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Relative extrema of Legendre functions of the second kind

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Abstract. Let $\mu_{k,n}$ denote the relative maxima of $|Q_n(x)|$, the Legendre function of the second kind, ordered so that $\mu_{k+1,n}$ occurs to the left of $\mu_{k,n}$. They by analogy with a theorem of Szegö for Legendre polynomials, $\mu_{k,n+1} < \mu_{k,n}$, $k = 1,\ldots,n$, $n = 1,2,\ldots$.

Keywords. Inequalities; monotonicity; Legendre functions of the second kind.

1. Introduction

In addition to the Legendre polynomials $P_n(x)$, the differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \tag{1}$$

has a second solution that occurs in some applications. This solution is called $Q_n(x)$ and can be given by many formulas. For x outside $-1 \leqslant x \leqslant 1$, one formula is

$$Q_n(x) = \frac{1}{2}\int_{-1}^{1} \frac{P_n(t)}{x-t}\,\mathrm{d}t. \tag{2}$$

See [3] and [5] for the formulas used below. Formula (2) can be rewritten as

$$\begin{aligned} Q_n(x) &= \frac{1}{2}\int_{-1}^{1} \frac{P_n(t)-P_n(x)}{x-t}\,\mathrm{d}t + \frac{P_n(x)}{2}\int_{-1}^{1}\frac{\mathrm{d}t}{x-t} \\ &= P_n(x)Q_0(x) - W_{n-1}(x) \end{aligned} \tag{3}$$

where $W_{n-1}(x)$ is a polynomial of degree $n-1$ given by

$$W_{n-1}(x) = \frac{1}{2}\int_{-1}^{1} \frac{P_n(x)-P_n(t)}{x-t}\,\mathrm{d}t. \tag{4}$$

For complex z off $-1 \leqslant z \leqslant 1$,

$$Q_0(z) = \tfrac{1}{2}\log\frac{z+1}{z-1} \tag{5}$$

with the argument determined by $Q_0(z) > 0$ when $z > 1$.

For $-1 < x < 1$, $Q_n(x)$ is defined as the average of $Q(z)$ when z approaches x from the upper and lower half plane:

$$Q_n(x) = \lim_{\varepsilon \to 0^+} \frac{Q_n(x+i\varepsilon) + Q_n(x-i\varepsilon)}{2}. \tag{6}$$

This gives

$$Q_n(x) = P_n(x)Q_0(x) - W_{n-1}(x), \quad -1 < x < 1 \tag{7}$$

with

$$Q_0(x) = \tfrac{1}{2}\log\frac{1+x}{1-x}. \tag{8}$$

From (2) it is easy to find the three term recurrence relation satisfied by $Q_n(x)$. Recall that $P_n(x)$ satisfies

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x), \quad n = 0, 1, \ldots. \tag{9}$$

Then

$$\begin{aligned}
&(2n+1)xQ_n(x) - (n+1)Q_{n+1}(x) - nQ_{n-1}(x) \\
&= \frac{(2n+1)}{2}\int_{-1}^{1} \frac{(x-t+t)P_n(t)}{x-t}\,dt - \frac{1}{2}\int_{-1}^{1} \frac{(n+1)P_{n+1}(t) + nP_{n-1}(t)}{x-t}\,dt \\
&= \frac{(2n+1)}{2}\int_{-1}^{1} P_n(t)\,dt + \frac{1}{2}\int_{-1}^{1} \frac{(2n+1)tP_n(t) - (n+1)P_{n+1}(t) - nP_{n-1}(t)}{x-t}\,dt \\
&= 0 \text{ when } n \geqslant 1, \\
&= 1 \text{ when } n = 0,
\end{aligned}$$

so

$$(2n+1)xQ_n(x) = (n+1)Q_{n+1}(x) + nQ_{n-1}(x), \quad n = 1, 2, \ldots. \tag{10}$$

Since $Q_n(x)$ as a function of x satisfies the same differential equation as $P_n(x)$, and as a function of n the same difference equation, one can ask if $Q_n(x)$ also satisfies the mixed differential-difference relations $P_n(x)$ satisfies. Here are two such

$$(1+x)[P'_{n+1}(x) - P'_n(x)] = (n+1)[P_{n+1}(x) + P_n(x)] \tag{11}$$

$$(1-x)[P'_{n+1}(x) + P'_n(x)] = (n+1)[P_n(x) - P_{n+1}(x)]. \tag{12}$$

The corresponding identities for $Q_n(x)$ are

$$(1+x)[Q'_{n+1}(x) - Q'_n(x)] = (n+1)[Q_n(x) + Q_{n+1}(x)] \tag{13}$$

$$(1-x)[Q'_{n+1}(x) + Q'_n(x)] = (n+1)[Q_n(x) - Q_{n+1}(x)]. \tag{14}$$

Since

$$Q_n(-x) = (-1)^{n+1}Q_n(x) \tag{15}$$

it will be sufficient to prove either (13) or (14). The symmetry identity (15) is easy to check directly when $n = 0$, and it then follows from

$$Q_1(x) = xQ_0(x) - 1$$

and (10). To prove (13), write

$$(1+x)[Q'_{n+1}(x) - Q'_n(x)] = \frac{(1+x)}{2}\int_{-1}^{1} \frac{-P_{n+1}(t) + P_n(t)}{(x-t)^2}\,dt$$

$$= \frac{(1+x)}{2} \int_{-1}^{1} [-P_{n+1}(t) + P_n(t)] \frac{d}{dt}\left(\frac{1}{x-t}\right) dt$$

$$= \frac{(1+x)}{2} \left[\frac{P_n(t) - P_{n+1}(t)}{x-t} \right]_{-1}^{1} + \frac{(1+x)}{2} \int_{-1}^{1} \frac{P'_{n+1}(t) - P'_n(t)}{x-t} dt$$

$$= (-1)^{n+1} + \frac{1}{2} \int_{-1}^{1} (x-t) \frac{[P'_{n+1}(t) - P'_n(t)]}{x-t} dt$$

$$+ \frac{1}{2} \int_{-1}^{1} \frac{(1+t)[P'_{n+1}(t) - P'_n(t)]}{x-t} dt$$

$$= (-1)^{n+1} + \frac{P_{n+1}(t) - P_n(t)}{2} \bigg|_{-1}^{1} + \frac{(n+1)}{2} \int_{-1}^{1} \frac{[P_n(t) + P_{n+1}(t)]}{x-t} dt$$

$$= (n+1)[Q_n(x) + Q_{n+1}(x)].$$

This proves (13) for x off $-1 \leqslant x \leqslant 1$, and for $-1 < x < 1$ it then follows by averaging via (6).

2. Zeros and maxima of $Q_n(x)$

Many properties of the Legendre polynomial continue to hold for the Legendre function $Q_n(x)$, but some are changed slightly. While the differential equation (1), and the differential difference relations (13) and (14) are exactly the same, the parity is opposite so the symmetry (15) is changed from

$$P_n(-x) = (-1)^n P_n(x) \tag{16}$$

and the recurrence relation (10) only holds for $n = 1, 2, \ldots$, instead of for $n = 0, 1, \ldots$. Another fact that changes slightly is the number of zeros in $-1 < x < 1$. $P_n(x)$ has n simple zeros, and the zeros of $P_n(x)$ and $P_{n+1}(x)$ separate each other. Clearly $Q_0(x)$ vanishes to first order when $x = 0$, and since $Q_1(x)$ is convex and vanishes at least once between 0 and each of ± 1, it has two simple zeros in $-1 < x < 1$. We will now show by induction that $Q_n(x)$ has $n + 1$ zeros in $-1 < x < 1$. One standard way of doing this is to use what is called the Christoffel–Darboux formula in orthogonal polynomials. In the present setting, consider

$$D_n(x, y) = \sum_{k=.1}^{n} (2k+1) Q_k(x) Q_k(y). \tag{17}$$

Then

$$(x-y) D_n(x, y) = Q_0(x) Q_1(y) - Q_1(x) Q_0(y)$$
$$+ (n+1)[Q_{n+1}(x) Q_n(y) - Q_n(x) Q_{n+1}(y)]$$
$$= -(x-y) Q_0(x) Q_0(y) + [Q_0(y) - Q_0(x)]$$
$$+ (n+1)[Q_{n+1}(x) Q_n(y) - Q_n(x) Q_{n+1}(y)].$$

This gives

$$\sum_{k=0}^{n} (2k+1) Q_k(x) Q_k(y) = \frac{Q_0(y) - Q_0(x)}{x-y}$$
$$+ (n+1) \left[\frac{Q_{n+1}(x) Q_n(y) - Q_n(x) Q_{n+1}(y)}{x-y} \right].$$

Next, letting $y \to x$ gives

$$\sum_{k=0}^{n}(2k+1)[Q_k(x)]^2 = -Q_0'(x)+(n+1)[Q_{n+1}'(x)Q_n(x)-Q_n'(x)Q_{n+1}(x)]. \quad (19)$$

But

$$Q_0'(x) = \frac{1}{2(1+x)} + \frac{1}{2(1-x)} = \frac{1}{1-x^2}$$

so

$$\frac{1}{1-x^2} + \sum_{k=0}^{n}(2k+1)[Q_k(x)]^2 = (n+1)[Q_{n+1}'(x)Q_n(x) - Q_n'(x)Q_{n+1}(x)]. \quad (20)$$

The left hand side is positive for $-1<x<1$, so the standard argument now works to show that between each pair of adjacent zeros of $Q_n(x)$ there is a zero of $Q_{n+1}(x)$, and between each pair of adjacent zeros of $Q_{n+1}(x)$ there is a zero of $Q_n(x)$. The arguments are the same, so only one will be given. Let $x_{k,n}$ denote the zeros of $Q_n(x)$ on $(-1,1)$ with $x_{k+1,n}<x_{k,n}$ being adjacent zeros. The zeros are simple because $Q_n(x)$ satisfies (1). Let $R_n(x)$ denote the left hand side of (20). Since $R_n(x)>0$, $-1<x<1$,

$$-R_n(x_{k,n}) = (n+1)Q_n'(x_{k,n})Q_{n+1}(x_{k,n}) < 0$$

and

$$-R_n(x_{k+1,n}) = (n+1)Q_n'(x_{k+1,n})Q_{n+1}(x_{k+1,n}) < 0.$$

Therefore

$$Q_n'(x_{k,n})Q_n'(x_{k+1,n})Q_{n+1}(x_{k,n})Q_{n+1}(x_{k+1,n}) > 0.$$

But $x_{k,n}$ and $x_{k+1,n}$ are adjacent simple zeros of $Q_n(x)$, so

$$Q_n'(x_{k,n})Q_n'(x_{k+1,n}) < 0,$$

which implies that

$$Q_{n+1}(x_{k,n})Q_{n+1}(x_{k+1,n}) < 0.$$

Thus $Q_{n+1}(x)$ has a zero between $x_{k+1,n}$ and $x_{k,n}$. This implies that $Q_{n+1}(x)$ has at least n zeros on $-1<x<1$. To show that there are at least $n+2$; and so exactly $n+2$, for more would imply $Q_n(x)$ has more than $n+1$ zeros on $-1<x<1$; it is sufficient to show that $Q_{n+1}(x)$ has a zero between $x_{1,n}$ and 1. If not, then $Q_{n+1}(x_{1,n})>0$. The graph of $Q_n(x)$, $n=0,1,\ldots,5$ in [3, figure 59, p. 110] suggests that not only is $Q_{n+1}(x_{1,n})<0$, but that $Q_{n+1}(x)<Q_n(x)$ for $x_{1,n}<x<1$. This can be proven by induction from (10). For it can be rewritten as

$$(n+1)[Q_n(x) - Q_{n+1}(x)] = n[Q_{n-1}(x) - Q_n(x)] + (1-x)(2n+1)Q_n(x).$$

Then for $x_{1,n-1}<x_{1,n}<x<1$, we have

$$Q_{n-1}(x) - Q_n(x) > 0$$

by the induction assumption, and $Q_n(x)>0$ since $Q_n(x)\to+\infty$ as $x\to 1^-$ from (3). This implies that $Q_n(x)>Q_{n+1}(x)$, $x_{1,n}<x<1$, which is what we wanted to prove. Thus $x_{1,n}<x_{1,n+1}$, and so

$$x_{k+1,n+1} < x_{k,n} < x_{k,n+1}, \quad k=1,2,\ldots,n+1 \quad (21)$$

and $Q_n(x)$ has $n+1$ simple zeros on $-1<x<1$.

Let $-1 < y_{k,n} < 1$ with $y_{k+1,n} < y_{k,n}$ denote the points where

$$Q_n'(y_{k,n}) = 0. \tag{22}$$

There is at least one such point in each interval $x_{k+1,n} < x < x_{k,n}$, and in fact there is exactly one, and none in $x_{1,n} < x < 1$. For, from the differential equation (1)

$$(1 - x^2)y'(x) = (1 - x_{k+1,n}^2)y'(x_{k+1,n}) - n(n+1)\int_{x_{k+1,n}}^{x} y(t)\,dt \tag{23}$$

so $(1 - x^2)y'(x)$ is a monotone function for $x_{k+1,n} < x < x_{k,n}$, and so can vanish at most once. This takes care of the case $x_{k+1,n} < x < x_{k,n}$. For $x_{1,n} < x < 1$, $Q_n'(x)$ has no zeros since $(1 - x^2)Q_n'(x)$ is monotone in x from (23), $Q_n'(x)$ is positive at $x = x_{1,n}$, and

$$\lim_{x \to 1} (1 - x^2)Q_n'(x) = 1. \tag{24}$$

To show that

$$y_{k+1,n+1} < y_{k,n} < y_{k,n+1} \tag{25}$$

one argues in the same way as in the proof of (21). This argument is given for general orthogonal polynomials in [1]. Formula (20) is all that is needed, so this argument will not be included here.

With all of this as background we can now prove Theorem 1. If

$$\mu_{k,n} = |Q_n(y_{k,n})| \tag{26}$$

denotes the absolute values of the relative extrema of $Q_n(x)$ on $-1 < x < 1$, then

$$\mu_{k+1,n} < \mu_{k,n}, \quad k = 1, 2, \ldots, \left[\frac{n}{2}\right] \tag{27}$$

and

$$\mu_{k,n+1} < \mu_{k,n}, \quad k = 1, \ldots, n, \quad n = 1, 2, \ldots. \tag{28}$$

Proof. The proof of (27) is an exact copy of the same proof for Legendre polynomials, [5, Chapter VII]. Set

$$f(x) = [Q_n(x)]^2 + \frac{(1 - x^2)[Q_n'(x)]^2}{n(n+1)}.$$

Then

$$f'(x) = \frac{2Q_n'(x)}{n(n+1)}[(1 - x^2)Q_n''(x) - xQ_n'(x) + n(n+1)Q_n(x)]$$

$$= \frac{2x[Q_n'(x)]^2}{n(n+1)}.$$

Thus $f(x)$ is increasing for $0 < x < 1$, and since

$$f(y_{k,n}) = \mu_{k,n}^2$$

this implies (27).

The proof of (28) is also a copy of one of the proofs of the same inequalities for Legendre polynomials. See Szegö [6]. Multiply (13) and (14) together to obtain

$$(1-x^2)[[Q'_{n+1}(x)]^2-[Q'_n(x)]^2]=(n+1)^2[[Q_n(x)]^2-[Q_{n+1}(x)]^2]. \quad (29)$$

Set

$$\begin{aligned} f(x) &= (1-x^2)[Q'_n(x)]^2+(n+1)^2[Q_n(x)]^2 \\ &= (1-x^2)[Q'_{n+1}(x)]^2+(n+1)^2[Q_{n+1}(x)]^2. \end{aligned} \quad (30)$$

Then

$$\begin{aligned} f'(x) &= 2Q'_n(x)[(1-x^2)Q''_n(x)-xQ'_n(x)+(n+1)^2Q_n(x)] \\ &= 2Q'_n(x)[xQ'_n(x)+(n+1)Q_n(x)] \end{aligned} \quad (31)$$

from the first equality in (30), and from the second

$$\begin{aligned} f'(x) &= 2Q'_{n+1}(x)[(1-x^2)Q''_{n+1}(x)-xQ'_{n+1}(x)+(n+1)^2Q_{n+1}(x)] \\ &= 2Q'_{n+1}(x)[xQ'_{n+1}(x)-(n+1)Q_{n+1}(x)]. \end{aligned} \quad (32)$$

These two strongly suggest that

$$xQ'_n(x)+(n+1)Q_n(x)=c_nQ'_{n+1}(x) \quad (33)$$

and

$$xQ'_{n+1}(x)-(n+1)Q_{n+1}(x)=c_nQ'_n(x) \quad (34)$$

for a constant c_n. To see this add (13) and (14) to obtain

$$2Q'_{n+1}(x)-2xQ'_n(x)=2(n+1)Q_n(x),$$

which is (33) with $c_n=1$. Subtracting gives (34) with $c_n=1$. Thus

$$f'(x)=2Q'_n(x)Q'_{n+1}(x).$$

Since

$$f(y_{k,n})=\mu_{k,n}^2$$

and

$$f(y_{k,n+1})=\mu_{k,n+1}^2,$$

to prove (28) it is sufficient to show that $f'(x)<0$, $y_{k,n}<x<y_{k,n+1}$. But this is clear from the facts proven above about the zeros of $Q'_n(x)$.

Here is one consequence of Theorem 1.

COROLLARY 2.

If $0<x<1$, *then* $Q_n(x)\geqslant -1$ *with equality only when* $n=1$ *and* $x=0$.

Proof. For each fixed n and for $0\leqslant x<1$,

$$Q_n(x)>-\mu_{1,n}$$

by (27), and $-\mu_{1,n}>-\mu_{1,1}$ if $n>1$ by (28).

When looking at either of the proofs of the monotonicity of the extrema of $P_n(x)$ as a function of n (given in [6] and [1]), it seems that the normalization of Legendre polynomials by

$$P_n(1) = 1 \tag{35}$$

plays a central role. Since $Q_n(x) \to \infty$ as $x \to 1^-$, it is not clear that the normalization used above plays the same role for $Q_n(x)$ that (35) did for Legendre polynomials, but it does. There are two different aspects of this. One would like to have

$$\lim_{x\to 1^-} Q_n(x) - Q_{n+1}(x) = 0.$$

It is not true, but it almost is. For

$$Q_n(x) - Q_{n+1}(x) = Q_0(x)[P_n(x) - P_{n+1}(x)] + W_n(x) - W_{n-1}(x),$$

and

$$\lim_{x\to 1^-} Q_0(x)[P_n(x) - P_{n+1}(x)] = \lim_{x\to 1} \frac{(1-x)}{2}\left[\log\frac{1+x}{1-x}\right]\frac{P_n(x) - P_{n+1}(x)}{1-x}$$
$$= 0[P'_{n+1}(1) - P'_n(1)] = 0.$$

However

$$W_n(1) - W_{n-1}(1) = \frac{1}{2}\int_{-1}^{1} \frac{1 - P_{n+1}(t)}{1-t}\,dt - \frac{1}{2}\int_{-1}^{1} \frac{1 - P_n(t)}{1-t}\,dt$$
$$= \sum_{k=1}^{n+1} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} = \frac{1}{n+1},$$

so

$$\lim_{x\to 1^-} Q_n(x) - Q_{n-1}(x) = \frac{1}{n+1}. \tag{36}$$

The identity

$$\int_{-1}^{1} \frac{1 - P_n(t)}{1-t}\,dt = 2\left(\sum_{k=1}^{n} \frac{1}{k}\right) \tag{37}$$

can be proven in many ways. One is to prove it by induction using (9). My favorite way is given in [2]. One first evaluates

$$\int_{-1}^{1} P_n(t)(1-t)^{-\gamma}\,dt$$

for $\gamma < 1$, which can be done by integration by parts using the Rodrigues formula, then gets

$$\int_{-1}^{1} \frac{1 - P_n(t)}{(1-t)^{\gamma}}\,dt$$

and lets $\gamma \to 1^-$.

There is a better way of seeing that the normalization of $Q_n(x)$ is the right one (at least up to a constant independent of n). This comes from the asymptotic formulas for $P_n(\cos\theta)$ and $Q_n(\cos\theta)$ for θ in $0<\theta\leqslant\pi-\varepsilon$, $\varepsilon>0$. Hilb's formula is

$$P_n(\cos\theta)=\left(\frac{\theta}{\sin\theta}\right)^{1/2}J_0((n+\tfrac{1}{2})\theta)+O(n^{-3/2}) \tag{38}$$

uniformly for $0\leqslant\theta<\pi-\varepsilon$, $t>0$. See Szegö [5, Theorem 8.21.6].

For $Q_n(\cos\theta)$ the corresponding formula is

$$Q_n(\cos\theta)=-\frac{\pi}{2}\left(\frac{\theta}{\sin\theta}\right)^{1/2}Y_0((n+\tfrac{1}{2})\theta)+O(n^{-3/2})$$

uniformly for $0<\theta\leqslant\pi-\varepsilon$, $\varepsilon>0$. See Olver [4, (13.08) and Exercise 13.3].

Thus

$$\lim_{n\to\infty}\sigma_{k,n}=-\frac{\pi}{2}Y_0(y_{1,k}) \tag{39}$$

and

$$\lim_{n\to\infty}v_{k,n}=J_0(j_{1,k}). \tag{40}$$

$Y_0(x)$ is the usual second solution to the Bessel differential equation

$$x^2y''+xy'+x^2y=0,$$

$J_0(x)$ is the usual first solution, $y_{1,k}$ and $j_{1,k}$ are the zeros of their derivatives and $v_{k,n}$ is $P_n(t_{k,n})$ with $P_n'(t_{k,n})=0$, $t_{k+1,n}<t_{k,n}$, and $\sigma_{k,n}$ is $Q_n(y_{k,n})$, i.e. the same as $\mu_{k,n}$ when $\sigma_{k,n}>0$, but $-\mu_{k,n}$ where $\sigma_{k,n}<0$. Thus up to the factor $-\pi/2$ these are analogous. From this point of view it would be preferable to renormalize $Q_n(x)$, but it is probably too late to do so.

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