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A generalization of the Riemann zeta-function

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Abstract. A generalization of the Riemann zeta-function which has the form

$$\zeta_a(s)=\prod_p \frac{1}{1-p^{-s}+(p+a)^{-s}}$$

is considered. Analytical properties with respect to s and asymptotic behaviour when $a\to\infty$ are investigated. The corresponding L-function is also discussed. This consideration has an application in the theory of p-adic strings.

Keywords. Riemann zeta-function.

1. Introduction

The Riemann zeta-function in terms of the Euler product has the form

$$\zeta(s)=\prod_p \frac{1}{1-p^{-s}} \tag{1}$$

where p ranges over all primes ($p=2,3,5,\ldots$) and $s=\sigma+it$, $\sigma>1$. The importance of $\zeta(s)$ in the number theory is well-known, see for example [1]. In this note we consider a generalization of $\zeta(s)$ defined by

$$\zeta_a(s)=\prod_p \frac{1}{1-p^{-s}+(p+a)^{-s}} \tag{2}$$

where $a>0$. Usefulness of such a generalization in the context of the p-adic string theory was noted in Areféva *et al* 1988 (this reference also contains references to the corresponding physical papers).

It will be proved here that the product (2) is absolutely convergent for $\sigma>0$ and it defines an analytical function which does not vanish for $\sigma\geqslant\sigma_0>0$ and large a, depending on σ_0. So the function $\zeta_a(s)$ can be considered as a holomorphic approximation to the Riemann zeta-function. One can hope to get an information on the zeros of the Riemann zeta-function investigating the behaviour of $\zeta_a(s)$ when $a\to\infty$ in the critical strip $0<\sigma<1$ (compare density theorem in [1]). It will be shown that the asymptotic behaviour $\zeta_a(1)$ when $a\to\infty$ coincides with the one for $\prod_{p<a}(1-p^{-1})^{-1}$.

We discuss here also some problems whose solutions would be useful for the p-adic string theory. Finally a generalization of the L-function along the same line is suggested.

2. Properties of $\zeta_a(s)$

The first lemma states that for large a the denominator in eq. (2) has no zeros.

Lemma 1. *Let the following conditions for s and a be satisfied.*

Either 1) $\sigma \geqslant 1$, $t \in \mathbb{R}$, $a > 0$,

or 2) $1 > \sigma > \sigma_0$, $t \in \mathbb{R}$ *and* $a > 2(2^{\sigma_0} - 1)^{-1/\sigma_0}$

for a fixed constant σ_0, $0 < \sigma_0 < 1$.

Then

$$\left|\frac{1}{p^s} - \frac{1}{(p+a)^s}\right| < 1.$$

Proof. We have

$$\left|\frac{1}{p^s} - \frac{1}{(p+a)^s}\right| \leqslant \left|\frac{1}{p^s}\right| + \left|\frac{1}{(p+a)^s}\right| = \frac{1}{p^\sigma} + \frac{1}{(p+a)^\sigma} \leqslant \frac{1}{2^\sigma} + \frac{1}{(2+a)^\sigma}.$$

Then for $\sigma \geqslant 1$ one has

$$\frac{1}{2^\sigma} + \frac{1}{(2+a)^\sigma} < 1.$$

If $a > 2(2^{\sigma_0} - 1)^{-1/\sigma_0}$ we have

$$\frac{1}{2^{\sigma_0} - 1} < \left(\frac{a}{2}\right)^{\sigma_0} < \left(\frac{a}{2}\right)^{\sigma_0}\left(1 + \frac{2}{a}\right)^{\sigma_0} = \left(\frac{a}{2} + 1\right)^{\sigma_0} < \left(\frac{a}{2} + 1\right)^{\sigma}$$

and taking into account

$$\frac{1}{2^\sigma - 1} \leqslant \frac{1}{2^{\sigma_0} - 1}$$

one gets

$$\frac{1}{2^\sigma - 1} < \left(\frac{a}{2} + 1\right)^\sigma \quad \text{or} \quad 1 + \frac{1}{\left(1 + \frac{a}{2}\right)^\sigma} < 2^\sigma$$

or finally

$$\frac{1}{2^\sigma} + \frac{1}{(2+a)^\sigma} < 1.$$

Theorem 1. *Let s and a be as in Lemma 1. Then the product (2) is absolutely convergent. This convergence is uniform on compact subsets of the s region and defines an analytic function $\zeta_a(s)$ which has no zeros for $\sigma \geqslant \sigma_0$.*

Proof. We have

$$\left|\frac{1}{p^s} - \frac{1}{(p+a)^s}\right| = \left|s\int_0^a \frac{du}{(p+u)^{s+1}}\right| \leqslant \frac{|as|}{p^{\sigma+1}} \tag{3}$$

when $p \to \infty$.

Let $y_p = y_p(s,a) = p^{-s} - (p+a)^{-s}$. Then the product (2) has the form

$$\prod_p \frac{1}{1 - y_p(s,a)} \tag{4}$$

and from (3) it follows that

$$\sum_p |y_p(s,a)| < \infty. \tag{5}$$

From Lemma 1, we also know that

$$|y_p(s,a)| < 1. \tag{6}$$

Now from (5) and (6) it follows that the product (4) is absolutely convergent.

In fact this convergence is uniform on compact subsets of the s region because one has

$$C_K = \sup_{\substack{p \\ s\in K}} |y_p(a,s)| < 1$$

where K is a compact subset of the region $\sigma \geqslant \sigma_0$. This completes the proof of Theorem.

Next we are going to consider the asymptotic behaviour of $\zeta_a(1)$ as $a \to \infty$.

Theorem 2. *Let $\sigma > 1$. Then we have*

$$\lim_{a\to\infty} \zeta_a(s) = \zeta(s).$$

Proof. We have

$$\log \zeta_a(s) - \log \zeta(s) = \sum_{p,m\geqslant 1} \frac{1}{m} f_a(m,s)$$

where

$$f_a(m,s) = \left(\frac{1}{p^s} - \frac{1}{(p+a)^s}\right)^m - \left(\frac{1}{p^s}\right)^m.$$

Hence

$$|f_a(m,s)| \leqslant \frac{1}{(p+a)^\sigma}\left(\sum_{j=0}^{m-1}\left(\left|\frac{1}{p^s} - \frac{1}{(p+a)^s}\right|^{m-j-1}\frac{1}{p^{j\sigma}}\right)\right)$$

$$\leqslant \frac{1}{(p+a)^\sigma}\sum_{j=0}^{m-1}\left(\frac{2}{p^\sigma}\right)^{m-j-1} p^{-j\sigma}$$

$$\leqslant \frac{2^{m-1}}{(p+a)^\sigma} \sum_{j=0}^{m-1} \left(\frac{1}{p^\sigma}\right)^{m-1}$$
$$= \frac{2^{m-1}}{(p+a)^\sigma} (mp^{-(m-1)\sigma}).$$

This proves that $\log \zeta_a(s) - \log \zeta(s) \to 0$ as $a \to \infty$.

Theorem 3. *Let $a \to \infty$. Then we have*

$$\zeta_a(1) = \prod_p \frac{1}{\left(1 - \frac{1}{p} + \frac{1}{p+a}\right)} = \prod_{p \leqslant a} \frac{1}{1 - \frac{1}{p}} + O(1). \tag{7}$$

Proof. Write

$$\prod_p \frac{1}{\left(1 - \frac{1}{p} + \frac{1}{p+a}\right)} = \prod_{p \leqslant a} \frac{1}{\left(1 - \frac{1}{p} + \frac{1}{p+a}\right)} \prod_{p > a} \frac{1}{\left(1 - \frac{1}{p} + \frac{1}{p+a}\right)}$$
$$= \prod\nolimits_1 (a) \prod\nolimits_2 (a).$$

We will consider $\prod_1(a)$ and $\prod_2(a)$ separately. We have

$$\log \prod\nolimits_2 (a) = \sum_{p > a} \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{a}{p(p+a)}\right)^m$$
$$< \sum_{p > a} \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{a}{p^2}\right)^m < \frac{c_1}{\log a'}$$

because, it is known (by prime number theorem for example) that

$$\sum_{p > a} \frac{1}{p^2} < \frac{c}{a \log a}.$$

So, we have

$$\prod\nolimits_2 (a) < \exp \frac{c_2}{\log a}. \tag{8}$$

Then let us consider the asymptotic behaviour of the function

$$g(a) = \prod_1 (a) \cdot \prod_{p \leqslant a} \left(1 - \frac{1}{p}\right) = \prod_{p \leqslant a} \frac{1}{1 + \frac{p}{(p-1)(p+a)}}.$$

One has

$$|\log g(a)| = \left| \sum_{\substack{p \leqslant a \\ m \geqslant 1}} (-1)^{m-1} \frac{1}{m} \left(\frac{p}{(p-1)(p+a)}\right)^m \right.$$

$$\leqslant \sum_{\substack{p \leqslant a \\ m \geqslant 1}} \left(\frac{p}{(p-1)(p+a)} \right)^m \leqslant \sum_{\substack{p \leqslant a \\ m \geqslant 1}} \cdot \left(\frac{2}{a} \right)^m$$

$$= \sum_{p \leqslant a} \frac{2}{a} \frac{1}{\left(1 - \frac{2}{a}\right)} \leqslant \frac{C_3}{a} \sum_{p \leqslant a} 1 \leqslant \frac{C_4}{\log a}$$

because it is known [1] that

$$\sum_{p \leqslant a} 1 \equiv \pi(a) \sim \frac{a}{\log a}. \tag{9}$$

Therefore one gets

$$g(a) \leqslant \exp \frac{C_4}{\log a}. \tag{10}$$

Now we have

$$\zeta_a(1) = \prod_2 (a) g(a) \cdot \prod_{p \leqslant a} \left(1 - \frac{1}{p} \right)^{-1}$$

and from (8) and (10)

$$\zeta_a(1) = \prod_{p \leqslant a} \frac{1}{1 - \frac{1}{p}} + O(1)$$

since

$$\prod_{p \leqslant a} \frac{1}{1 - \frac{1}{p}} \sim e^{\gamma} \log a$$

where γ is the Euler constant. This proves Theorem 3.

3. Discussion

Theorem 3 relates to the asymptotics of the functions $\zeta_a(s)$ and $f_a(s) = \prod_{p \leqslant a} (1 - p^{-s})^{-1}$ for $s = 1$. In fact there is a relation between the asymptotics of these two functions for any fixed $s \neq 1$, $0 < \sigma \leqslant 1$. We consider s real and $0 < s < 1$ for simplicity. We have

$$\log f_a(s) \sim \sum_{p \leqslant a} \frac{1}{p^s}.$$

Then we use the Abel identity

$$\sum_{p \leqslant a} \frac{1}{p^s} = \frac{\pi(a)}{a^s} + s \int_2^a \pi(t) \frac{\mathrm{d}t}{t^{1+s}} \tag{11}$$

which is true for any $s\in\mathbb{C}$. Here $\pi(a)$ is the number of prime numbers not exceeding a. By (9) we have for $a\to\infty$

$$\int_2^a \pi(t)\frac{\mathrm{d}t}{t^{1+s}}\sim\int_2^a\frac{\mathrm{d}t}{t^s\log t}\sim\frac{a^{1-s}}{(1-s)\log a}$$

and from (11)

$$\sum_{p\leqslant a}\frac{1}{p^s}\sim\frac{1}{1-s}\frac{a^{1-s}}{\log a} \tag{12}$$

Therefore we have the following

Theorem 4. *Let $0<s<1$. Then we have the following asymptotic equality when $a\to\infty$.*

$$\log\prod_{p\leqslant a}(1-p^{-s})^{-1}\sim\frac{1}{1-s}\frac{a^{1-s}}{\log a}. \tag{13}$$

For the function $\zeta_a(s)$ we have

$$\log\zeta_a(s)\sim\sum_p\left(\frac{1}{p^s}-\frac{1}{(p+a)^s}\right).$$

Now one gets (for $0<s<1$)

$$\sum_p\left(\frac{1}{p^s}-\frac{1}{(p+a)^s}\right)=s\int_2^\infty\pi(t)\left[\frac{1}{t^{s+1}}-\frac{1}{(t+a)^{s+1}}\right]\mathrm{d}t. \tag{14}$$

Since $\pi(x)\sim(x/\log x)$ as $x\to\infty$, we have,

$$\log\zeta_a(s)\sim s\int_2^\infty\frac{t}{\log t}\left(\frac{1}{t^{s+1}}-\frac{1}{(t+a)^{s+1}}\right)\mathrm{d}t.$$

We split the integral into 3 parts say $I_1+I_2+I_3$ where

$$I_1=\int_2^{a\varepsilon}\cdots,\ I_2=\int_{a\varepsilon}^{a\varepsilon^{-1}}\cdots\text{ and } I_3=\int_{a\varepsilon^{-1}}^\infty\cdots$$

where $0<\varepsilon<1$ is any fixed constant. It is easy to see that

$$I_1=O\left(\frac{(a\varepsilon)^{1-s}}{\log a}\right)\text{ and } I_3=O\left(a\int_{a\varepsilon^{-1}}^\infty\frac{t}{\log t}\frac{(s+1)}{t^{s+2}}\mathrm{d}t\right)=O\left(\frac{a(a\varepsilon^{-1})^{-s}}{\log a}\right)$$

and so I_1 and I_3 are $O\left(\frac{a^{1-s}}{\log a}\right)$ as $\varepsilon\to 0$. Now

$$I_2\sim\int_{a\varepsilon}^{a\varepsilon^{-1}}\frac{t}{\log a}\left(\frac{1}{t^{s+1}}-\frac{1}{(t+a)^{s+1}}\right)\mathrm{d}t=\frac{a^{1-s}}{\log a}\int_\varepsilon^{\varepsilon^{-1}}t\left(\frac{1}{t^{s+1}}-\frac{1}{(t+1)^{s+1}}\right)\mathrm{d}t$$

as can be seen by the substitution $t=au$. Thus we end up with

$$I_2 \sim \frac{a^{1-s}}{\log a} \int_0^{\infty} t \left(\frac{1}{t^{s+1}} - \frac{1}{(t+1)^{s+1}} \right) dt.$$

It is not hard to prove that the last integral is asymptotic to $[s(1-s)]^{-1} (a^{1-s}/\log a)$ so that we end up with an asymptotic relation which we state as a theorem.

Theorem 5. *For fixed s in $0 < s < 1$ we have as $a \to \infty$,*

$$\log f_a(s) \sim \frac{a^{1-s}}{(1-s)\log a}.$$

Remark. It is not very difficult to extend this for complex s with real part lying in the interval $(0, 1]$.

For finding the leading terms in asymptotic expansion for the functions

$$\prod_{p \leqslant a} \frac{1-p^{s-1}}{1-p^{-s}} \text{ and } \prod_p \frac{1-p^{s-1}+(p+a)^{s-1}}{1-p^{-s}+(p+a)^{-s}}$$

we can use above mentioned formulas. It would be interesting to find also the next terms in these expansions. For $0 < s < 1$ one has

$$\log \prod_{p \leqslant a} \frac{1-p^{s-1}}{1-p^{-s}} \sim \left[\frac{1}{1-s} \frac{a^{1-s}}{\log a} - \frac{1}{s} \frac{a^s}{\log a} \right]$$

but in fact we need next terms in the asymptotic expansion.

Note also that the analogous analysis can be applied to the L-function of the form

$$L_a(s, \chi) = \prod_p \frac{1}{1 - \chi(p)[p^{-s} - (p+a)^{-s}]} \tag{15}$$

where $\chi(p)$ is the Dirichlet character. To the product (15) the proof of the theorem 1 can be applied and we have also

$$\lim_{a \to \infty} L_a(s, \chi) = L(s, \chi) = \prod_p \frac{1}{1 - \chi(p)p^{-s}}$$

for $\sigma > 1$. It would be interesting to investigate the behaviour $L_a(1, \chi)$. The natural hypothesis is that

$$\lim_{a \to \infty} L_a(1, \chi) = L(1, \chi) = \prod_p \frac{1}{1 - \chi(p)p^{-1}}$$

for non principal characters.

It would be interesting to investigate also the questions about the analytical continuation of limit as $a \to \infty$ of $\zeta_a(s)$ in right half plane $\operatorname{Re} s > 0$ and the question about functional equation. Such problems are also interesting for the function

$$\zeta(a, s) = \prod_p \frac{1}{1 - (p+a)^{-s}}$$

which is a multiplication analogue of the Hurwitz zeta-function

$$\zeta(a, s) = \sum_{n=1}^{\infty} \frac{1}{(n+a)^s}.$$

References

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[2] Areféva I Ya, Dragovič B, Volovich I V, *On the Adelic string Amplitudes*, Preprint, Institute of Physics, Beograd (1988)