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On unbounded subnormal operators

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Abstract. A minimal normal extension of unbounded subnormal operators is established and characterized and spectral inclusion theorem is proved. An inverse Cayley transform is constructed to obtain a closed unbounded subnormal operator from a bounded one. Two classes of unbounded subnormals viz analytic Toeplitz operators and Bergman operators are exhibited.

Keywords. Unbounded subnormal operator; Cayley transform; Toeplitz and Bergman operators; minimal normal extension.

1. Introduction

Recently there has been some interest in unbounded operators that admit normal extensions viz unbounded subnormal operators defined as follows:

DEFINITION 1.1

Let S be a linear operator (not necessarily bounded) defined in $D(S)$, a dense subspace of a Hilbert space H. S is called a *subnormal operator* if it admits a normal extension $(N, D(N), K)$ in the sense that there exists a Hilbert space K, containing H as a closed subspace (the norm induced by K on H is the given norm on H) and a normal operator N with domain $D(N)$ in K such that $Sh = Nh$ for all $h \in D(S)$.

These operators appear to have been introduced in [12] following Foias [4]. An operator could be subnormal internally admitting a normal extension in H; or it could admit a normal extension in a larger space. As is well known, a symmetric operator always admits a self-adjoint extension in a larger space, contrarily a formally normal operator may fail to be subnormal ([2], [11]). Recently Stochel and Szafraniec ([12], [13]) obtained a Halmos–Bram type characterization of unbounded subnormal operators.

Here we discuss the existence and characterization of minimal normal extension N of an unbounded subnormal S. This is followed by the spectral inclusion theorem $\sigma(N) \subset \sigma(S)$. In §3, we set up a Cayley transform between a bounded subnormal and an unbounded one. We also exhibit two large classes of unbounded subnormals viz Bergman operators and analytic Toeplitz operators.

Let us recall [16, Ex. 5.39 p. 127] that given an operator T with domain $D(T)$ in a Hilbert space H, a closed subspace M of H is *invariant under* T if $T(D(T) \cap M) \subset M$. M is *reducing under* T if $T(M \cap D(T)) \subset M$, $T(M^{\perp} \cap D(T)) \subset M^{\perp}$ and $D(T) =$

$[M \cap D(T)] + [M^{\perp} \cap D(T)]$. Note that restriction of a normal operator to a reducing subspace is normal.

2. Minimal normal extension

DEFINITION 2.1

A normal extension $(N, D(N), K)$ of a subnormal operator $(S, D(S), H)$ is a *minimal normal extension* (MNE) if for any normal extension $(N_1, D(N_1), K_1)$ of S, $S \subset N_1 \subset N$ and K_1 is reducing under N implies $K_1 = K$ and $N_1 = N$.

In [13, p. 51] a normal extension N in K of S $(SD(S) \subset D(S))$ is called 'minimal' if $D = \{N^{*j}N^i x : x \in D(S),\ i, j = 0, 1, 2, \dots\}$ is linearly dense in K. The second half of the following theorem shows that it is in fact a MNE. The class of *C^∞-vectors* for an operator T in H is $C^\infty(T) = \bigcap_{n=1}^{\infty} D(T^n)$.

Theorem 2.2. (a) *A subnormal operator admits a minimal normal extension.*
(b) *Let S be a subnormal operator with dense domain $D(S)$ in a Hilbert space H. Let $(N, D(N), K)$ be a normal extension of S. Let D be the linear span of $\{N^{*i}N^j x : i, j = 1, 2, \dots;\ x \in C^\infty(S)\}$.*

(i) *If D is dense in K, then N is a* MNE
(ii) *If N is a* MNE *and $D(N) = D + (D(N) \cap D^{\perp})$, then D is dense in K.*

Proof. (a) Let $\mathscr{E}$ be the class of all normal extensions $\alpha = (N_\alpha, D(N_\alpha), K_\alpha)$ of a subnormal operator S in a Hilbert space H with domain $D(S)$. $\mathscr{E}$ is partially ordered by $\alpha \leqslant \beta = (N_\beta, D(N_\beta), K_\beta)$ if $N_\alpha \subset N_\beta$ and K_α is a reducing subspace for N_β. Note that for $\alpha \leqslant \beta$, the restriction $N_\beta|_{K_\alpha}$ of N_β on K_α with domain $D(N_\beta|_{K_\alpha}) = K_\alpha \cap D(N_\beta)$ is a normal operator in K_α which is an extension in K_α itself of the normal operator N_α. Since a normal operator is maximally normal [10, p. 350], $N_\alpha = N_\beta|_{K_\alpha}$ so that $D(N_\alpha) = K_\alpha \cap D(N_\beta)$. We shall apply Zorn's lemma to $\mathscr{E}$.

Let $\mathscr{C}$ be a chain in $\mathscr{E}$. Let $K = \cap\{K_\alpha | \alpha \in \mathscr{C}\}$, $D = \cap\{D(N_\alpha) | \alpha \in \mathscr{C}\}$. For $\alpha \in \mathscr{C}$, let $P_K^\alpha : K_\alpha \to K$ and for $\gamma \leqslant \alpha$, $P_\gamma^\alpha : K_\alpha \to K_\gamma$ be orthogonal projections. Now, let $\alpha \in \mathscr{C}$ be fixed. Since $\mathscr{C}$ is a chain, $K = \cap\{K_\gamma | \gamma \leqslant \alpha, \gamma \in \mathscr{C}\}$ and $D = \cap\{D(N_\gamma) | \gamma \leqslant \alpha, \gamma \in \mathscr{C}\}$.

Claim. K is a reducing subspace for the normal operator N_α. For this, note that $P_K^\alpha = \mathrm{glb}\{P_\gamma^\alpha | \gamma \in \mathscr{C}\} = \mathrm{glb}\{P_\gamma^\alpha | \gamma \in \mathscr{C}, \gamma \leqslant \alpha\}$, as in [15, p. 124]. Now consider the weak bounded commutant of N_α viz $\{N_\alpha\}' = \{S \in B(K_\alpha) | SN_\alpha \subset N_\alpha S\}$, $B(K_\alpha)$ denoting the set of all bounded linear operators on K_α. By Fuglede–Putnam theorem for unbounded normal operators [10, p. 365], $\{N_\alpha\}' = \{S \in B(K_\alpha) | SN_\alpha \subset N_\alpha S, SN_\alpha^* \subset N_\alpha^* S\} = \{N_\alpha, N_\alpha^*\}'$. Let E be the spectral measure for the bounded normal operator $(1 + N_\alpha^* N_\alpha)^{-1}$. For $k = 0, 1, 2, \dots$. let $w_0(0)$, $w_k = (1/k + 1, 1/k]$, and $N_{\alpha,k} = N_\alpha E(w_k)$ which are bounded normal operators. Then as shown in the proof of Theorem 2·1 in [8], $\{N_\alpha\}' = \{N_{\alpha,k} | k = 0, 1, 2, \dots.\}'$ (usual commutant in $B(K_\alpha)$ of a collection of bounded operators) which is a von Neumann algebra. Now by [16, p. 128], reducing subspaces of N_α correspond (via usual way of range projections) to projections in $\{N_\alpha\}'$. Hence for $\gamma \leqslant \alpha$, $P_\gamma^\alpha \in \{N_\alpha\}'$. Since projections in a von Neumann algebra form a complete lattice [15, p. 124], $P_K^\alpha \in \{N_\alpha\}'$; and hence K is a reducing subspace for N_α.

Now for $\gamma \leqslant \alpha$, $P_K^\alpha(D(N_\alpha)) = K \cap D(N_\alpha)$ since $D(N_\alpha) = [K \cap D(N_\alpha)] + [K^{\perp} \cap D(N_\alpha)]$.

Hence $P_K^\alpha(D(N_\alpha)) \subset K_\gamma \cap D(N_\alpha) = D(N_\gamma)$. Thus $P_K^\alpha(D(N_\alpha)) \subset \cap\{D(N_\gamma) | \gamma \in \mathscr{C}, \gamma \leqslant \alpha\} = D$. This implies that D is dense in K. For, given $x \in D^\perp$ ($\perp$ in K), for all $y \in D(N_\alpha)$, $\langle x, y\rangle = \langle P_K^\alpha x, y\rangle = \langle x, P_K^\alpha y\rangle = 0$, hence $x = 0$. Define an operator N in K with domain $D(N) = D$ as $Nx = N_\alpha x$. Then N is a well defined closed operator. To show that N is normal, consider an operator C in K with domain $D(C) = D$ as $Cx = N_\alpha^* x$ (adjoint in K_α). Then $C \subset N^*$ (adjoint in K). Now given $x \in D(N^*N)$, the functional $y \in D \to \langle N^*x, N^*y\rangle = \langle Cx, Cy\rangle = \langle N_\alpha^* x, N_\alpha^* y\rangle = \langle N_\alpha x, N_\alpha y\rangle = \langle Nx, Ny\rangle$ is continuous on D as $Nx \in D(N^*)$. Thus $N^*x \in D(N^{**}) = D(N)$ as N is closed. Thus $x \in D(NN^*)$ and $D(N^*N) \subset D(NN^*)$. In fact, $N^*N \subset NN^*$; and so $N^*N = NN^*$ both being self-adjoint (as N is closed). (Note that normality of N also implies $N = N_\alpha|_K$.)

The normal extension $(N, D(N), K)$ is a lower bound of $\mathscr{C}$. Now Zorn's lemma completes the proof.

(b) (i) Let the linear span D of $\{N^{*j}N^i x | x \in C^\infty(S); i,j = 1,2,\ldots\}$ be dense in K. Let $(N_0, D(N_0), K_0)$ be a normal extension of S such that $(N_0, D(N_0), K_0) \leqslant (N, D(N), K)$ (partial order as in the proof of (a)). Let $x \in C^\infty(S)$. Then for all $i = 1,2,\ldots, SC^\infty(S) \subset C^\infty(S)$ gives that $N^i x = S^i x \in C^\infty(S) \subset C^\infty(N) \cap K_0$. Now for any positive integer k, by the normality of N^k, $D(N^k) = D((N^k)^*) = D((N^*)^k)$ which implies that $C^\infty(N) = C^\infty(N^*)$. Thus $N^{*j}N^i$ are defined for all $i, j = 1,2,\ldots$. Further, since K_0 is invariant under N^*, $N^{*j}N^i x \in K_0$. Thus $D \subset K_0$. Hence $K_0 = K$, $N_0 = N$ showing that N is MNE.

(ii) Let $(N, D(N), K)$ be a MNE of S satisfying the given condition. Let $K_0 = \bar{D}$ (closure in K). By definition of D, $N^*D \subset D$, $ND \subset D$. These give $N(D(N) \cap K_0^\perp) \subset K_0^\perp$, $N^*(D(N) \cap K_0^\perp) \subset K_0^\perp$. Further, the given condition is equivalent to $D(N) = [D(N) \cap K_0] + [D(N) \cap K_0^\perp]$. We show that $N(D(N) \cap K_0) \subset K_0$. Let $x \in D(N) \cap K_0$. Then for all $y \in D(N) \cap K_0^\perp$, $\langle Nx, y\rangle = \langle x, N^*y\rangle = 0$. As $D(N) \cap K_0^\perp$ is dense in $K_0^\perp$, $Nx \in K_0$. Thus K_0 is reducing for N. Then $N|_{K_0}$ is a normal extension of S contained in N. By the minimality of N, $K_0 = K$. This completes the proof of the theorem.

The following is a spectral inclusion theorem analogous to the one for bounded subnormal. Its proof is patterned along Halmos [5, p. 157].

Theorem 2.3. *Let S be a subnormal operator in a Hilbert space H with domain $D(S)$ and a minimal normal extension N. Then $\sigma(N) \subset \sigma(S)$.*

Proof. Let $\lambda \notin \sigma(S)$. Then $(\lambda - S)^{-1}$ is a bounded operator on H. We can assume $\lambda = 0$ and $\|S^{-1}\| = 1$. Now for $0 < \varepsilon < 1$, consider $E_\varepsilon = \{x \in C^\infty(N) | \ \|N^n x\| < \varepsilon^n \|x\|$ for $n = 1,2,\ldots\}$. For $x \in E_\varepsilon$, $y \in H$,

$$\begin{aligned} |\langle x, y\rangle| &= |\langle x, S^n S^{-n} y\rangle| \\ &= |\langle N^{*n} x, S^{-n} y\rangle| \\ &\leqslant \varepsilon^n \|x\| \, \|y\| \text{ for all } n. \end{aligned}$$

As $\varepsilon < 1$, $\langle x, y\rangle = 0$. Thus $H \subset E_\varepsilon^\perp$ ($\perp$ in K). Let $N = \int z \, \mathrm{d}E(z)$ be the spectral theorem for N. Then $E_\varepsilon = E(\Delta_\varepsilon)K$ where $\Delta_\varepsilon = \{z \in \mathbb{C} : |z| \leqslant \varepsilon\}$. Hence E_ε, and so $E_\varepsilon^\perp$ is a reducing subspace of N. Now $N|_{E_\varepsilon^\perp}$ being normal, the minimality of N implies that $E_\varepsilon = K$. Hence $E(\Delta_\varepsilon)K = E_\varepsilon = \{0\}$. Thus $\phi = \Delta_\varepsilon \cap \operatorname{supp} E = \Delta_\varepsilon \cap \sigma(N)$; and so $0 \notin \sigma(N)$.

Notice that, in above notations, bdry $\sigma(S) \subset \sigma_\pi(S) \subset \sigma_\pi(N) = \sigma(N)$ (σ_π denotes the approximate point spectrum) and component of $\mathbb{C}\backslash\sigma(N)$ is either contained in $\sigma(S)$ or is disjoint from $\sigma(S)$.

COROLLARY 1

Let S be a subnormal operator. Then

(i) $\sigma(S) \neq \phi$.
(ii) *S is bounded iff* $\sigma(S)$ *is bounded.*
(iii) *S is essentially self-adjoint iff* $\sigma(S)$ *is real.*

COROLLARY 2

A symmetric operator has nonempty spectrum.

Remarks 2·4. (i) Let S be an operator in a Hilbert space H. A vector $x \in C^\infty(S)$ is an *analytic vector for* S if there exists a $t > 0$ such that

$$\sum_{n=1}^{\infty} \frac{\|S^n x\| t^n}{n!} < \infty.$$

Let $A(S)$ be the collection of all analytic vectors for S. If S is subnormal admitting a normal extension N such that $D(S) = D(N) \cap H$, then $A(S)$ is dense in H. Indeed, in this case, $A(S) = A(N) \cap H$. Hence taking the orthogonal complement in K, $A(S)^\perp = H^\perp$ as $A(N)$ is dense in K.
(ii) A symmetric operator in H admitting a normal extension N in (possibly a larger space) K satisfying $D(S) = D(N) \cap H$ is essentially self-adjoint. For, in view of (i), the well-known Nelson theorem [16, p. 261] applies.
(iii) Normal extensions of an unbounded subnormal operator satisfying the above spectral inclusion (distinguished normal extensions) have been discussed recently in [6]. Thus a MNE is distinguished, though a distinguished extension need not be minimal. For example let N_1 be a MNE of S in K_1. Let N_2 be a normal operator in K_2 with $\sigma(N_2) \subset \sigma(N_1)$. Take $N = N_1 \oplus N_2$ a normal extension of S in $K_1 \oplus K_2 = K$. Then $N_0 = N|_{E(\sigma(S))K}$, (where E is the spectral measure of N) is distinguished normal extension as in [6] which is not minimal.
(iv) Ôta [7] showed that if T is a densely defined closed operator in a Hilbert space H such that $TD(T) \subset D(T^*)$, then T is bounded. This has the following implication.

PROPOSITION

Let S be a closed subnormal operator in H with dense domain $D(S)$ *such that* $SD(S) \subset D(S)$. *Then S is bounded.*

This follows from $D(S) \subset D(S^*)$ [14].

We are thankful to Prof. Ôta for bringing this to our notice.

(v) Ôta [7] has also another interesting result, viz if T is a densely defined closed operator in a Hilbert space H such that the range of T is contained in its domain and if T is unbounded, then the numerical range $W(T) = \{\langle Tx, x\rangle | x \in \mathcal{D}(T), \|x\| = 1\}$ is the entire complex plane. The following is an analogous result for spectrum.

PROPOSITION

Let T be a densely defined closed operator in a Hilbert space H such that $TD(T) \subset D(T)$. If $\sigma(T)$ is not the whole of complex plane, then T is bounded.

Proof. If $\lambda \notin \sigma(T)$, then $S = (T - \lambda 1)^{-1}$ is a bounded operator satisfying $S(T - \lambda 1) \subset (T - \lambda 1)S = 1$. Thus $(T - \lambda 1)D(T) \subset D(T) = H$. Closed graph theorem shows that T is bounded.

3. A Cayley transform

The problem of self-adjoint extension (within the space) of a symmetric operator is discussed via Cayley transform [15, Ch. 8] which provides a correspondence between certain partial isometries and symmetric operators that admit self-adjoint extensions. We extend this so as to associate an unbounded subnormal operator with a bounded one.

Theorem 3.1. *Let S be a bounded subnormal operator on H with a bounded normal extension N on K. If*

(i) *$1 - N$ is one-to-one and*
(ii) *$1 \in \sigma(N)$, $\sigma(N) \backslash \{1\} \subset \{z \in \mathbb{C} : |z| < 1\}$*

then $\psi(N)|_H$ is an unbounded closed subnormal operator where $\psi(N)$ is the normal operator in K defined via the spectral theorem by the function $\psi(z) = i(1 + z)(1 - z)^{-1}$.

Proof. Define N' in K with domain $D(N') = R(1 - N)$ by $N'x = i(1 + N)(1 - N)^{-1}x$. Then N' is densely defined.

Claim (a). $\overline{N'} = \psi(N)$.
For, given $x \in D(N')$, $(1 - N)y = x$, and so

$$\int |(1 + z)(1 - z)^{-1}|^2 \, \mathrm{d}E_{x,x} = \int |(1 + z)^2 (1 - z)^{-2}| (1 - z)(1 - \bar{z}) \, \mathrm{d}E_{y,y} < \infty,$$

and for all $u \in K$

$$\langle N'x, u \rangle = i\langle (1 + N)y, u \rangle = i \int (1 + z)(1 - z)^{-1} \, \mathrm{d}E_{x,u}$$
$$= \langle \psi(N)x, u \rangle.$$

Hence $N' \subset \psi(N)$. As $\psi(N)$ is closed, $\overline{N'} \subset \psi(N)$. Now let $N_0 = N'^*|_{D(\psi(N^*)\psi(N))}$. Then $G(N_0)$ is dense in $G(N'^*)$, $G(\cdot)$ denoting the graph of the operator. Indeed, note that $G(N'^*)$ is closed in $K \times K$. Let $(u, N'^*u) \in G(N'^*)$, $(u, N'^*u) \perp G(N_0)$. Then for all $x \in D(\psi(N)^* \psi(N))$,

$$0 = \langle (u, N'^*u), (x, N'^*x) \rangle$$
$$= \langle u, x \rangle + \langle N'^*u, \psi(N)^*x \rangle \text{ (as } \psi(N)^* \subset N'^*)$$
$$= \langle u, x \rangle + \langle u, \psi(N)\psi(N)^*x \rangle. \qquad (\alpha)$$

Here we have used the following that can be easily verified.

Lemma. Let A and B be densely defined linear operators in a Hilbert space with B closed and $D(B)=D(B^*)$. *If* $A\subset B$, *then for all* $u\in D(A^*)$, $y\in D(B)$, $\langle A^*u,y\rangle=\langle u,By\rangle$.

Thus in (α), since $R(1+\psi(N)^*\psi(N))$ is dense in K, $u=0$. Then $G(N_0)$ is dense in $G(N'^*)$. Now, let $y\in D(N'^*)$. Then for some sequence (y_i) in $D(\psi(N)^*\psi(N))$, $y_i\to y$ and $N_0y_i-N'^*y=N'^*y_i-N'^*y\to 0$. Since $\|\psi(N)y_i-\psi(N)y_j\|=\|\psi(N)^*y_i-\psi(N)^*y_j\|=\|N'^*y_i-N'^*y_j\|$, $(\psi(N)y_i)$ converges to some $u\in K$. Since $\psi(N)$ is closed, $(y,u)\in G(\psi(N))$, $y\in D(\psi(N))$. Thus $D(N'^*)\subset D(\psi(N))=D(\psi(N)^*)$; hence $D(N'^*)=D(\psi(N)^*)$, $N'^*=\psi(N)$ and so $\overline{N'}=\psi(N)$.

Claim (b). H is invariant under $\psi(N)$ (and N'). For if, $x\in D(\psi(N))\cap H$, $y\in H^\perp$, then since 1 is not an eigenvalue, $E(\{1\})=0$; and so

$$\begin{aligned}\langle\psi(N)x,y\rangle&=i\int_{\sigma(N)}(1+z)(1-z)^{-1}\mathrm{d}E_{x,y}\\&=i\int_{\sigma(N)-\{1\}}(1+z)(1-z)^{-1}\,\mathrm{d}E_{x,y}\\&=i\sum_k\int(1+z)z^k\,\mathrm{d}E_{x,y}\\&=i\sum_k\langle(1+N)N^kx,y\rangle=0\end{aligned}$$

as $(1+N)\,N^kx\in H$. Thus $\psi(N)x\in H$. This establishes our claim (b).

It is easy to see that $\psi(N)|H$ with domain $D(\psi(N)|_H)=D(\psi(N))\cap H$ is a closed operator.

Remark 3.2. Note that if $R(1-S)$ (range of $(1-S)$) is dense in H, then $S'=i(1+S)(1-S)^{-1}$ with domain $D(S')=R(1-S)$ and $S_0=N'|_H$ and hence are subnormals (not necessarily closed) in H.

4. Examples

4.1 *Unbounded analytic Toeplitz operators*

Let

$$U=\{z\in\mathbb{C}:|z|<1\},\quad \Gamma=\{z\in\mathbb{C}:|z|=1\}.$$

Let ϕ be a measurable function on Γ and $D_\phi=\{f\in H^2(U):\phi f\in L^2(\Gamma)\}$. Define T_ϕ in $H^2(U)$ with domain D_ϕ as $T_\phi f=P(\phi f)$, where $P:L^2(\Gamma)\to H^2(U)$ is the projection. The Toeplitz operator T_ϕ is an analytic Toeplitz operator if ϕ is analytic. Such a T_ϕ admits a normal extension M_ϕ with domain $D(M_\phi)=\{f\in L^2(\Gamma):\phi f\in L^2(\Gamma)\}$, $M_\phi f=\phi f$. Thus, in this case, if D_ϕ is dense in $H^2(U)$, then T_ϕ is subnormal. Note that it is bounded iff ϕ is bounded. We exhibit below a class of function ϕ for which T_ϕ is a closed unbounded subnormal operator.

(i) $\phi(z)=(1-z)^{-1}$. Then $D_\phi = R(1-S)$ where S is the unilateral shift. Hence $D_\phi = \ker(1-S^*)^\perp = H^2(U)$. Also T_ϕ is closed. For, if $(f_n, T_\phi f_n)\to(f,g)$, then (identifying $H^2(U)$ with a closed subspace of $L^2(\Gamma)$), there exists a subsequence (f_{nk}) of (f_n) each of whose subsequence converges a.e. to f on Γ. Since $f_{n_k}(z)(z-1)^{-1}\to g\in L^2(\Gamma)$, $(z-1)g=f$ a.e. Hence $g=T_\phi f$ in $H^2(U)$.

(ii) A similar argument can be applied for $\phi(z)=(z-\lambda_1)^{-n_1}(z-\lambda_2)^{-n_2}\cdots(z-\lambda_k)^{-n_k}$ with $|\lambda_i|\geqslant 1$, $n_i=1,2,\ldots$.

(iii) As discussed in [6], functions $\phi\in H^2(U)$ define unbounded analytic Toeplitz operators.

Unbounded Teoplitz operators also arise quite naturally in representation of certain topological algebras by unbounded operators.

Consider Arens algebra [1] $L^w(\Gamma)=\bigcap_{1\leqslant p<\infty}L^p(\Gamma)\neq L^\infty(\Gamma)$ with pointwise operations. It is a Frechet * algebra with the topology of L^p-convergence for each p, $1\leqslant p<\infty$. The Hardy-Arens algebra $H^w(U)=\bigcap_{1\leqslant p<\infty}H^p(U)\neq H^\infty(U)$ [9, Ch. 17, Ex. 10] can be regarded as a closed subalgebra of $L^w(\Gamma)$. For a $\phi\in H^w(U)$, D_ϕ is dense in $H^2(U)$ since $H^\infty(U)\subset D_\phi$ and $H^\infty(U)$ is dense in $H^2(U)$. In fact, as in (i) above, T_ϕ is closed. It is easily seen that $\phi\to T_\phi$ is a representation of $H^w(U)$ by unbounded subnormal operators in $H^2(U)$ which is the restriction of the unbounded * representation $\phi\to M_\phi$ of $L^w(\Gamma)$ into normal operators in $L^2(\Gamma)$.

4.2 *Unbounded Bergman operators*

Let G be a bounded domain in $\mathbb{C}$. For $1\leqslant p<\infty$, consider the Bergman spaces $L^p_a(G)=\{f\in L^p(G): f \text{ is analytic on } G\}$ with $\|\cdot\|_p$ norm. Let $L^w_a(G)=\bigcap_{1\leqslant p<\infty}L^p_a(G)$. For $g\in L^w_a(G)$, define S_g in $L^2_a(g)$ with domain $D(S_g)=\{f\in L^2_a(G): gf\in L^2_a(G)\}$ as $S_g f = gf$. Again S_g is densely defined if $L^\infty_a(G)$ is dense in $L^2_a(G)$, in particular, if G is a Caratheodory domain [3, Ch. 3] in which case $L^2_a(G)=P^2(G)$, the $L^2_a(G)$-closure of polynomials. In this way, one gets a large class of unbounded subnormals.

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