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Fractional integral formulae involving a general class of polynomials and the multivariable *H*-function

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Abstract. We obtain two fractional integral formulae involving a general class of polynomials and the multivariable H-function. On account of the most general nature of the polynomials and the multivarible H-function involved herein, our findings provide interesting unifications and extensions of a number of (known and new) results. We have mentioned here only two such results.

Keywords. Riemann–Liouville operator; Erdélyi-Kober operator; multivariable H-function; general class of polynomials; fractional integral formulae.

1. Introduction and definitions

The familiar fractional integral operator (FIO) is defined and represented in the present paper as

$$ {}_cI_x^\nu\{f(x)\} = \frac{1}{\Gamma(\nu)} \int_c^x (x-t)^{\nu-1} f(t)\,\mathrm{d}t, \quad \mathrm{Re}(\nu) > 0 \tag{1.1} $$

The special case of the above operator (when $c = 0$) is well known in the literature as Riemann–Liouville fractional integral operator and is written as $I_x^\nu\{f(x)\}$.

Also the FIO investigated by Erdélyi-Kober is defined and represented as [2]

$$ I_x^{\eta,\nu}\{f(x)\} = \frac{x^{-\eta-\nu+1}}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} t^{\eta-1} f(t)\,\mathrm{d}t, \quad \mathrm{Re}(\nu) > 0, \eta > 0 \tag{1.2} $$

which is obviously a generalization of the Riemann–Liouville FIO.

The multivariable H-function has been defined by Srivastava and Panda [8]. We shall use the following contracted form [7, p. 251, eq. (C.1)]:

$$ H[z_1,\ldots,z_r] = H^{0,N:M',N';\cdots;M^{(r)},N^{(r)}}_{P,Q:P',Q';\cdots;P^{(r)},Q^{(r)}} $$
$$ \times\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\left|\begin{matrix} (a_j;\alpha'_j,\ldots,\alpha_j^{(r)})_{1,P}:(c'_j,\gamma'_j)_{1,P'};\cdots;(c_j^{(r)},\gamma_j^{(r)})_{1,P^{(r)}} \\ (b_j;\beta'_j,\ldots,\beta_j^{(r)})_{1,Q}:(d'_j,\delta'_j)_{1,Q'};\cdots;(d_j^{(r)},\delta_j^{(r)})_{1,Q^{(r)}} \end{matrix}\right.\right] \tag{1.3} $$

to denote the H-function of r complex variables $z_1,\ldots,z_r$. All the Greek letters are assumed to be positive real numbers for standardization purposes; the definition of the multivariable H-function will, however, be meaningful even if some of these quantities are zero. Various special cases and details of this function can be found in the paper referred to above.

Srivastava has also introduced the general class of polynomials (see [3] and [4]):

$$S_n^m[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} x^k, \quad n = 0, 1, 2, \ldots, \tag{1.4}$$

where m is an arbitrary positive integer and the coefficients $A_{n,k}$ $(n, k \geqslant 0)$ are arbitrary constants, real or complex. By suitably specializing the coefficients $A_{n,k}$, the above general class of polynomials can be reduced to a large spectrum of polynomials as cited in the papers referred to above ([3] and [4]).

2. Results required

The following results will be required in the sequel:

(I)
$$ {}_cI_x^{\nu}\{x^{\lambda}\} = \sum_{s=0}^{\infty} \frac{(-1)^s \Gamma(\lambda+1)(x-c)^{s+\nu} x^{\lambda-s}}{\Gamma(\nu)\Gamma(\lambda-s+1)\cdot(s+\nu)\cdot s!}, \quad \mathrm{Re}(\lambda) > -1 \tag{2.1}$$

(II)
$$\sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(\nu)\cdot(s+\nu)\cdot s!} H_{P,Q+1:*}^{0,N\ :*} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} (a_j; \alpha'_j, \ldots, \alpha_j^{(r)})_{1,P} & :* \\ (s-k; u_1, \ldots, u_r), (b_j; \beta'_j, \ldots, \beta_j^{(r)})_{1,Q} & :* \end{matrix} \right]$$

$$= H_{P,Q+1:*}^{0,N\ :*} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} (a_j; \alpha'_j, \ldots, \alpha_j^{(r)})_{1,P} & :* \\ (-\nu-k; u_1, \ldots, u_r), (b_j; \beta'_j, \ldots, \beta_j^{(r)})_{1,Q} & :* \end{matrix} \right] \tag{2.2}$$

the asterisk (*) in (2.2) indicates that the parameters at these places are the same as the parameters of the multivariable H-function in (1.3).

(III)
$$I_x^{\eta,\nu}\{x^{\lambda}\} = \frac{\Gamma(\eta+\lambda)}{\Gamma(\eta+\lambda+\nu)} x^{\lambda}, \quad \mathrm{Re}(\lambda) > -\eta. \tag{2.3}$$

The results given above are all consequences of the binomial series

$$(1-t)^{\lambda} = \sum_{s=0}^{\infty} \frac{(-\lambda)_s t^s}{s!} (|t| < 1)$$

and would follow from it on suitable integrations.

3. The fractional integral formulae

Recently Srivastava and Garg [5, p. 688] have obtained two integrals involving a general class of polynomials and the multivariable H-function. Here we obtain the following fractional integral formulae involving them:

$$ {}_cI_x^{\nu}\{x^{\rho}(x+b)^{\sigma}S_n^m[ax^u(x+b)^v]H[z_1x^{u_1}(x+b)^{v_1},\ldots,z_rx^{u_r}(x+b)^{v_r}]\} $$

$$ = b^{\sigma}x^{\rho}\sum_{s,l=0}^{\infty}\sum_{k=0}^{[n/m]}\left[\frac{(-n)_{mk}A_{n,k}a^kb^{vk-l}(-1)^s(x-c)^{s+\nu}x^{uk+l-s}}{\Gamma(\nu)\cdot(s+\nu)\cdot s!\,l!\,k!}\right]\cdot H^{0,N+2\;\;:*}_{P+2,Q+2:*} $$

$$ \times\left[\begin{matrix} z_1b^{v_1}x^{u_1}\\ \vdots\\ z_rb^{v_r}x^{u_r}\end{matrix}\middle|\begin{matrix}(-\sigma-vk;v_1,\ldots,v_r),(-\rho-uk-l;u_1,\ldots,u_r),\;\;(a_j;\alpha_j',\ldots,\alpha_j^{(r)})_{1,P}:*\\ (l-\sigma-vk;v_1,\ldots,v_r),(s-\rho-uk-l;u_1,\ldots,u_r),\;(b_j;\beta_j',\ldots,\beta_j^{(r)})_{1,Q}:*\end{matrix}\right] \quad (3.1) $$

provided that

(i) $\mathrm{Re}(\nu)>0$; $\min\{u_i,v_i,u,v\}>0,\quad i=1,\ldots,r;\;\left|\arg\left(\frac{x}{b}\right)\right|<\pi,$

(ii) $\mathrm{Re}(\rho)+\sum_{i=1}^{r}u_i\min_{1\leqslant j\leqslant M^{(i)}}\left\{\mathrm{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right\}>-1,$

(iii) $\Omega_i>0$, $|\arg z_i|<\frac{1}{2}\Omega_i\pi,\quad\forall i\in\{1,2,\ldots,r\},$
where

$$ \Omega_i=-\sum_{j=N+1}^{P}\alpha_j^{(i)}-\sum_{j=1}^{Q}\beta_j^{(i)}+\sum_{j=1}^{N^{(i)}}\gamma_j^{(i)}-\sum_{j=N^{(i)}+1}^{P^{(i)}}\gamma_j^{(i)}+\sum_{j=1}^{M^{(i)}}\delta_j^{(i)}-\sum_{j=M^{(i)}+1}^{Q^{(i)}}\delta_j^{(i)} $$

(iv) the series occurring on right-hand side of (3.1) is absolutely convergent.

Proof. To establish (3.1), we first express the general class of polynomials occurring on left-hand side of it in the series form given by (1.4) and replace multivariable H-function by its Mellin-Barnes contour integral, collect the powers of x and $(x+b)$ and apply the binomial expansion

$$ (x+b)^{\sigma}=b^{\sigma}\sum_{l=0}^{\infty}\binom{\sigma}{l}\left(\frac{x}{b}\right)^l,\quad\left|\frac{x}{b}\right|<1. \quad (3.2) $$

Further, making use of the result (2.1) and interpreting the resulting Mellin-Barnes contour integral as the H-function of r variables, we shall arrive at (3.1).

Following (3.1) and using the result (2.3) in place of (2.1), we easily arrive at the following fractional integral formula for Erdélyi-Kober operator defined by (1.2):

$$ I_x^{\eta,\nu}\{x^{\rho}(x+b)^{\sigma}S_n^m[ax^u(x+b)^v]H[z_1x^{u_1}(x+b)^{v_1},\ldots,z_rx^{u_r}(x+b)^{v_r}]\} $$

$$ = b^{\sigma}x^{\rho}\sum_{l=0}^{\infty}\sum_{k=0}^{[n/m]}\left[\frac{(-n)_{mk}A_{n,k}a^kb^{vk-l}x^{uk+l}}{l!\,k!}\right]H^{0,N+2\;\;:*}_{P+2,Q+2:*} $$

$$ \left[\begin{matrix} z_1b^{v_1}x^{u_1}\\ \vdots\\ z_rb^{v_r}x^{u_r}\end{matrix}\middle|\begin{matrix}(-\sigma-vk;v_1,\ldots,v_r),(1-\eta-\rho-uk-l;u_1,\ldots,u_r),(a_j;\alpha_j',\ldots,\alpha_j^{(r)})_{1,P}\;\;:*\\ (l-\sigma-vk;v_1,\ldots,v_r),(-\nu+1-\eta-\rho-uk-l;u_1,\ldots,u_r),(b_j;\beta_j',\ldots,\beta_j^{(r)})_{1,Q}:*\end{matrix}\right] \quad (3.3) $$

where $\eta>0$ and

$$ \mathrm{Re}(\rho)+\sum_{i=1}^{r}u_i\min_{1\leqslant j\leqslant M^{(i)}}\left\{\mathrm{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right\}>-\eta. $$

The remaining conditions of validity being the same as given in (3.1).

4. Special cases and applications

If we take $c = 0$ in our main integral formula (3.1) and use the result (2.2), we arrive at the following fractional integral formula:

$$I_x^{\nu}\{x^{\rho}(x+b)^{\sigma}S_n^m[ax^u(x+b)^v]H[z_1x^{u_1}(x+b)^{v_1},\ldots,z_rx^{u_r}(x+b)^{v_r}]\}$$

$$= b^{\sigma}x^{\rho+\nu}\sum_{l=0}^{\infty}\sum_{k=0}^{[n/m]}\left[\frac{(-n)_{mk}A_{n,k}a^kb^{vk-l}x^{uk+l}}{l!k!}\right]H^{0,N+2\ \ :*}_{P+2,Q+2:*}$$

$$\times\left[\begin{matrix}z_1b^{v_1}x^{u_1}\\ \vdots\\ z_rb^{v_r}x^{u_r}\end{matrix}\left|\begin{matrix}(-\sigma-vk;v_1,\ldots,v_r),(-\rho-uk-l;u_1,\ldots,u_r),(a_j;\alpha_j',\ldots,\alpha_j^{(r)})_{1,P}\ :*\\ (l-\sigma-vk;v_1,\ldots,v_r),(-\nu-\rho-uk-l;u_1,\ldots,u_r),(b_j;\beta_j',\ldots,\beta_j^{(r)})_{1,Q}:*\end{matrix}\right.\right] \quad (4.1)$$

The conditions of validity of (4.1) are directly obtainable from that of (3.1).

Again, if we take $n = 0$ (the polynomial $S_0^m[x]$ will reduce to $A_{0,0}$) in (4.1), we get a result which is in essence the same as obtained earlier by Srivastava and Goyal [6, p. 644].

The importance of our main integral formulae lies in their manifold generality. Firstly, in view of the generality of the polynomials $S_n^m[x]$, on specializing the coefficients $A_{n,k}$ and making a free use of the special cases of $S_n^m[x]$ listed in a paper by Srivastava [4], our results can be reduced to a large number of integral formulae involving simpler polynomials.

Secondly, by specializing the various parameters and variables in the multivariable H-function, we can obtain from our results, several integral formulae involving a remarkably wide variety of useful functions (or products of several such functions), which are expressible in terms of E, F, G and H-functions of one and several variables.

We shall give here only one such formula for the sake of illustration. Thus, if we take $a = u = 1$, $v = 0$, $m = 1$ and $A_{n,k} = [\Gamma(1+\alpha+\lambda n)]/[n!\Gamma(1+\alpha+\lambda k)]$ in (4.1), the polynomial $S_n^1[x]$ reduces to $z_n^{\alpha}(x^{1/\lambda};\lambda)$ (known as Konhauser biorthogonal polynomials [1, p. 304] and we easily get

$$I_x^{\nu}\{x^{\rho}(x+b)^{\sigma}z_n^{\alpha}(x^{1/\lambda};\lambda)H[z_1x^{u_1}(x+b)^{v_1},\ldots,z_rx^{u_r}(x+b)^{v_r}]\}$$

$$= b^{\sigma}x^{\rho+\nu}\sum_{l=0}^{\infty}\sum_{k=0}^{n}\frac{(-n)_k\Gamma(1+\alpha+\lambda n)b^{-l}x^{k+l}}{l!k!n!\Gamma(1+\alpha+\lambda k)}H^{0,N+2\ \ :*}_{P+2,Q+2:*}$$

$$\times\left[\begin{matrix}z_1b^{v_1}x^{u_1}\\ \vdots\\ z_rb^{v_r}x^{u_r}\end{matrix}\left|\begin{matrix}(-\sigma;v_1,\ldots,v_r),(-\rho-k-l;u_1,\ldots,u_r),(a_j;\alpha_j',\ldots,\alpha_j^{(r)})_{1,P}\ :*\\ (l-\sigma;v_1,\ldots,v_r),(-\nu-\rho-k-l;u_1,\ldots,u_r),(b_j;\beta_j',\ldots,\beta_j^{(r)})_{1,Q}:*\end{matrix}\right.\right] \quad (4.2)$$

The conditions of validity of (4.2) are directly obtainable from those of (3.1). Further, on taking $\lambda = 1$ in (4.2), we get the corresponding fractional integral formula involving Laguerre polynomials $L_n^{(\alpha)}(x)$.

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