(i)}+1}^{Q(i)}\delta_j^{(i)}
$$

$$
-[u_i+v_i(l_1+1)+w_i(l_2+1)]>0,
$$

$|\arg z_i|<(\tfrac{1}{2})U_i\pi$ or $U_i=0$ and $z_i>0$, $i=1,\ldots,r$;

$$
0<l_j<1, |\arg\lambda_j|<(1-l_j)\frac{\pi}{2}, j=1,2 \text{ or } l_1=l_2=1 \text{ and } \lambda_1>0,\ \lambda_2>0.
$$

It may be remarked here that some of the exponents $l_1, l_2, u, v, w, u', v', w', u_i, v_i, w_i$, $i=1,\ldots,r$ in (2) can decrease to zero provided both sides of the resulting equation have a meaning.

Proof. We first express the general class of polynomials occurring in the left hand side of (2) in the series form [4, p. 1, eq. (1)] and the multivariable H-function involved therein in terms of its Mellin-Barnes contour integral [5, p. 251, eq. (C.1)]. Now by making use of a known formula [2, eq. (1.6)] easily obtainable from a well-known result [3, p. 122, eq. (2.2)], proceeding in a manner as given in a recent paper [2] and reinterpreting the resulting contour integral in terms of the H-function of $r+2$ variables, we arrive at the desired result after a little simplification. The details of the proof can be seen in [2].

If we take $\eta = l_1 = 0$, $l_2 = 1$, $u = v = 0$, $w = 1$, $n' = 0$, $u_i = v_i = 0$, $i = 1, \ldots, r$ in (2), we get a known result due to Srivastava and Singh [7, p. 169, eq. (2.10)] expressed in a different form.

Also if in the left hand side of (2) we put $N = P = Q = 0$, $M^{(i)} = Q^{(i)} = 1$, $N^{(i)} = P^{(i)} = 0$, $d_1^{(i)} = 0$, $\delta_1^{(i)} = 1$, $u_i = 1$, $v_i = w_i = 0$ and let z_i tends to zero ($i = 2, \ldots, r$), the H-function of r variables occurring therein reduces to the H-function of Fox [1], and we arrive at a result obtained by us [2] after a little simplification.

2. Special cases

(i) If we take $x = u = 1$, $v = w = 0$, $m = 2$, $A_{n,k} = (-1)^k$ and $y = u' = 1$, $v' = w' = 0$, $m' = 1$, $A'_{n',k'} = \binom{n'+\delta}{n'} \frac{(\delta+\zeta+n'+1)_{k'}}{(\delta+1)_{k'}}$ in (2), $S_n^2[1/s]$ occurring therein breaks into the Hermite polynomials [7, p. 158, eq (1.4)], $S_{n'}^1[1/s]$ reduces to the Jacobi polynomials [7, p. 159, eq. (1.6)] and we get the following result

$$
\begin{aligned}
L^{-1}\Big\{ & s^{-(\eta+n/2)}(s^{l_1}+\lambda_1)^{-\sigma}(s^{l_2}+\lambda_2)^{-\rho} H_n\left[\frac{\sqrt{s}}{2}\right] P_{n'}^{(\delta,\zeta)}\left[1-\frac{2}{s}\right] \\
& \times H[z_1 s^{-u_1}(s^{l_1}+\lambda_1)^{-v_1}(s^{l_2}+\lambda_2)^{-w_1}, \ldots, z_r s^{-u_r}(s^{l_1}+\lambda_1)^{-v_r}(s^{l_2}+\lambda_2)^{-w_r}]\Big\} \\
= t^{\eta+l_1\sigma+l_2\rho-1} & \sum_{k=0}^{[n/2]} \sum_{k'=0}^{n'} \frac{(-n)_{2k}(-n')_{k'}}{k!\,k'!}(-1)^k \binom{n'+\delta}{n'} \frac{(\delta+\zeta+n'+1)_{k'}}{(\delta+1)_{k'}} t^{k+k'} \\
& \times \mathrm{H}^{O,N+2\;:M',N';\ldots;M^{(r)},N^{(r)};1,0;1,0}_{P+2,Q+3:P',Q';\ldots;P^{(r)},Q^{(r)};0,1;0,1} \left[\begin{matrix} z_1 t^{u_1+v_1 l_1+w_1 l_2} \\ \vdots \\ z_r t^{u_r+v_r l_1+w_r l_2} \\ \lambda_1 t^{l_1} \\ \lambda_2 t^{l_2} \end{matrix} \right| \\
& (1-\sigma; v_1, \ldots, v_r, 1, 0), (1-\rho; w_1, \ldots, w_r, 0, 1), (a_j; \alpha'_j, \ldots, \alpha_j^{(r)}, 0, 0)_{1,P} \\
& (1-\sigma; v_1, \ldots, v_r, 0, 0), (1-\rho; w_1, \ldots, w_r, 0, 0), (1-\eta-l_1\sigma-l_2\rho-k-k'; \\
& \qquad --- \qquad : \\
& u_1+v_1 l_1+w_1 l_2, \ldots, u_r+v_r l_1+w_r l_2, l_1, l_2), (b_j; \beta'_j, \ldots, \beta_j^{(r)}, 0, 0)_{1,Q} : \\
& \left. \begin{matrix} (c'_j, \gamma'_j)_{1,P'}; \ldots; (c_j^{(r)}, \gamma_j^{(r)})_{1,P^{(r)}}; ---; --- \\ (d'_j, \delta'_j)_{1,Q'}; \ldots; (d_j^{(r)}, \delta_j^{(r)})_{1,Q^{(r)}}; (0,1); (0,1) \end{matrix} \right] \qquad (3)
\end{aligned}
$$

provided the conditions easily obtainable from (2) are satisfied. (ii) If we take $N=P=Q=0$, $M^{(i)}=Q^{(i)}=2$, $N^{(i)}=P^{(i)}=0$, $d_1^{(i)}=\frac{1}{4}+(v_i/2)$, $d_2^{(i)}=\frac{1}{4}-(v_i/2)$, $\delta_1^{(i)}=\delta_2^{(i)}=\frac{1}{2}$, $u_i=1$, $v_i=w_i=0$ and replace z_i by $z_i/2$ in (3), we arrive at the following interesting result involving the product of r different modified Bessel functions of the second kind:

$$L^{-1}\left\{s^{-(\eta+n/2+r/2)}(s^{l_1}+\lambda_1)^{-\sigma}(s^{l_2}+\lambda_2)^{-\rho}H_n\left[\frac{\sqrt{s}}{2}\right]P_{n'}^{(\delta,\zeta)}\left[1-\frac{2}{s}\right]\prod_{i=1}^{r}K_{v_i}\left[\frac{z_i}{s}\right]\right\}$$

$$=2^{-3r/2}\prod_{i=1}^{r}(z_i)^{-1/2}\frac{t^{\eta+l_1\sigma+l_2\rho-1}}{\Gamma(\sigma)\Gamma(\rho)}\sum_{k=0}^{[n/2]}\sum_{k'=0}^{n'}\frac{(-n)_{2k}(-n')_{k'}}{k!\,k'!}$$

$$\times(-1)^k\binom{n'+\delta}{n'}\frac{(\delta+\zeta+n'+1)_{k'}}{(\delta+1)_{k'}}t^{k+k'}$$

$$\times H^{0,0:2,0;\ldots;2,0;1,1;1,1}_{0,1:0,2;\ldots;0,2;1,1;1,1}\left[\begin{matrix}z_1 t\\ \vdots\\ z_r t\\ \lambda_1 t^{l_1}\\ \lambda_2 t^{l_2}\end{matrix}\middle| \begin{matrix}---\\ (1-\eta-l_1\sigma-l_2\rho-k-k';1,\ldots,1,l_1,l_2):\end{matrix}\right.$$

$$\left.\begin{matrix}---\;;\ldots;\;---\;;(1-\sigma,1);(1-\rho,1)\\ \left(\frac{1}{4}\pm\frac{v_i}{2},\frac{1}{2}\right);\ldots;\left(\frac{1}{4}\pm\frac{v_r}{2},\frac{1}{2}\right);(0,1)\;\;;(0,1)\end{matrix}\right] \tag{4}$$

where

$$\mathrm{Re}(s)>0, z_i>0, \mathrm{Re}(\eta+l_1\sigma+l_2\rho\pm v_i+\tfrac{1}{2})>0,\ i=1,\ldots,r;$$

$$0<l_j<1,\ |\arg\lambda_j|<(1-l_j)\frac{\pi}{2},\ j=1,2 \text{ or } l_1=l_2=1 \text{ and } \lambda_1>0, \lambda_2>0.$$

The inverse Laplace transforms of the product of numerous other polynomials (which are special cases of S_n^m and $S_{n'}^{m'}$) and special functions involving one or more variables (which are particular cases of the multivariable H-function) can also be obtained from our main result.

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