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A descriptive version of Ambrose's representation theorem for flows

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Abstract. We prove here an analogue of Ambrose–Kakutani representation theorem for measurable flows. No measure is required and no points are dropped. This helps us to generalize a theorem due to Shelah and Weiss and answer a question due to A Ramsay.

Keywords. Measurable flow; flow built under a function; ergodic measures; Polish topology.

1. Introduction

Recently there has been an attempt to see how much of ergodic theory can be done in the setting of a general measurable structure. In [6], Weiss confined himself to the case of actions of $\mathbb{Z}$, i.e. the iterates of a single Borel automorphism of a standard Borel space. It is natural to see how many of the known things admit a generalization to a measurable setting in the case of actions of other groups.

In the case of action of $\mathbb{R}$, Ambrose [1] proved the existence of a prototype for ergodic flows preserving a measure, the so-called flow built under a function. This result has been extended to the case of non-singular flows by Dani [2] and Krengel [3]. However, in all these cases, the result is proved modulo the null sets of a measure on the space under consideration. We prove here that the flow built under a function is a prototype for all measurable flows satisfying a certain condition (see theorem 1). It turns out that every jointly measurable flow (on a standard Borel space) which does not admit a fixed point satisfies the condition of our theorem.

Some interesting consequences of our theorem are (i) every jointly measurable flow on a standard Borel space is a flow of homeomorphisms on a complete separable metric space (see theorem 2 and its corollary). This answers a question of Ramsay [5] for flows indexed by the real line. (ii) Every jointly measurable flow without fixed points admits, for each α, $0 \leqslant \alpha \leqslant 1$, a Borel set A_α such that every orbit spends precisely α proportion of time in A_α. (iii) Every jointly measurable flow (on a standard Borel space) whose orbit space does not admit a Borel cross-section is strictly ergodic (i.e. non-transitive and ergodic) with respect to some continuous finite measure on the Borel space. This extends to the case of $\mathbb{R}$ actions a result of Shelah and Weiss for Z-actions.

Since this paper was submitted for publication, [4] has been written which gives a connected account of ergodic theory in general measurable structures.

2.

Let $(X,\mathscr{B})$ be a standard Borel space. Let $T_t, t\in\mathbb{R}$ be a jointly measurable flow of Borel automorphisms on X, i.e. for each $t\in\mathbb{R}$, T_t is a Borel automorphism of X such that

(i) the map $(t,x)\to T_t x$ from $\mathbb{R}\times X\to X$ is measurable,
(ii) $T_0 x = x$ for all x,
(iii) $T_{t+s}x = T_t(T_s x)$.

(Here $\mathbb{R}$ is given the usual Borel structure and $\mathbb{R}\times X$ the product Borel structure.) The purpose of this paper is to prove the following theorem and give some of its applications.

Theorem 1. *Let $(X,\mathscr{B})$, $T_t, t\in\mathbb{R}$, be as above. Assume that there is a set $B\in\mathscr{B}$ such that for all $x\in X$ the sets $\{t\in\mathbb{R}: T_t x\in B\}$ and $\{t\in\mathbb{R}: T_t x\notin B\}$ have positive Lebesgue measure. Then there is a Borel set $\Omega_0\subseteq X$ such that for every $x\in X$ the set $\{t: T_t x\in\Omega_0\}$ is non-empty and discrete, hence either finite or countably infinite.*

Proof. The method of proof is adapted from Ambrose's original paper where a measure preserving flow on a finite measure space is shown to be isomorphic to a flow built under a function (see [1]).

(i) Let $A = X - B$ and let U denote the characteristic function of A. Write

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\int_0^\varepsilon U(T_t x)\,\mathrm{d}t.$$

Now for any $s\in\mathbb{R}$

$$\begin{aligned}\varphi_\varepsilon(T_s x) &= \frac{1}{\varepsilon}\int_0^\varepsilon U(T_t T_s x)\,\mathrm{d}t\\ &= \frac{1}{\varepsilon}\int_0^\varepsilon U(T_{t+s}x)\,\mathrm{d}t\\ &= \frac{1}{\varepsilon}\int_s^{s+\varepsilon} U(T_t x)\,\mathrm{d}t \to U(T_s x),\end{aligned}$$

as $\varepsilon\to 0$ for almost every s with respect to Lebesgue measure. This is because of Lebesgue's density theorem. By hypothesis, therefore, for every x, there exists s_1, s_2 such that $\varphi_\varepsilon(T_{s_1}x)\to 0$ and $\varphi_\varepsilon(T_{s_2}x)\to 1$ as $\varepsilon\to 0$.

(ii) The function $\xi(s) = \varphi_\varepsilon(T_s x)$, for fixed ε and x is Lipschitz with constant $2/\varepsilon$. Indeed

$$\begin{aligned}\xi(s)-\xi(l) &= \frac{1}{\varepsilon}\int_0^\varepsilon U(T_{t+s}x)\,\mathrm{d}t - \frac{1}{\varepsilon}\int_0^\varepsilon U(T_{t+l}x)\,\mathrm{d}t\\ &= \frac{1}{\varepsilon}\left(\int_s^{s+\varepsilon} U(T_t x)\,\mathrm{d}t - \int_l^{l+\varepsilon} U(T_t x)\,\mathrm{d}t\right)\\ &= \frac{1}{\varepsilon}\left(\int_s^{l} U(T_t x)\,\mathrm{d}t - \int_{s+\varepsilon}^{l+\varepsilon} U(T_t x)\,\mathrm{d}t\right).\end{aligned}$$

Since U is bounded by one we see that

$$|\xi(s)-\xi(l)| \leqslant \frac{2}{\varepsilon}|l-s|.$$

(iii) Let

$$A_n=\left\{x:\frac{1}{1/n}\int_0^{1/n} U(T_s x)<\tfrac{1}{4}\right\},$$

$$B_n=\left\{x:\frac{1}{1/n}\int_0^{1/n} U(T_s x)>\tfrac{3}{4}\right\}.$$

Let A_n^*, B_n^* be the smallest sets invariant under the flow and containing A_n and B_n respectively. Let $Q_n = A_n^* \cap B_n^*$. The sets A_n and B_n are Borel and from the manner in which they are defined we see that for any x the sets $\{t: T_t x \in A_n\}$, $\{t: T_t x \in B_n\}$ are open in $\mathbb{R}$. We see from this that $A_n^* = \cup T_r A_n$, $B_n^* = \cup T_r B_n$ where the unions are taken over r running over rational numbers. Thus A_n^*, B_n^* are also Borel sets. The sets A_n^*, B_n^* are invariant under the flow, hence Q_n is invariant under the flow . In view of the conclusion under 1, each $x \in X$ belongs to some Q_n. Write

$$\begin{aligned}
P_1 &= Q_1\\
P_2 &= Q_2 - Q_1\\
&\ \vdots\\
P_n &= Q_n - (Q_1 \cup Q_2 \cup \cdots \cup Q_{n-1})\\
&\ \vdots
\end{aligned}$$

The sets $P_1, P_2, P_3, \ldots$ are pairwise disjoint, measurable, invariant under the flow, and their union is all of X.

(iv) Let

$$\Omega_n^1=\left\{x:\varphi_{1/n}(x)=\tfrac{1}{2}, \varphi_{1/n}(T_s x)<\tfrac{1}{2} \text{ for } 0<s<\frac{1}{8n}\right\}$$

$$\Omega_n^2=\left\{x:\varphi_{1/n}(x)=\tfrac{1}{2}, \varphi_{1/n}(T_s x)>\tfrac{1}{2} \text{ for } 0<s<\frac{1}{8n}\right\}$$

$$\Omega_n=\Omega_n^1 \cup \Omega_n^2.$$

In view of the definition of Ω_n, it is clear that for any x the set $\{t: T_t x \in \Omega_n\} = E_n(x)$ is discrete; two points of $E_n(x)$ being separated by a distance at least $1/8n$. We show that the orbit of every $x \in P_n$ intersects Ω_n. Let $x \in P_n$. Then there exists s_1 and s_2 such that

$$\varphi_{1/n}(T_{s_1} x) < \tfrac{1}{4} \quad \text{and} \quad \varphi_{1/n}(T_{s_2} x) > \tfrac{3}{4}.$$

The function $\varphi_{1/n}(T_s x)$ as a function of s is Lipschitz, hence continuous. Hence there exists a largest s_0 between s_1 and s_2 such that $\varphi_{1/n}(T_{s_0} x) = \frac{1}{2}$. We show that $T_{s_0} x \in \Omega_n$. Now s_0 is strictly between s_1 and s_2. If $s_0 < s_1$ we see that

(i) $\varphi_{1/n}(T_s x) < \frac{1}{2}$ for $s_0 < s < s_1$ i.e.

$$\varphi_{1/n}(T_s T_{s_0} x) < \tfrac{1}{2} \quad \text{for } 0 < s < s_1 - s_0.$$

(ii) Since $\varphi_{1/n}(T_{s_0}x) = \frac{1}{2}$ and $\varphi_{1/n}(T_{s_1}x) < \frac{1}{4}$, we see that

$$\frac{1}{4} < |\varphi_{1/n}(T_{s_0}x) - \varphi_{1/n}(T_{s_1}x)| < 2n|s_0 - s_1|$$

i.e. $1/8n < s_1 - s_0$.

Thus (i) and (ii) show that $T_{s_0}x \in \Omega_n^1$, hence in Ω_n. If $s_0 < s_2$, then we see similarly that $T_{s_0}x \in \Omega_n^2$, hence in Ω_n. Thus orbit of x intersects Ω_n. Obviously the set $E_n(x) = \{t: T_t x \in \Omega_n\}$ is discrete. If we set

$$\Omega_0 = \bigcup_{n=1}^{\infty} (P_n \cap \Omega_n)$$

then Ω_0 is measurable and for any $x \in X$ the set $\{t: T_t x \in \Omega_0\}$ is non-empty and discrete. This proves the theorem.

3.

(i) We continue with the assumptions of theorem 1. Let $\Omega_0 \subseteq X$ be a Borel set such that for each $x \in X$ the set $E(x) = \{t: T_t x \in \Omega_0\}$ is non-empty and discrete. The subset of $\mathbb{R} \times X$ defined by $\{(t, x): T_t x \in \Omega_0\}$ is a Borel set such that each x section, is non-empty and discrete in $\mathbb{R}$. The functions

$$i(x) = \inf\{t: T_t x \in \Omega_0\},$$
$$j(x) = \sup\{t: T_t x \in \Omega_0\},$$

are then measurable functions. The sets

$$I = \{x: i(x) > -\infty\}$$
$$J = \{x: j(x) < \infty\}$$

are measurable and invariant under the flow. Further

$$i(T_t x) = i(x) - t \tag{1}$$

and

$$j(T_t x) = j(x) - t. \tag{2}$$

If we write $k(x) = T_{i(x)}x$ for $x \in I$ then $k(T_t x) = k(x)$ because of (1). The function $k: I \to I$ is thus measurable and constant on orbits. This means that if we restrict the flow to I then its orbit space admits a Borel cross-section. Thus on the set $I \cup J$ the orbit space of the flow admits a Borel cross-section. It is easy to see that on the set $K = I \cup J$, the flow is isomorphic to a flow built under a function, indeed, under a constant function.

(ii) Now let $\Omega = (X - K) \cap \Omega_0$. Define for $w \in \Omega$,

$$Sw = T_{s(w)}w,$$

where $s(w)$ is the smallest $t > 0$ such that $T^{s(w)}w \in \Omega$. Since $w \notin K, i(w) = -\infty$ and $j(w) = \infty$, so that S is one-one and onto and Borel as a function from Ω to Ω. We define $f(w) = s(w)$. Then it is clear that the flow T_t, $t \in \mathbb{R}$ is isomorphic to the flow built under the function f with base space Ω and base transformation S. Thus we have the following corollary of theorem 1.

COROLLARY

Every flow $T_t, t \in \mathbb{R}$ on a standard Borel space satisfying the assumptions of theorem 1 is isomorphic to a flow built under a function.

Remark. A result of Shelah and Weiss [6] states that if the orbit space of a Borel automorphism S on a standard Borel space does not admit a Borel cross-section then the space admits a continuous probability measure which is quasi-invariant and ergodic under S. It is easy to see that the corollary above permits us to extend this result to the case of jointly measurable flows on a standard Borel space.

4.

Let $(\Omega, \mathscr{B}_0)$ be a standard Borel space, $S:\Omega \to \Omega$ a Borel automorphism, and, $f:\Omega \to \mathbb{R}$ a positive measurable function such that for $w \in \Omega$ the sums $\sum_{k=0}^{\infty} f(S^k w)$ and $\sum_{k=0}^{\infty} f(S^{-k} w)$ are infinite. One can then define a flow $T_t, t \in \mathbb{R}$ on the set

$$X = \{(w,t): 0 \leqslant t < f(w)\}$$

the so-called flow built under the function f with base transformation S and base space Ω. The set X is given the Borel structure it inherits as a subset of $\Omega \times \mathbb{R}$.

A theorem proved in Weiss [6] and attributed by him to Ramsay and Mackey may be stated as follows:

Theorem. *Given a Borel automorphism S_1 on Ω and a countable collection τ of sets in $\mathscr{B}_0$, there is a Polish topology on Ω such that*

(i) *S_1 is a homeomorphism under this topology*
(ii) *every member of τ is open in this topology*
(iii) *the Borel sets generated by this topology from precisely the class $\mathscr{B}_0$.*

One can use this theorem to give a Polish topology $\mathscr{T}$ on Ω such that S is a homeomorphism under $\mathscr{T}, f$ is continuous, and, Borel sets of $\mathscr{T}$ agree with the class $\mathscr{B}_0$. Let Ω be given such a topology $\mathscr{T}$. We next equip $\Omega \times \mathbb{R}$ with product topology, (where $\mathbb{R}$ is given the usual topology). Let $\bar{X} = \{(w,t): 0 \leqslant t \leqslant f(w)\}$ be the closure of X in $\Omega \times \mathbb{R}$. Define $g:\bar{X} \to X$ by

$$\begin{aligned} g(w,t) &= (w,t) \quad \text{if } 0 \leqslant t < f(w) \\ &= (Sw, 0) \quad \text{if } t = f(w) \end{aligned}$$

and let $\mathscr{T}_1$ be the largest topology on X which makes g continuous. It is clear that the flow $T_t, t \in \mathbb{R}$ (built under f with base transformation S and base space Ω) is a jointly continuous flow of homeomorphisms under this topology.

PROPOSITION

$\mathscr{T}_1$ is Polish.

Proof. (Suggested by M S Raghunathan). There is no loss of generality if we assume that $f(w) = 1$ for all $w \in \Omega$. For otherwise the transformation

$$J(w, s) = (w, s/f(w))$$

from X onto $\Omega x[0, 1)$ will reduce the problem to the case where $f(w) = 1$ for all w.

Let Y denote the countable cartesian product of $\mathbb{R}$ with itself with product topology. Let $f: \Omega \to Y$ be a homeomorphic imbedding of Ω in Y, as is possible by Urysohn's metrization theorem. Let W denote the cartesian product $\prod_{i \in Z} Y_i$ where for each integer i, $Y_i = Y$.

Give W the product topology and imbed Ω into W by the map

$$\xi(x) = (f(S^n x))_{n \in Z}.$$

Then $\xi(Sx) = S_1(\xi(x))$ where S_1 denotes the backward shift on W, i.e. the nth co-ordinate of $S_1 w$ is the $(n+1)$th co-ordinate of w. Now Y and W are real linear spaces. Further they are closed sets in Y_c and W_c, where Y_c and W_c are formed from the complex numbers $\mathbb{C}$ the same way the spaces Y and W were formed from $\mathbb{R}$. For $0 \leqslant t < 1$ define the map $w \to w^t$ from $W \to W$ as follows:

If $w = (w_n)_{n \in Z}$, then the nth co-ordinate of w^t is given by $(w^t)_n = (1-t)w_n + tw_{n+1}$. Clearly $w^0 = w$ and $w^t \to S_1 w$ as $t \to 1$. Now consider the map $h: W \times [0, 1) \to W_c \times \mathbb{C}$ given by $h(w, t) = (w_t + (1 - \exp(2\pi i t))iw, 1 - \exp(2\pi i t))$. For each $t, h(., t)$ is a homeomorphism of W into $W_c \times \mathbb{C}$. Further h is one-one, continuous and satisfies

(i) $h(w, 0) = (w, 0)$ for all $w \in W$,

(ii) $\lim_{t \to 1} h(w, t) = (S_1 w, 0)$ for all $w \in W$.

The restrictions h to $W \times [0, a]$ and $W \times [a, 1)$ are even homeomorphisms provided $0 < a < 1$. The map $k(x, t) = h(\xi(x), t), (x, t) \in \Omega x[0, 1)$ is a homeomorphism of $\Omega \times [0, 1)$ onto its image under k provided $\Omega \times [0, 1)$ is equipped with topology $\mathscr{T}_1$. Now if we show that the image of $\Omega \times [0, 1)$ under the map k is G_δ in $W_c \times \mathbb{C}$, then it will follow that the topology $\mathscr{T}_1$ is Polish. However

$$k(\Omega \times [0, 1)) = k(\Omega \times [0, \tfrac{1}{2}]) \cup k(\Omega \times [\tfrac{1}{2}, 1)).$$

Each term on the right hand side is G_δ because k is a homeomorphism on $\Omega \times [0, \frac{1}{2}]$ and $\Omega \times [\frac{1}{2}, 1)$, where these sets are equipped with product topology. Thus the image under k of $\Omega \times [0, 1)$ is G_δ. q.e.d.

We have thus proved:

Theorem 2. *If T_t, $t \in \mathbb{R}$, is a jointly measurable flow on a standard Borel space $(X, \mathscr{B})$ and if this flow is isomorphic to a flow built under a function then X admits a Polish topology under which $T_t, t \in \mathbb{R}$ is a jointly continuous flow of homeomorphisms and the Borel sets of the topology form precisely the class $\mathscr{B}$.*

COROLLARY

Let $T_t, t\in\mathbb{R}$, be a jointly measurable flow on a standard Borel space $(X, \mathscr{B})$, then there is a Polish topology $\mathscr{T}$ on X such that

(i) *the flow $T_t, t\in\mathbb{R}$ is jointly continuous on $\mathbb{R}\times X$.*

(ii) *the Borel sets generated by $\mathscr{T}$ form precisely the class $\mathscr{B}$.*

Proof. Let F denote the fixed points of the flow and $Y = X - F$. The flow restricted to Y has no fixed points and is thus isomorphic to a flow built under a function by corollary of theorem 1 mentioned in § 2. We can apply theorem 2 to the restriction of the flow to Y. Further the flow when restricted to F is jointly continuous with respect to any topology on F. The corollary therefore follows.

This corollary answers a question raised by Ramsay [5] at least for flows indexed by the real line.

Remark 1. Every jointly measurable flow $T_t, t\in\mathbb{R}$ which does not admit a fixed point satisfies the condition of the theorem of § 1. To see this let $A_1, A_2, A_3, \ldots$ be a set of generators of $\mathscr{B}$. We first observe that for each $x\in X$, there is an A_i such that the sets $\{t: T_t x\in A_i\}$ and $\{t: T_t x\notin A_i\}$ both have positive Lebesgue measure. For otherwise, for every A_i, one of the sets $\{t: T_t x\in A_i\}$ and $\{t: T_t x\notin A_i\}$ has zero Lebesgue measure and the other has full Lebesgue measure. We write

$$C_i = \begin{cases} A_i^c & \text{if } \{t: T_t x\in A_i \text{ has zero Lebesgue measure}\} \\ A_i & \text{otherwise.} \end{cases}$$

Then for almost every t, $T_t x\in \bigcap_{i=1}^{\infty} C_i$. But the set $\bigcap_{i=1}^{\infty} C_i$ is either empty or a singleton set by virtue of the collection $A_1, A_2, A_3 \ldots$ being generators for $\mathscr{B}$. Thus x is a fixed point contrary to assumption. Now let

$$E_i = \{x: T_t x\in A_i \text{ for a } t \text{ set of positive Lebesgue measure.}\}$$

$$F_i = \{x: T_t x\notin A_i \text{ for a } t \text{ set of positive Lebesgue measure.}\}$$

The sets E_i, F_i are invariant sets under the flow $T_t, t\in\mathbb{R}$ and they are measurable. We let $G_i = F_i\cap E_i$ and set $H_1 = G_1$, $H_2 = G_2 - G_1, \ldots, H_n = G_n - \bigcup_{k=1}^{n-1} G_k \ldots$. The sets H_1 are invariant, disjoint and their union is all of X. Let $B = (A_1\cap H_1)\cup(A_2\cap H_2)\cup\cdots\cup(A_n\cap H_n)\cup\cdots$ Then B is in $\mathscr{B}$ and for every $x\in X$, the sets $\{t: T_t x\in B\}$ and $\{t: T_t x\notin B\}$ have Lebesgue measure positive. The flow $T_t, t\in\mathbb{R}$, without fixed points, thus satisfied the condition of theorem 1.

Remark 2. Using the isomorphism with a flow built under a function we can show that given a jointly measurable flow $T_t, t\in\mathbb{R}$ without fixed points and a real number α, $0\leqslant\alpha\leqslant 1$, there exists a measurable set B such that for every x the orbit of x spends α proportion of time in B, more precisely, $1/N$ Lebesgue measure

$$\{t: T_t x\in B, 0\leqslant t\leqslant N\}\to\alpha \quad \text{as } N\to\infty.$$

Remarks 1 and 2 were suggested by M G Nadkarni.

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