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Closest point of the cut locus to submanifold

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Abstract. This paper deals with some of the matching and non-matching properties of two minimal geodesics from a cut point, which is not a focal point, to the points of the submanifold. Closest cut point to the submanifold has been obtained under some additional assumptions.

Keywords. Cut-locus; closest point; submanifold; Riemannian metric; tangent bundles; Jacobi fields.

1. Introduction

Let M be a complete connected Riemannian manifold of class C^∞ and L be a compact connected imbedded submanifold of M of class C^∞. Let $N(L)$ be the normal bundle of L which is a subbundle of tangent bundle TM of M. The exponential map of the Riemannian manifold M restricted to $N(L)$ is a map $e{:}N(L)\to M$ of class C^∞. Let $d{:}M\times M\to R$ be the Riemannian distance function, then to any point $z\in M$ there is at least one point $x\in L$ such that $d(z,x)=\inf\{d(z,x')\,|\,x'\in L\}$ holds and x is said to be a point nearest to z in L. Let $z\notin L$ and consider a geodesic $c{:}R\to M$ of the Riemannian manifold such that $c(0)=x\in L, c(t)=z$ for some $t>0$ and that the restriction of c to $[0,t]$ yields a minimal geodesic from x to z. Then the tangent vector $\dot{c}(0)$ of c is in the normal space N_xL of L at x by a basic observation [2]. Since M is complete such a geodesic c always exists and consequently the map e is surjective. As to the injectivity of the map e some further concepts are essential which can be summarized as follows:

If the tangent linear map

$$T_ve{:}T_vN(L)\to T_{e(v)}M$$

of e at $v\in N(L)$ is not injective then v is called a focal point of L in the normal bundle $N(L)$ and $e(v)$ is said to be a focal point of L in M. The set of focal points v of L is said to be a focal locus of L in the normal bundle $N(L)$ and the set of focal points $e(v)$ of L is called the focal locus of L in M. In the special case when the submanifold L reduces to a single point $y\in M$ and consequently the normal bundle $N(L)$ coincides with the tangent space T_yM, the focal points of L are said to be points conjugate to y and the focal locus of L is called conjugate locus of y.

2. Cut locus of submanifold

Let $c:R \to M$ be a geodesic such that $c(0) = x \in L$, $\dot{c}(0) = w \in N_x L$, $\|w\| = 1$. Since the restricted exponential map e is injective in a sufficiently small neighbourhood of the zero section in the normal bundle $N(L)$ of the submanifold L [2], consequently x is the unique nearest point of L to $z = c(t_0)$ for $0 < t_0 \leqslant t$ where t is sufficiently small positive value, and $c \restriction [0, t]$ is the unique minimal geodesic. Let s_w be the supremum of such values $t > 0$ which is always well defined, since M is complete. If s_w is finite, then $c \restriction [0, s_w]$ is a minimal geodesic. The point $v' = (s_w w)$ is called a cut point of L in the normal bundle and $e(v') = c(s_w w)$ is called a cut point of L in M. The set of cut points of L in $N(L)$ is called the cut locus of L in $N(L)$ and the set formed by the cut points of L in M is called the cut locus of L in M.

A straightforward generalization of some basic facts established in the special case when L reduces to a single point [2] yields the following:

If $v' = s_w w \in N(L)$ is the cut point of the submanifold L then at least one of the following two assertions is true.

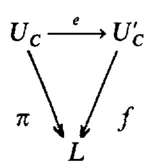
(i) The point $v' = s_w w$ is the first focal point of the submanifold L along c.
(ii) There exist at least two minimizing geodesics from two different points of the submanifold L to $e(s_w w)$.

Consider now the open neighbourhood U_C of the zero section in the normal bundle $N(L)$ given by the following definition:

$$U_C = \{tw \mid 0 \leqslant t \leqslant s_w, \ \|w\| = 1, \ w \in N_x L\}.$$

In other words U_C is the neighbourhood of the zero section which is bounded by the cut locus of L in $N(L)$. (i), (ii) and some simple observations yield that the restriction of e to U_C is a diffeomorphism. Consequently $U'_C = e(U_C)$ is an open neighbourhood of the submanifold L in M.

Consider the map $f: U'_C \to L$ which maps a point $y' \in U'_C$ to the unique point $y \in L$ which is nearest to y'. Let π be the canonical projection of the normal bundle $N(L)$, then the following commutative diagram is obtained:



Consequently f is a smooth retraction of the neighbourhood U'_C to the submanifold.

3. Riemannian metric of the tangent bundles and Jacobi fields

Let M be an n-dimensional complete connected Riemannian manifold with metric g. Then a metric $\bar{g}$ of its tangent bundle TM is given by [1]

$$\bar{g}(Z, Z) = g(Tp_M(Z), Tp_M(Z)) + g(K(Z), K(Z))$$

for any $Z\in T\dot{T}M$, where $Tp_M:TTM\to TM$ is the tangent linear map corresponding to the canonical projection $p_M:TM\to M$ and $K:TTM\to TM$ is the connector map. It is well known that for every $v\in TM$ the fibre T_vTM of the canonical projection $p_{TM}:TTM\to TM$ has a unique orthogonal decomposition

$$T_vTM\to H_vTM\oplus V_vTM,$$

where the subspaces

$$H_vTM=\mathrm{Ker}(K\upharpoonright T_vTM),$$
$$V_vTM\,\mathrm{Ker}(Tp_M\upharpoonright T_vTM)$$

are called the horizontal and vertical subspaces of T_vTM respectively. Moreover the following conditions are satisfied:

(a) Both of the canonical isomorphisms

$$h_v^{-1}=Tp_M\upharpoonright H_vTM:H_vTM\to T_xM$$

and $i_v=K\upharpoonright V_vTM:V_vTM\to T_xM$ are isometries for any $v\in TM$, where $x=p_M(v)$.
(b) The canonical projection $p_M:TM\to M$ is a Riemannian submersion.

Let $v:V\to TM$ be a vector field on a neighbourhood V of $x\in M$, then for the covariant derivative of v in the direction $v_1\in T_xM$ we have $v_2=D_{v_1}\,v(x)=K\cdot Tv(v_1)$ where Tv is the tangent linear map defined by v. Since $Tv(v_1)\in T_vTM$

$$Tv(v_1)=h_v(v_1)+i_v^{-1}(v_2)$$

holds by the above decomposition. More generally for any $Z\in TTM$, $Z=h_v(v_1)+i_v^{-1}(v_2)$ holds, where $v_1=Tp_M(Z), v_2=K(Z)$ and $v=p_{TM}(Z)$. Also any $Z\in TTM$, $p_{TM}(Z)\neq 0$, defines uniquely a Jacobi field

$$J:[0,\infty)\to TM$$

along the geodesic $c:[0,\infty)\to M$ such that $\dot{c}(0)=v=p_{TM}(Z)$, and the following initial conditions are considered.

$$J(0)=v_1=Tp_M(Z),$$
$$\dot{J}(0)=v_2=K(Z).$$

Now let L be an m-dimensional compact connected submanifold of M where $m<n$. Let $N(L)$ be the normal bundle of L. Let $v:V\to N(L)$ be a section of the normal bundle $N(L)$ on a neighbourhood V of $x\in L$, then for the covariant derivative of v in the direction $v_1\in T_xL$ we have

$$v_2=D_{v_1}v(x)=K\cdot Tv(v_1)$$

holds as before. But considering the orthogonal decomposition $T_xM=T_xL\oplus N_xL$ of the tangent space T_xM, $v_2=A_v(v_1)+\tilde{v}_2$ is valid where $A_v:T_xL\to T_xL$ is the Weingarten map corresponding to the normal $v(x)\in N_xL$ and $\tilde{v}_2=\tilde{D}_{v_1}\,v(x)\in N_xL$. The inclusions $N(L)\subset TM$ and $TN(L)\subset TTM$ permit to consider an induced connector map $\tilde{K}: TN(L) \to N(L)$ given by $\tilde{K}(Z) = K(Z) - A_v(v_1)$ for any $Z \in TN(L)$ where $v = p_{TM} \upharpoonright TN(L)(Z) \in N_x L$ and $v_1 = Tp_M \upharpoonright TN(L)(Z) \in T_x L$. Thus the induced covariant derivative of v in the direction v_1 is given by $\tilde{v}_2 = \tilde{D}_{v_1} v(x) = \tilde{K}\ Tv(v_1) \in N_x L$. Thus the canonical metric $\tilde{g}$ of the normal bundle $N(L)$ can be given as

$$\tilde{g}(Z, Z) = g(Tp_M(Z), Tp_M(Z) + g(\tilde{K}(Z), \tilde{K}(Z)))$$

for any $Z \in TN(L)$.

It is easy to see that for every $v \in N(L)$ the fibre $T_v N(L)$ of the canonical projection

$$p_{TM} \upharpoonright TN(L) = TN(L) \to N(L)$$

has a unique orthogonal decomposition

$$T_v N(L) = \tilde{H}_v N(L) \oplus \tilde{V}_v N(L), \quad \text{where the subspaces}$$

$$\tilde{H}_v N(L) = \operatorname{Ker}(\tilde{K} \upharpoonright T_v N(L)),$$

$$\tilde{V}_v N(L) = \operatorname{Ker}(Tp_M \upharpoonright T_v N(L))$$

are called the horizontal and vertical subspaces of $T_v NL$ respectively. Moreover the following conditions are satisfied.

(i) Both of the isomorphisms

$$\tilde{h}_v^{-1} = Tp_M \upharpoonright \tilde{H}_v N(L): \tilde{H}_v N(L) \to T_x: L$$

and $\tilde{i}_v = \tilde{K} \upharpoonright \tilde{V}_v N(L): \tilde{V}_v N(L) \to N_x L$ are isometries for any $v \in N(L)$, where $x = p_M(v) \in L$;

(ii) The canonical projection

$$p_M \upharpoonright N(L): N(L) \to L$$

is a Riemannian submersion.

It may be noted that for any $Z \in TN(L) \subset TTM$ we have already two orthogonal decompositions. In the first case the components of $Z, Z_{\text{hor}} = h_v(v_1)$ and

$$Z_{\text{vert}} = i_v^{-1}(v_2)$$

are orthogonal in the metric g of TM. In the second case Z has the following components

$$\tilde{Z}_{\text{hor}} = Z_{\text{hor}} + i_v^{-1}(A_v(v_1))$$

and

$$\tilde{Z}_{\text{vert}} = Z_{\text{vert}} - i_v^{-1}(A_v(v_1)),$$

which are evidently orthogonal with respect to the metric $\tilde{g}$ of the normal bundle $N(L)$.

Let $\Psi: TN(L) \to TM$ be the map defined by $\Psi(Z) = v_1 + v_2 - A_v(v_1) \in TM$ where $Z \in TN(L)$, $x = p_M(v)$. Then it is easy to see that for any $v \in N(L)$ the map $\Psi \upharpoonright T_v N(L)$: $T_v N(L) \to T_x M$ is an isometry if the metrics $\tilde{g}$ and g are considered. Now a characterization of the so-called L-Jacobi fields will be given below.

It is clear that if $v \in N(L)$ is non-zero and $\rho: [0, \infty) \to N(L)$ is a ray of $N(L)$ defined by $\rho(t) = t \cdot v$, for $t \in [0, \infty)$, then $\dot{\rho}(1) \in T_v N(L)$ holds. Let $Z \in TN(L)$ such that

$v = p_{TM}(Z) \neq 0$ and $\tilde{g}(\dot{\rho}(1), Z) = 0$ holds. Then Z defines uniquely an L-Jacobi field [2] $J:[0,\infty) \to TM$ along the geodesic $c:[0,\infty) \to M$ such that $\dot{c}(0) = v = p_{TM}(Z) \in N_x L$, and the following initial conditions are considered:

$$J(0) = v_1 = Tp_M(Z) \in T_x L,$$
$$J'(0) = v_2 = K(Z).$$

Notice that the conditions $g(v, v_1) = 0$ and $g(v, v_2) = \tilde{g}(\dot{\rho}(1), Z) = 0$ imply that $g(\dot{c}, J) = 0$ is valid for every parameter along the geodesic c, on the other hand

$$\tilde{K}(Z) = v_2 - A_v(v_1) \in N_x L \quad \text{holds as well.}$$

Lemma 3.1. Let L be a smooth connected submanifold of a complete connected Riemannian manifold M. Let $e:N(L) \to M$ be the restricted exponential map. If $v \in N(L)$ is a non-zero vector and ρ is the ray of $N(L)$ defined by v, then

$$\tilde{g}(\dot{\rho}(1), Z) = g(T_v e(\dot{\rho}(1)), T_v e(Z))$$

holds for any $Z \in T_v N(L)$, where $T_v e$ denotes the tangent linear map of e at v.

Proof. Let $Z \in T_v N(L)$. Then Z can be represented by a smooth variation α of the geodesic $c = e(\rho)$:

$$\alpha(s,t) = e(s, v(t)), \quad \text{for} \quad s,t \in R \quad \text{where} \quad Z = \left.\frac{d\alpha(s,t)}{ds}\right|_{s=0}.$$

In other words, $Z \in T_v N(L)$ defines a Jacobi field J along the geodesic c such that $\dot{c}(0) = v = p_{TM}(Z)$

$$v_1 = Tp_M(Z)$$
$$J' = v_2 = K(Z)$$

are the initial conditions and

$$J(1) = T_v e(Z), \quad \dot{c}(1) = T_v e(\dot{\rho}(1))$$

are also valid. The Jacobi equation $J'' + R(J, \dot{c})\dot{c} = 0$ implies that $g(\dot{c}, J'') = 0$ and so $g(\dot{c}, J) = c_1 t + c_2$ holds. Applying the definition of the metric $\tilde{g}$

$$\tilde{g}(\dot{\rho}(1), Z) = g(0, v_1) + g(v, v_2 - A_v(v_1)) = g(\dot{c}(0), J'(0))$$

is valid after a short computation. Substituting $t = 1$ and

$$c_1 = g(\dot{c}(0), J'(0)) = \tilde{g}(\dot{\rho}(1), Z),$$
$$c_2 = g(\dot{c}(0), J(0)) = 0,$$

we get the desired result

$$g(\dot{c}(1), J(1)) = \tilde{g}(\dot{\rho}(1), Z).$$

COROLLARY

Let L be a connected submanifold of a complete connected Riemannian manifold M. Let $e: N(L) \to M$ be the restricted exponential map. If $v \in N(L)$ is a non-zero vector then the kernel of the map $T_v e$ is orthogonal to $\dot{\rho}(1)$ where ρ is the ray defined by v.

Proof. Let $Z \in \operatorname{Ker} T_v e$, then $g(T_v e(\dot{c}(1), T_v e(Z))) = 0$. Using the above lemma we get the result.

4. Closest point of the cut locus to submanifold

Let $c: [0, a] \to M$ be a normal geodesic such that $c(0) = z \in L, \dot{c}(0) \in N_z L$. Let x be a cut point of L along c. If c' is another geodesic from the cut point x to a different point $z_1 \in L$ and orthogonal to L such that c' does not match smoothly with c at x, then x is not the closest cut point of L. This fact is stated in the following theorem.

Theorem 4.1. *Let M be a complete connected Riemannian manifold of class C^∞ and L be a compact connected submanifold of M of class C^∞. Let $e(v) = x \in M$, $v \in N(L)$ be a point of the cut locus $C(L)$ to L which is not a focal point of L. If c_1 and c_2 are two different minimizing geodesics from x to L such that they do not match smoothly at x. Then there is a cut point of L in any neighbourhood of x which is nearer to L than x.*

We have used the following lemma [8] in proving the theorem.

Lemma. Let M be a complete connected Riemannian manifold and L be a compact connected submanifold of M. Let $c: [0, a] \to M$ be a minimal geodesic from $c(a)$ to L. If c' is the part of c then c' minimizes the distance uniquely from its end point $c'(b)$ to the points of L for any value of the parameter $b < a$.

Proof of the theorem. For the sake of convenience the zero-section of the normal bundle $N(L)$ is identified with the submanifold L. Let $p_L: N(L) \to L$ be the projection map. Since $N(L)$ is a vector bundle, it is locally trivial, that is, every point y of the zero-section of $N(L)$ has a neighbourhood U such that $p_L^{-1}(U)$ is isomorphic with $U \times N_y L$. There is a Riemannian metric on $N(L)$ by basic results studied in § 3 and with respect to this metric $T_v e$ preserves the length of vectors tangent to the rays of $N(L)$ by lemma (3.1). Let $v \in N_y L$ be a non-zero vector where $y \in U$, then the locus of the end points of such v with fixed length will be the sphere of dimension $n - m - 1$. Consider with v the family of vectors of the same length as v in $p_L^{-1}(U)$ then corresponding to these vectors there is a union of the spheres which forms a piece of a hypersurface say K and the tangent space $T_v K$ at v is orthogonal to v which is proved as follows:

Let $\rho: [0, \infty) \to N(L)$ be a ray of $N(L)$ defined by $\rho(t) = t \cdot v$, $t \in [0, \infty)$ then $\dot{\rho}(1) = v$. Consider a geodesic variation $w_t(s), t \in [0, 1]$, $s \in (-\varepsilon, \varepsilon)$ of v considered as a geodesic in $N(L)$ such that $w_t = (0) = v$ and varied geodesics are of the same lengths and orthogonal to U. Then the variation vector field

$$W = \left. \frac{\partial w_t(0)}{\partial s} \right|_{t=1}$$

will be a Jacobi field tangent to the hypersurface K at v and is a non-zero field at v, since v is not a focal point of L in $N(L)$. This Jacobi field will be orthogonal to v at $t=1$. This process can be applied for any element of T_vK. Since each variation vector field which is an element of T_vK is orthogonal to v therefore T_vK will be orthogonal to v, i.e. for $X\in T_vK, \tilde{g}(\dot{\rho}(1), X)=0$ holds, where $\tilde{g}$ is the induced metric of $N(L)$. Now we define geodesic $c_1\colon [0,1]\to M$ such that $c_1(0)=z_1\in L, \dot{c}_1(0)\in N_{z_1}L, c_1(1)=x=e(v)$. Consider for c_1 a family of neighbouring geodesics each orthogonal to L then under the restricted exponential map e each member of this family is the image of non-zero vectors taken in $p_L^{-1}(U)$ corresponding to v. Since each vector in $p_L^{-1}(U)$ is of the same length as that of v, their images under the map e will be of the same lengths too. Since $e(v)=x$ is not a focal point of L in M, the image $e(K)$ will be a piece of hypersurface containing x in M. Since T_vK is orthogonal to $\dot{\rho}(1)$, therefore by lemma (3.1), $T_ve(T_vK)=T_{e(v)}e(K)$ will be orthogonal to $\dot{c}(1)$, i.e. for $Z=T_ve(x)\in T_{e(v)}e(K)$,

$$\dot{c}_1(1)=T_ve(\dot{\rho}(1)),$$

$$g(\dot{c}_1(1),Z)=g(T_ve(\dot{\rho}(1)),T_ve(X))$$

$$=\tilde{g}(\dot{\rho}(1),X)=0,$$

where g is the Riemannian metric of M. Consequently the hypersurface $e(K)$ is orthogonal to c_1. Similar result holds for the geodesic c_2 passing orthogonally through the point $z_2\in L$ to x. Since c_1 and c_2 do not match smoothly at x, the two tangent hyperplanes at x do intersect in the neighbourhood of x. From this there exists a point x' close to x and geodesics c_1' and c_2' neighbouring to c_1 and c_2 and shorter than them respectively such that c_i' join to L. But this is adequate to conclude that there is a cut point x'' to L closer to L than x. In fact if the length of either c_1' or c_2' is greater than the distance of x' to L, there must be a cut point on that geodesic even before it reaches x'. If length $c_1'=$ length $c_2'=$ distance of x' to L, then x' is a cut point. Now there are two possibilities:

(i) If $z_1\neq z_2$, then the minimizing geodesics from x' have different foot points. Consequently they are different. But then x' is a cut point of L nearer to L than x.

(ii) If $z_1=z_2$, then since c_1 and c_2 are minimizing geodesics and not matching smoothly at x, there will be two different subspaces of codimension 1 in T_xM each orthogonal to c_1 and c_2 respectively at x and both the subspaces intersect in the neighbourhood of x. Therefore there is a point x' in the neighbourhood of x such that there are two different minimizing geodesics from x' to L and x' is nearer to L than x. Consequently x' is a cut point of L nearer to L than x.

Theorem 4.2. *Let M be a complete connected Riemannian manifold and L be a compact connected submanifold of M. Let the cut locus of L be non-empty and let $e(v)$ be a closest point of the cut locus to L. If $e(v)$ is not a focal point of L. Then there are at most two different points of L which are at minimal distance from $e(v)$.*

Proof. Let $e(v)$ be not a focal point of L, then there are at least two different minimizing geodesics from L to the cut point $e(v)$. We have to show that there are at most two different points of L which are foot points of minimizing geodesics from L to $e(v)$. Let c_1 and c_2 be any two minimal geodesics through $e(v)$ to two different points z_1

and z_2 of L. If c_1 and c_2 do not match smoothly at $e(v)$, then there is a point x' which is nearer to L than $e(v)$. But then by theorem (4.1) the point x' will be a cut point of L, which contradicts the fact that $e(v)$ is the closest cut point of the submanifold L. Thus if c_1 and c_2 are two minimizing geodesics from two different points of L to $e(v)$ and $e(v)$ is the closest cut point of L then they match smoothly at $e(v)$. It remains to prove that there are at most two different points of L which are at minimal distance from $e(v)$. To prove this we assume that z_3 is a third foot point of a minimal geodesic passing through $e(v)$. But then neither minimizing geodesics c_1, c match nor c, c_2 match at $e(v)$ in consequence of the fact that c_1 and c_2 match smoothly at $e(v)$. Hence assertion of the theorem follows.

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