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A tribute to a work of C P Ramanujam

R V GURJAR
School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

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Abstract. In this expository article we describe some recent results on the existence and general properties of complex homology 2-cells and contractible surfaces. The motivation for these can be directly traced back to C P Ramanujam's important paper, 'A topological characterization of the affine plane as an algebraic variety'. We also discuss the related results of M Miyanishi, T Sugie and T Fujita.

Keywords. Affine surface; fundamental group; rational surface.

1. Introduction

This is an expository article that centres around a theorem of C P Ramanujam and recent developments arising out of it. In trying to solve the cancellation problem for an affine space, '$V \times \mathbb{C} \approx \mathbb{C}^3$ implies $V \approx \mathbb{C}^2$ (as an affine variety)', C P Ramanujam proved the following beautiful result in [11].

Let V be an affine, non-singular, topologically contractible surface/$\mathbb{C}$ which is simply-connected at infinity. Then V is isomorphic to $\mathbb{C}^2$ as an affine variety.

Actually, from the proof it follows that it suffices to assume only that V is affine, non-singular, the co-ordinate ring $\Gamma(V)$ is a UFD and V is simply-connected at ∞. To quote from Prof. David Mumford:

'Perhaps C P Ramanujam's most perfect piece of work in algebraic geometry is his proof that a smooth affine complex surface, which is contractible and simply-connected at ∞, is isomorphic to the plane $\mathbb{C}^2$. The proof of this is not simple and uses many techniques; in particular, it shows how well he knew his way about in the classical geometry of surfaces! What is equally astonishing is his very striking counter-example showing that the hypothesis "Simply-connected at ∞ cannot be dropped. The position of this striking example in a general theory of 4-manifolds and particularly in a general theory of the topology of algebraic surfaces is yet to be understood'. See [10].

Some recent results of A R Shastri, M Miyanishi and myself have considerably increased the understanding of smooth contractible affine surfaces. We will now describe these new results in a cronological order.

In my Ph.D. thesis I applied C P Ramanujam's method for the following question: Let $f: \mathbb{C}^2 \to V$ be a proper morphism onto a smooth affine variety/$\mathbb{C}$. Is $V \approx \mathbb{C}^2$ as an affine variety? I was able to prove that assuming more generally that V is only normal, that V is contractible and has finite fundamental group at infinity. If further

V is smooth, then either the fundamental group at infinity is trivial or it is the binary icosahedral group of order 120. As a corollary, it follows that if degree $f \nmid 120$, then $V \approx \mathbb{C}^2$. Since the method used by C P Ramanujam has been important in the proof of the above result and subsequent papers, [2], [4], [5], [12], we will describe it briefly.

One embedds an affine, normal surface $V \subset \bar{V}$ where $\bar{V}$ is a normal, projective surface which contains V as a Zariski-dense open subset such that $\bar{V}$ is smooth outside the singularities of V and $\bar{V} - V$ is a divisor with normal crossing. Let $\bar{V} - V = \bigcup_{i=1}^{r} C_i$. Here C_i are the irreducible components of $\bar{V} - V$. When V is smooth and contractible (more generally, when $H_i(V; \mathbb{Z}) = (0)$ for all $i > 0$), each $C_i \approx \mathbb{P}^1$, there is no cycle in the dual graph of $\bigcup_{i=1}^{r} C_i$ and the determinant $|C_i \cdot C_j| = \pm 1$. One can further assume (without loss of generality) that if a component C_i has self-intersection -1, then it meets at least three other components. One can construct a "nice" tubular neighbourhood T of UC_i in $\bar{V}$ (in fact, a fundamental system of such neighbourhoods) such that ∂T is a compact, C^∞ 3-manifold. In his important paper on the topology of normal 2-dimensional singularities, D Mumford gave a generators and relations presentation of $\pi_1(\partial T)$ in terms of the intersection matrix $(C_i \cdot C_j)$. It is this presentation which has proved to be extremely important in C P Ramanujam's proof (as also in the study of rational singularities, ...). When $|C_i \cdot C_j| = \pm 1$, ∂T is actually a homology 3-sphere (which is the boundary of the contractible, C^∞ 4-manifold $V - \mathring{T}$). D Mumford proved the following purely group theoretical lemma:

Let G_1, G_2, G_3 be non-trivial groups and $a_i \in G_i$ be three elements. Then the group

$$\frac{G_1 * G_2 * G_2}{\langle a_1 a_2 a_3 \rangle}$$

is non-trivial.

A R Shastri and myself studied C P Ramanujam's method closely. In order to remove the restriction in the above result viz. $120 \nmid \deg f$, it was necessary to generalize Mumford's group-theoretical lemma. Shastri found such a generalization which is as follows:

Let $G_1, \ldots, G_n$ be any non-trivial groups, $a_i \in G_i$ be any elements. Then (i) for $n \geqslant 4$, $G_1 * \cdots * G_n / \langle a_1 \cdot a_2 \cdots a_n \rangle$ is infinite, (ii) for $n \geqslant 3$, $G_1 * G_2 * G_3 / \langle a_1 \cdot a_2 \cdot a_3 \rangle$ is non-trivial and (iii) $G_1 * G_2 * G_2 / \langle a_1 \cdot a_2 \cdot a_3 \rangle$ is finite $\Rightarrow G_i$ are cyclic groups generated by $a_i, i = 1, 2, 3$.

Using this lemma and C P Ramanujam's method, we could prove that a smooth, contractible surface cannot have a non-trivial finite fundamental group at infinity.

Around the same time, M Miyanishi, T Sugie and T Fujita proved the cancellation theorem for $\mathbb{C}^2$. Their proof was quite different and uses the theory of logarithmic Kodaira dimension of pseudo-effective divisors. Because of the importance of this theory for their proof and the papers [1, 8, 9] and [3], we will quickly describe the statement of this decomposition.

Let X be a smooth, projective surface$/k = \bar{k}$ and D a pseudo-effective divisor on X i.e. $D \cdot H \geqslant 0$ for every ample divisor H on X. Then $\exists$ Q-divisors P and N having the properties

(i) $D \underset{\text{numerically}}{\approx} P + N$.
(ii) P is pseudo-ample i.e. $P \cdot C \geqslant 0$ for every curve C on X.
(iii) N is a non-negative Q-linear combination of irreducible curves such that the intersection form on the components of N is negative definite.
(iv) $P \cdot C = 0$ for every irreducible component of $\operatorname{supp} \cdot N$.
(v) if D is effective, P is also effective.

Zariski had proved the existence of such a decomposition for the case when D is effective. T Fujita generalized it to the case of pseudo-effective divisors. S Iitaka has introduced the notion of logarithmic Kodaira dimension, which we now describe.

Let V be a smooth, quasi-projective surface/$\mathbb{C}$. Embed $V \subset \bar{V}$ with $\bar{V}$ a smooth, projective surface such that $\bar{V} - V$ is a divisor with normal crossings. Let D be the reduced divisor $\bar{V} - V$. Define

$$\bar{K}(V) = \max_n \dim \Phi_{|n(K_{\bar{V}} + D)|}(\bar{V}),$$

where $\Phi_{|n(K_{\bar{V}} + D)|}$ is the rational map given by the linear system $|n(K_{\bar{V}} + D)|$. Now the characterization of $\mathbb{C}^2$ due to Miyanishi, Sugie and Fujita is as follows.

Let V be a smooth, affine surface/$\mathbb{C}$ such that

(i) the co-ordinate ring $\Gamma(V)$ is a UFD.
(ii) $\Gamma(V)^* = \mathbb{C}^*$ (i.e. no non-trivial units in $\Gamma(V)$).
(iii) $\bar{K}(V) = -\infty$ i.e. $|m(K_{\bar{V}} + D)| = \phi$ for all $n \geqslant 1$. Then $V \approx \mathbb{C}^2$.

From this characterization, it is easy to see that if $f: \mathbb{C}^2 \to V$ is a proper, surjective morphism onto a smooth affine surface, then $V \approx \mathbb{C}^2$. Cancellation theorem for $\mathbb{C}^2$ also follows from the above characterization. The theory of Zariski–Fujita decomposition has proved to be very important for the classification of non-complete algebraic surfaces, as has been amply demonstrated by the work of Y Kawamata, T Fujita, M Miyanishi, S Tsunoda and others.

So the difference in C P Ramanujam's characterization of $\mathbb{C}^2$ and Miyanishi, Sugie, Fujita's characterization of $\mathbb{C}^2$ is in the hypothesis about the behaviour at infinity of V. The relationship between these two hypotheses was something of a puzzle for me. But more about it later.

M Miyanishi also proved the following nice result in [8].

Let $f: \mathbb{C}^2 \to V$ be a proper, surjective morphism onto a normal, affine surface. Then $V \approx \mathbb{C}^2/G$ where G is a "small" finite subgroup of $GL(2, \mathbb{C})$ acting linearly on $\mathbb{C}^{2\prime}$. His method again used the theory of logarithmic Kodaira dimension. Then A R Shastri and myself proved the same result using the topological method of C P Ramanujam (i.e. using the fundamental group at infinity). Later on Shastri and I proved the converse of the above result of Miyanishi which is at the same time a generalization of C P Ramanujam's result. We proved:

Let V be a normal, contractible surface which has finite fundamental group at infinity. Then $V \approx \mathbb{C}^2/G$ where G is a "small" finite subgroup of $GL(2, \mathbb{C})$ acting linearly on $\mathbb{C}^2$.

Because of the usefulness of the notion of fundamental group at infinity, it was desirable to have a complete classification of configurations of curves at infinity such that the fundamental group at infinity ($\pi_1(\partial T)$ in the notation used earlier) is finite. This was done by A R Shastri in [12]. In addition to C P Ramanujam's technique, Shastri's proof uses results like Dehn's lemma and loop theorem from 3-dimensional topology! As a corollary of his classification, Shastri proved that any affine, smooth surface with finite fundamental group at infinity is rational. I also found a proof of this rationality without making use of Shastri's classification.

As mentioned earlier, the relationship between Ramanujam's method and the purely geometric method of Miyanishi, Sugie, Fujita was a mystery I wanted to solve. Finally such a connection was discovered in [3]. The result is the following:

'Let V be a smooth, affine surface with finite fundamental group at infinity. Then $\bar{K}(V) = -\infty$'.

From $\bar{K}(V) = -\infty$ again the rationality of V follows easily. Thus it appears that Ramanujam's hypothesis is stronger than that of Miyanishi, Sugie, Fujita. But the proof of the above result uses Shastri's classification mentioned above, which in turn uses C P Ramanujam's method. So Ramanujam's characterization of $\mathbb{C}^2$ does not follow from Miyanishi, Sugie, Fujita's characterization! I believe that the use of explicit classification of Shastri can be avoided in the proof of the above implication.

2. New examples of homology 2-cells

In his paper [11], C P Ramanujam found an example of a smooth, contractible surface with infinite fundamental group at infinity. The example can be described easily. In $\mathbb{CP}^2$, let C be the irreducible cubic $\{ZY^2 - X^3 = 0\}$ and D a smooth conic which intersects C in two points p, q with intersection multiplicities 5 and 1 resp. Assume p, q are not inflexion points of C. Blow-up $\mathbb{P}^2$ at q and let X be the surface obtained. Let C', D' be the proper transforms of C, D resp. Then $V = X - (C' U D')$ is the required example.

For a long time, this was the only non-trivial example of a smooth contractible surface. There was another reason to have more such example. This was because of a question Van de Ven asked in [13].

'Let V be a smooth, affine surface such that $H_i(V; \mathbb{Z}) = (0)$ for $i > 0$. Is V rational?'

We will call such a surface a complex homology 2-cell.

In order to see if more examples exist (and for other questions) Miyanishi and I decided to study affine surfaces systematically using the invariant $\bar{K}$. It turns out that for $\bar{K} = 0$ such surfaces do not exist. But when $\bar{K} = 1$, U Kawamata has proved that V admits a $\mathbb{C}^*$-fibration. Using the homology conditions on V, one can analyse the possible singular fibres of such a $\mathbb{C}^*$-fibration. Using all this, finally Miyanishi and I produced infinitely many non-isomorphic complex homology 2-cells and contractible surfaces. In fact, all homology 2-cells and contractible smooth surfaces V with $\bar{K}(V) = 1$ arise in the way we describe in [3]. In [3] an example of a smooth, contractible surface with $\bar{K} = 2$ is also constructed. The construction is somewhat similar to C P Ramanujam's example. This leads us to an unsolved question

Question: Are there infinitely many non-isomorphic smooth contractible surfaces V with $\bar{K}(V) = 2$?

3. Rationality of complex homology cells

Recently A R Shastri and I have proved that all smooth, complex homology 2-cells are rational. See [6, 7]. More generally, we prove the following.

Theroem. *Let X be an irreducible, smooth, projective surface/$\mathbb{C}$, with the geometric genus $P_g(X) = 0$. Let D be a reduced (not necessarily connected) curve on X which has normal crossings. Suppose each connected component of* Supp *D is simply-connected and the irreducible components of D generate the divisor class group* Pic(X). *Then X is rational.*

I will briefly describe the method. Y Miyaoka has proved an inequality between Chern-numbers for non-complete smooth surfaces (for complete case, it was also conjectured by Van de Ven). This inequality is one of the main steps in the proof of above theorem. Also the insight we gained about configurations of rational curves with unimodular determinant from C P Ramanujam's paper was very important. Detailed properties of Zariski decomposition worked out by T Fujita also play an important role in the proof. The proof is split up into two parts, to deal with surfaces of general type and elliptic type. Surprisingly, the elliptic case gave much more trouble in settling. The details of the proof are too technical to describe here.

We get the following interesting consequence of the rationality of complex homology 2-cells:

COROLLARY

For a complex homology 2-cell V, the Chow-groups of 0-cycles and 1-cycles are trivial. All algebraic vector bundles on V are trivial.

This is a generalization of the theorem of C S Seshadri on the triviality of vector bundles on $\mathbb{C}^2$. We can therefore ask the following analogue of Serre's question for contractible varieties:

Question: If V is a smooth, contractible affine variety, are all the algebraic vector bundles on V trivial?

Finally, to give a proof of the cancellation theorem for $\mathbb{C}^2$ using C P Ramanujam's method remains a cherished dream for me.

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