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Generalized epimorphism theorem

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Abstract. Let $R[X, Y]$ be a polynomial ring in two variables over a commutative ring R and let $F \in R[X, Y]$ such that $R[X, Y]/(F) = R[Z]$ (a polynomial ring in one variable). In this set-up we prove that $R[X, Y] = R[F, G]$ for some $G \in R[X, Y]$ if either R contains a field of characteristic zero or R is a seminormal domain of characteristic zero.

Keywords. Epimorphism theorem; polynomial ring; seminormal domain; characteristic zero.

1. Introduction

Let k be a field of characteristic zero. Let $k[X, Y]$ be a polynomial ring in two variables over k and $F \in k[X, Y]$ such that $k[X, Y]/(F) = k[Z]$ (a polynomial ring in one variable). In this set-up the famous epimorphism theorem of Abhyankar and Moh ([2], Theorem 1.2) says that $k[X, Y] = k[F, G]$ for some $G \in k[X, Y]$. Russell and Sathaye had obtained the following analog of the epimorphism theorem ([6], Theorem 2.6.2): If R is a locally factorial Krull domain of characteristic zero and $F \in R[X, Y]$ such that $R[X, Y]/(F) = R[Z]$, then $R[X, Y] = R[F, G]$. Therefore one asks the following natural question:

Is the foregoing result valid for an arbitrary commutative domain R of characteristic zero?

In this paper we answer this question affirmatively under the assumption that R is seminormal. We prove:

Theorem A. *Let R be a seminormal commutative domain of characteristic zero. Let I be an ideal of $R[X, Y]$ such that $R[X, Y]/I = R[Z]$. Then I is a principal ideal say generated by F and $R[X, Y] = R[F, G]$ for some $G \in R[X, Y]$.*

Moreover we give an example (Example 3.8) to show that I need not be principal if R is not seminormal. When R contains a field of characteristic zero we prove the following (weaker) epimorphism theorem:

Theorem B. *Let R be a commutative ring containing a field of characteristic zero. Let $F \in R[X, Y]$ such that $R[X, Y]/(F) = R[Z]$. Then $R[X, Y] = R[F, G]$ for some $G \in R[X, Y]$.*

2. Preliminaries

Throughout this paper all rings will be commutative.

In this section we set up notations and state some results for later use.

R will denote a commutative ring.

$R^{[n]}$: polynomial ring in n variables over R.

R^n: free R-module of rank n.

For a finitely generated R-algebra A,

$\Omega_{A/R}$: universal module of R-differentials of A.

For a prime ideal $\mathfrak{P}$ of R,

$k(\mathfrak{P})$: $R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}}$.

DEFINITION

A reduced ring R is said to be *seminormal* if it satisfies the condition: for b, $c\in R$ with $b^3=c^2$, there is an $a\in R$ with $a^2=b$, $a^3=c$.

Lemma (2.1). *Let R be a noetherian ring and let $s\in R$ be a non-zero divisor. Let M be a finitely generated R-module. If M_s is a projective R_s-module of rank d and M/sM is R/sR-projective of rank d then M is R-projective of rank d.*

Proof. Without loss of generality we can assume that R is local.

Since M/sM is R/s-projective and R is local there exists a surjective R-linear map β: $R^d\twoheadrightarrow M$ ($d=\text{rank } M/sM$). Let $N=\ker\beta$. Since M_s is R_s-projective of rank d and β is surjective we get $N_s=0$. But s is a non-zero-divisor of R and $N\subset R^d$, therefore $N_s=0\Rightarrow N=0$ and β is an isomorphism.

Lemma (2.2). *Let R be a noetherian ring and I be an ideal of $R^{[n]}$ such that $R^{[n]}/I\approx R^{[n-1]}$ as R-algebras. Then for an ideal $\mathfrak{G}$ of R, $I\cap\mathfrak{G}R^{[n]}=\mathfrak{G}I$. Moreover if I is a principal ideal of $R^{[n]}$ say generated by F, then*

(i) *F is a non-zero-divisor of $R^{[n]}$.*

(ii) *F is a algebraically independent over R,* i.e. $R[F]\approx R^{[1]}$.

(iii) $R[F]\cap\mathfrak{G}R^{[n]}=\mathfrak{G}R[F]$ *for any ideal* $\mathfrak{G}$ of R.

Proof. Since for any non-negative integer l, $R^{[l]}$ is a free R-module, the exact sequence $0\to I\to R^{[n]}\xrightarrow{\alpha}R^{[n-1]}\to 0$; α: R-algebra homomorphism of R-modules gives rise to the exact sequence

$$0\to I\bigotimes_R R/\mathfrak{G}\to R^{[n]}\bigotimes_R R/\mathfrak{G}\xrightarrow{\alpha\otimes_R 1_{R/\mathfrak{G}}}R^{[n-1]}\bigotimes_R R/\mathfrak{G}\to 0$$

proving that the canonical map $I/\mathfrak{G}I\to I+\mathfrak{G}R^{[n]}/\mathfrak{G}R^{[n]}$ of $R/\mathfrak{G}^{[n]}$-modules is an isomorphism. Hence $I\cap\mathfrak{G}R^{[n]}=\mathfrak{G}I$.

Now we assume that $I=(F)$.

(i) It is easy to see that $F\notin \mathfrak{m}R^{[n]}$ for any maximal ideal $\mathfrak{m}$ of R. This shows that F is a non-zero-divisor of $R^{[n]}$.

(ii) Suppose $c_0+c_1F+\cdots+c_rF^r=0$ where $c_i\in R\,\forall i$, $0\leqslant i\leqslant r$. Then $0=\alpha(c_0+c_1F+\cdots+c_rF^r)=c_0$ i.e. $F(c_1+c_2F+\cdots+c_rF^{r-1})=0$. Therefore, as by (i) F is a non-zero-

divisor, $c_1 + c_2F + \cdots c_rF^{r-1} = 0$ showing that $c_1 = 0$. Repeating this argument we see that $c_i = 0 \forall i, 0 \leqslant i \leqslant r$.
(iii) Let $\bar{F}$ be the image of F in $R/\mathfrak{G}^{[n]}$ $(= R^{[n]}/\mathfrak{G}R^{[n]})$. Then obviously $R/\mathfrak{G}^{[n]}/(\bar{F}) \approx R/\mathfrak{G}^{[n-1]}$. Therefore by (ii) $\bar{F}$ is algebraically independent over $R/\mathfrak{G}$ and hence $R[F] \cap \mathfrak{G}R^{[n]} = \mathfrak{G}R[F]$.

Lemma (2.3). *Let R be a noetherian ring and let $\mathfrak{G}$ be the nilradical of R. Let I be an ideal of $R^{[n]}$ such that $R^{[n]}/I \approx R^{[n-1]}$ as R-algebras. If $I/\mathfrak{G}I$ is a projective $R/\mathfrak{G}^{[n]}$-module of (constant) rank 1 then I is a projective $R^{[n]}$-module of (constant) rank 1.*

Proof. Since $\mathfrak{G}$ is nilpotent, the canonical map $\mathrm{Pic}(R^{[n]}) \to \mathrm{Pic}(R/\mathfrak{G}^{[n]})$ is an isomorphism. Therefore there exists a projective $R^{[n]}$-module L of constant rank 1 such that $L/\mathfrak{G}L \approx I/\mathfrak{G}I$. Hence there exists a $R^{[n]}$-linear map $\psi: L \to I$ such that the induced map $\bar{\psi}: L/\mathfrak{G}L \to I/\mathfrak{G}I$ is an isomorphism.

We claim that ψ is an isomorphism.

Surjectivity of ψ: Since $\bar{\psi}$ is an isomorphism, we have $I = \psi(L) + \mathfrak{G}I$. But $\mathfrak{G}$ is nilpotent and hence $I = \psi(L)$.

Injectivity of ψ: Let $M = \ker \psi$. Then we get the following exact sequence of $R^{[n]}$-modules:

$$0 \to M \to L \xrightarrow{\psi} I \to 0.$$

As in Lemma 2.2 we see that I is a projective R-module. Therefore the above exact sequence gives rise to the following exact sequence:

$$0 \to M/\mathfrak{G}M \to L/\mathfrak{G}L \xrightarrow{\bar{\psi}} I/\mathfrak{G}I \to 0.$$

But $\bar{\psi}$ is an isomorphism. Therefore $M/\mathfrak{G}M = 0$ i.e. $M = \mathfrak{G}M$. The nilpotency of $\mathfrak{G}$ shows that $M = 0$.

Thus ψ is an isomorphism.

3. Main theorems

In this section we prove Theorem A and Theorem B which are quoted in the introduction. For the proof of these theorems we need some lemmas and a proposition. Lemma 3.1 is well known but for the lack of a proper reference we give a proof.

Lemma 3.1. Let R be a noetherian ring and S be a noetherian R-algebra. Let $\pi \in R$ be such that S_π is a flat R_π-algebra and $S/\pi\, S$ is a flat $R/\pi\, R$-algebra. Moreover assume that $\mathrm{Tor}_1^R(S, R/\pi R) = 0$. *Then S is a flat R-algebra.*

Proof. Let M and N be finitely generated R-modules and let $f: M \to N$ be a R-linear injective map. Then we want to show that the map $f \otimes 1_S: M \otimes_R S \to N \otimes_R S$ is injective. Let $K = \ker(f \otimes 1_S)$.

Since S_π is R_π-flat we have $K_\pi = 0$. Let $T = 1 + \pi R$ and $T' = 1 + \pi S$. Then since $\mathrm{Tor}_1^{R_T}(S_{T'}, R_T/\pi R_T) = \mathrm{Tor}_1^R(S, R/\pi R) \otimes_S S_{T'} = 0$ and $R_T/\pi R_T = R/\pi R$, $S_{T'}/\pi S_{T'} = S/\pi S$, by ([1], Theorem 3.2, p. 91) $S_{T'}$ is flat over R_T and hence $K_{T'} = 0$.

Thus $K_{T'} = 0$, $K_\pi = 0$. Therefore $K = 0$ showing that S is a flat R-algebra.

Lemma 3.2. Let R be a noetherian ring of finite Krull dimension (denoted by dim R). *Let F be an element of $R[X, Y]$ such that $R[X, Y]/(F) = R[Z]$ as R-algebras. Then $R[X, Y]$ is a flat $R[F]$-algebra.*

Proof. Let $\mathfrak{G}$ be the nilradical of R. Since $R[F]$ (F being algebraically independent over R) and $R[X, Y]$ are flat over R, for every module M over R we have $\mathrm{Tor}_1^{R[F]}(R[X, Y], M \otimes_R R[F]) = 0$. In particular for every ideal J of R we have $\mathrm{Tor}_1^{R[F]}(R[X, Y], R[F]/JR[F]) = 0$. Therefore, since $\mathfrak{G}$ is nilpotent, by ([1], Theorem 3.2, p. 91) $R[X, Y]$ is flat over $R[F]$ if $R[X, Y]/\mathfrak{G}R[X, Y]$ $(= R/\mathfrak{G}[X, Y])$ is flat over $R[F]/\mathfrak{G}R[F]$. So it is enough to prove the result when R is a reduced ring.

We prove the result by induction on dim R. Without loss of generality we can assume that R is local.

If dim $R = 0$ then R is a field, $R[F]$ is a principal ideal domain and $R[X, Y]$ is a domain. Therefore $R[X, Y]$ is $R[F]$-flat.

Now we assume that dim $R > 0$. Let $\pi \in R$ be a nonunit non-zero-divisor of R. Let $\bar{F}$ denote the image of F in $R/(\pi)[X, Y]$. Then $R/(\pi)[\bar{F}] = R[F]/\pi R[F]$. Since $\dim R/(\pi) < \dim R$ and $\dim R_\pi < \dim R$, by the induction hypothesis $R_\pi[X, Y]$ is flat over $R_\pi[F]$ and $R[X, Y]/\pi R[X, Y]$ is flat over $R[F]/\pi R[F]$. Moreover $\mathrm{Tor}_1^{R[F]}(R[X, Y], R[F]/\pi R[F]) = 0$. Therefore by Lemma 3.1 $R[X, Y]$ is a flat $R[F]$-algebra.

Thus the proof of Lemma 3.2 is complete.

We state a definition before stating the next lemma.

DEFINITION

An element F of $R[X, Y]$ is called a *residual variable* if for every prime ideal $\mathfrak{Y}$ of R, $k(\mathfrak{Y})[X, Y] = k(\mathfrak{Y})[\bar{F}]^{[1]}$ where $\bar{F}$ denotes the image of F in $k(\mathfrak{Y})[X, Y]$.

Lemma 3.3. Let R be a ring and $F \in R[X, Y]$ be such that $R[X, Y]/(F) = R[Z]$ as R-algebras. Assume that F is a residual variable. Then for every prime ideal $\mathfrak{Y}'$ of $R[F]$, $k(\mathfrak{Y}') \otimes_{R[F]} R[X, Y] = k(\mathfrak{Y}')^{[1]}$.

Proof. Let $\mathfrak{Y}' \cap R = \mathfrak{Y}$. Then $\mathfrak{Y}R[F] \subset \mathfrak{Y}'$ and by (2.2) $\mathfrak{Y} R[F] = \mathfrak{Y}R[X, Y] \cap R[F]$. Since F is a residual variable, we have $k(\mathfrak{Y})[X, Y] = k(\mathfrak{Y})[\bar{F}]^{[1]}$ where $\bar{F}$ denotes the image of F in $k(\mathfrak{Y})[X, Y]$. Moreover $k(\mathfrak{Y})[\bar{F}] = k(\mathfrak{Y}) \otimes_R R[F]$ and there exists a $R[F]$-algebra homomorphism $k(\mathfrak{Y})[\bar{F}] \to k(\mathfrak{Y}')$. Therefore

$$k(\mathfrak{Y}') \bigotimes_{R[F]} R[X, Y] = k(\mathfrak{Y}') \bigotimes_{k(\mathfrak{Y})[\bar{F}]} k(\mathfrak{Y})[\bar{F}] \bigotimes_{R[F]} R[X, Y]$$

$$= k(\mathfrak{Y}')^{[1]}.$$

Lemma 3.4. Let R be a ring and $F \in R[X, Y]$ be such that $R[X, Y]/(F) = R[Z]$ as R-algebras. Assume that F is a residual variable. Then $\Omega_{R[X,Y]/R[F]}$ is a free $R[X, Y]$-module of rank one.

Proof. We have the following right exact sequence of $R[X, Y]$-modules

$$\Omega_{R[F]/R} \bigotimes_{R[F]} R[X, Y] \xrightarrow{\theta} \Omega_{R[X,Y]/R} \rightarrow \Omega_{R[X,Y]/R[F]} \rightarrow 0.$$

Since $\Omega_{R[X,Y]/R}$ is a free $R[X, Y]$-module of rank two with a basis $\mathrm{d}X, \mathrm{d}Y$ and $\mathrm{Im}\,(\theta) = N$ is the cyclic submodule generated by $F_X \mathrm{d}X + F_Y \mathrm{d}Y$ where $F_X = \partial F/\partial X$ and $F_Y = \partial F/\partial Y$, it is enough to show that the ideal $(F_X, F_Y) = R[X, Y]$.

Suppose $\mathfrak{N}$ is a maximal ideal of $R[X, Y]$ such that $(F_X, F_Y) \subset \mathfrak{N}$. Let $\mathfrak{N} \cap R = \mathfrak{P}$. Then replacing R by $R_{\mathfrak{P}}$ and $\mathfrak{N}$ by $\mathfrak{N}_{\mathfrak{P}}$ we can assume that R is a local ring with the maximal ideal m, $\mathfrak{N}$ is a maximal ideal of $R[X, Y]$ with $\mathfrak{N} \cap R = m$ and $(F_X, F_Y) \subset \mathfrak{N}$. But then, since F is a residual variable, we have $R[X, Y] = (F_X, F_Y) + mR[X, Y] \subset \mathfrak{N}$ which is absurd. Hence $(F_X, F_Y) = R[X, Y]$.

Lemma 3.5. Let R be a noetherian ring such that no prime integer is a zero-divisor in R. Let $F \in R[X, Y]$ be such that $R[X, Y]/(F) = R[Z]$ as R-algebras. Then F is a residual variable.

Proof. Let $\mathfrak{P}$ be a prime ideal of R and let $\bar{F}$ denote the image of F in $k(\mathfrak{P})[X, Y]$. Then $k(\mathfrak{P})[X, Y]/(\bar{F}) = k(\mathfrak{P})[Z]$.

If $ht\ \mathfrak{P} = 0$, then since no prime integer is a zero-divisor in R, $k(\mathfrak{P})$ is a field of characteristic zero. Therefore by the Abhyankar-Moh epimorphism theorem ([2], Theorem 1.2) $k(\mathfrak{P})[X, Y] = k(\mathfrak{P})[\bar{F}]^{[1]}$.

If $ht\ \mathfrak{P} > 0$ then there exists a discrete valuation ring V of characteristic zero with the uniformizing parameter π and a ring homomorphism $\alpha\colon R \rightarrow V$ such that $\alpha^{-1}((\pi)) = \mathfrak{P}$ and the field extension $k(\mathfrak{P}) \rightarrow V/(\pi)$ (induced by α) is algebraic.

Let $\tilde{F}$ denote the image (through α) of F in $V[X, Y]$. Then $V[X, Y]/(\tilde{F}) = V[Z]$. Therefore by ([6], Theorem 2.6.2) $V[X, Y] = V[\tilde{F}]^{[1]}$ and hence $V/(\pi)[X, Y] = V/(\pi)[\bar{\tilde{F}}]^{[1]}$ where $\bar{\tilde{F}}$ is the image of $\tilde{F}$ in $V/(\pi)[X, Y]$.

Since we have the following commutative diagram of rings

$$\begin{array}{ccc} R & \xrightarrow{\alpha} & V \\ \downarrow & & \downarrow \\ k(\mathfrak{P}) & \longrightarrow & V/(\pi) \end{array}$$

and $V/(\pi)$ is algebraic over $k(\mathfrak{P})$, by ([4], Proposition 1.16) $k(\mathfrak{P})[X, Y] = k(\mathfrak{P})[\bar{F}]^{[1]}$.

Thus we prove that F is a residual variable.

PROPOSITION 3.6.

Let R be a ring and I be an ideal of $R^{[n]}$ such that $R^{[n]}/I \approx R^{[n-1]}$ as R-algebras. Then I is a projective $R^{[n]}$-module of (constant) rank 1. Moreover if there exists a projective R-module L of rank 1 such that $L \otimes_R R^{[n]} \approx I$ as $R^{[n]}$-modules then I is a free $R^{[n]}$-module of rank 1 i.e. I is a principal ideal (necessarily generated by a non-zero-divisor of $R^{[n]}$).

Proof. It is easy to see that under the hypothesis of the proposition there exists a subring R' of R which is finitely generated over the ring of integers and an ideal I' of $R'^{[n]}$ such that $R'^{[n]}/I' \approx R'^{[n-1]}$ and $I = I'R^{[n]} \approx I' \otimes_{R'} R = I' \otimes_{R'^{[n]}} R^{[n]}$. Therefore for proving the first part of the proposition we can assume without loss of generality that R is noetherian of finite Krull dimension.

We prove the result by induction on dim R.

Let dim $R = 0$. By Lemma 2.3 we can assume that R is reduced. But then R is a finite product of fields and hence, since $R^{[n]}/I \approx R^{[n-1]}$, I is a principal ideal (of height 1) generated by a non-zero-divisor. Therefore I is a free $R^{[n]}$-module of rank 1.

Now we assume that dim $R > 0$. Again by Lemma 2.3 we can assume that R is reduced. Let S be the set of non-zero-divisors of R. Then R_S is a finite product of fields and as before we conclude that I_S is a free $R_S^{[n]}$-module of rank 1. Therefore $\exists$ $s \in S$ such that I_s is a free $R_s^{[n]}$-module of rank 1. We may assume that s is a nonunit of R.

Since $I \cap sR^{[n]} = sI$, $I/sI \approx I + sR^{[n]}/sR^{[n]}$ as $R/(s)^{[n]}$-modules. Therefore, since $R^{[n]}/(I + sR^{[n]}) \approx R/(s)^{[n-1]}$ and $\dim R/(s) < \dim R$, by the induction hypothesis I/sI is a projective $R/(s)^{[n]}$-module. Since s is a non-zero-divisor of R, $I \subset R^{[n]}$ and I_s (resp. I/sI) is a projective $R_s^{[n]}$-module (resp. $R/(s)^{[n]}$-module) of rank 1, by Lemma 2.1 I is a projective $R^{[n]}$-module of (constant) rank 1.

Now assume that there exists a projective R-module L of rank 1 such that $L \otimes_R R^{[n]} \approx I$ as $R^{[n]}$-modules.

Since $R^{[n]}/I \approx R^{[n-1]}$ as R-algebras, we get the following right exact sequence of $R^{[n-1]}$-modules:

$$I/I^2 \to \Omega_{R^{[n]}/R}/I\Omega_{R^{[n]}/R} \to \Omega_{R^{[n-1]}/R} \to 0.$$

Since, for non-negative integer l, $\Omega_{R^{[l]}/R}$ is a free $R^{[l]}$-module of rank l and I/I^2 is a projective $R^{[n-1]}$-module (as I is projective over $R^{[n]}$ of rank 1) of rank 1 we see that the above sequence is also left exact and

$$\Omega_{R^{[n]}/R}/I\Omega_{R^{[n]}/R} \approx \Omega_{R^{[n-1]}/R} \oplus I/I^2.$$

Thus I/I^2 is a stably free $R^{[n-1]}$-module of rank 1 and therefore I/I^2 is free over $R^{[n-1]}$ of rank 1.

Let θ: $R^{[n-1]} \to R$ be a surjective R-algebra homomorphism. Then composite map

$$R \to R^{[n]} \twoheadrightarrow R^{[n]}/I \approx R^{[n-1]} \overset{\theta}{\twoheadrightarrow} R$$

is the identity automorphism of R.

Since $L \otimes_R R^{[n]} \approx I$, we get

$$\begin{aligned} L &= L \bigotimes_R R^{[n]} \bigotimes_{R^{[n]}} R^{[n-1]} \bigotimes_{R^{[n-1]}} R \approx I \bigotimes_{R^{[n]}} R^{[n-1]} \bigotimes_{R^{[n-1]}} R \\ &= I/I^2 \bigotimes_{R^{[n-1]}} R. \end{aligned}$$

But I/I^2 is a free $R^{[n-1]}$-module of rank 1. Hence L is a free R-module of rank 1 and therefore I is a free $R^{[n]}$-module of rank 1 i.e. I is a principal ideal.

Thus the proof of Proposition 3.6 is complete.

Now we prove Theorem A.

Theorem 3.7. *Let R be a ring such that R_{red} is seminormal and no prime integer is a zero-divisor in R_{red}. Let I be an ideal of $R[X, Y]$ such that $R[X, Y]/I = R[Z]$ (as R-algebras). Then I is a principal ideal say generated by F and $R[X, Y] = R[F]^{[1]}$.*

Proof. Since R_{red} is seminormal by ([7], Theorem 6.1) $\operatorname{Pic}(R) = \operatorname{Pic}(R^{[n]})$ for every n. Therefore by Proposition 3.6 I is a principal ideal say generated by F.

Let $\mathfrak{G}$ be the nilradical of R and let $\bar{F}$ be the image of F in $R/\mathfrak{G}[X, Y]$. If $R/\mathfrak{G}[X, Y] = R/\mathfrak{G}[\bar{F}]^{[1]}$ then it is easy to see that $R[X, Y] = R[F]^{[1]}$. Therefore we can assume that R is reduced. It is also easy to see that there exists a subring S of R which is finitely generated over the ring of integers such that $F \in S[X, Y]$ and $S[X, Y]/(F) = S[Z]$ as S-algebras. Note that S is a noetherian ring of finite Krull dimension.

Since $S \hookrightarrow R$ and R is reduced, by the hypothesis of the theorem, no prime integer is a zero-divisor in S. Therefore F is a residual variable in $S[X, Y]$ by Lemma 3.5. Hence $\Omega_{S[X,Y]/S[F]}$ is a free $S[X, Y]$-module of rank one by Lemma 3.4. Moreover by Lemma 3.3, for every prime ideal $\mathfrak{P}'$ of $S[F]$, $k(\mathfrak{P}') \otimes_{S[F]} S[X, Y] = k(\mathfrak{P}')^{[1]}$. $S[X, Y]$ is a (finitely generated) flat $S[F]$-algebra by Lemma 3.2. Therefore by ([3], Lemma 3.3) there exists a positive integer m such that $S[X, Y]^{[m]} = S[F]^{[m+1]}$.

Now $S[X, Y]^{[m]} = S[F]^{[m+1]}$ implies that $R[X, Y]^{[m]} = R[F]^{[m+1]}$. Since R is seminormal (we have assumed R to be reduced) by ([5], Theorem 2.6) $R[X, Y] = R[F]^{[1]}$.

Thus the proof of Theorem 3.7 is complete.

The following example shows that if R_{red} is not seminormal then $R[X, Y]/I = R[Z]$ need not imply that I is principal.

Example 3.8. Let k be a field of characteristic zero and let $\tilde{R} = k[[t]]$: a power series in one variable over k. Let $R = k[[t^2, t^3]]$, considered as a subring of $\tilde{R}$. It is obvious that $\tilde{R}$ is the normalization of R and R is not seminormal.

Let α: $R[X, Y] \to R[Z]$ be the R-algebra homomorphism defined as: $\alpha(X) = Z + t^3Z^2$ and $\alpha(Y) = t^2Z$. Let $I = \ker \alpha$. Then

(1) α is surjective

(2) I is not a principal ideal of $R[X, Y]$.

Proof. Since $\alpha(X - t^3X^2 + t^2XY^2 + Y^3) = Z$, α is surjective.

Let $\tilde{\alpha}$: $\tilde{R}[X, Y] \to \tilde{R}[Z]$ be the $\tilde{R}$-algebra homomorphism such that $\tilde{\alpha}(X) = \alpha(X) = Z + t^3Z^2$ and $\tilde{\alpha}(Y) = \alpha(Y) = t^2Z$. Let $\tilde{I} = \ker \tilde{\alpha}$. Then $\tilde{I}$ is a principal prime ideal of $\tilde{R}[X, Y]$ generated by $F(X, Y) = t^2X - Y - tY^2$. Moreover $\tilde{I} = I\tilde{R}[X, Y] (= I \otimes_R \tilde{R})$.

If I is a principal ideal of $R[X, Y]$ say generated by H then $H = uF$ $(= u(t^2X - Y - tY^2))$ where u is a unit in R and $ut \in R$ i.e. $t \in R$ which is a contradiction.

Thus we prove that I cannot be principal.

We conclude this section with the proof of Theorem B.

Theorem 3.9. *Let R be a ring containing a field k of characteristic zero. Let $F \in R[X, Y]$ such that $R[X, Y]/(F) = R[Z]$ as R-algebras. Then $R[X, Y] = R[F]^{[1]}$.*

Proof. As in Theorem 3.7, we can assume that R is reduced and R contains a noetherian subring S of finite Krull dimension such that $F \in S[X, Y]$ and $S[X, Y]/(F) = S[Z]$ as S-algebras. Moreover we can assume that S contains k. Repeating the same arguments we see that there exists a positive integer m such that $S[X, Y]^{[m]} = S[F]^{m+1}$. Now since S contains k (a field of characteristic zero) by ([5], Theorem 2.8), $S[X, Y] = S[F]^{[1]}$. Hence $R[X, Y] = R[F]^{[1]}$.

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