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The states of the character ring of a compact group

V S VARADARAJAN
Department of Mathematics, University of California, Los Angeles, California 90024, USA

Abstract. Deligne's generalization of the Hadamard–Vallée Poussin method in classical number theory is formulated as the representability of certain states of the character ring of a compact group, and the determination of all the representable states is carried out.

Keywords. Compact group; character ring; Hadamard–Vallée Poussin method; Deligne's generalization.

1. Introduction

The aim of this note is to make a few simple remarks on Deligne's generalization of the well-known method of Hadamard–de la Vallée Poussin in classical number theory ([2], § 2; see also the nice exposition by Moreno [4]). The Hadamard–de la Vallée Poussin method, as treated by Mertens, permits one to prove that the Riemann Zeta function and other L-functions do not vanish on the line $\mathrm{Re}(s) = 1$ (see [1]). Deligne's generalization leads to similar non-vanishing results for the L-functions that arise in non-classical situations. One may view the essential point in the proof of Deligne's main lemma as the assertion that a certain state of the complex character ring of a compact group is defined by an invariant measure on the group. This approach raises the questions that are discussed here.

Let G be a locally compact group and let $R_Z(G)$ be the ring of integral linear combinations of the irreducible finite dimensional unitary characters of G. Write $R(G) = \mathbf{C} \otimes R_Z(G)$. $R_Z(G)$ is canonically isomorphic to the Grothendieck ring of the virtual finite dimensional unitary representations of G; $R(G)$ is a commutative algebra over $\mathbf{C}$ with an involution $*(a \to a^{\mathrm{conj}})$. If G^B is the Bohr compactification of G, we have $R_Z(G) \cong R_Z(G^B)$ and $R(G) \cong R(G^B)$ canonically, so that one may always suppose G to be compact; we shall do so from now on. Then Deligne's generalization is a consequence of the following lemma (Lemma 2.1.5 of [2]):

Lemma 1. Let ν be an additive map of $R_Z(G)$ into $\mathbf{Z}$ with the following properties.

(a) $\nu(1) = 1$; $\nu(\tau) \leqslant 0$ *for all irreducible* $\tau \neq 1$,
(b) $\nu(\tau^*) = \nu(\tau)$,
(c) $\nu(\tau\tau^*) \geqslant 0$ *for all* $\tau \in R_Z(G)$.

Then $\nu(\tau) = 0$ for all irreducible τ, except possibly for a single $\tau = \tau_0$; such a τ_0 is necessarily of degree 1, $(\tau_0)^2 = 1$, and $\nu(\tau_0) = -1$.

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It is actually sufficient to require (c) only for unitary characters ξ, i.e., *actual*, not virtual characters ξ, but possibly reducible. Indeed, if this is so and $\xi \in R_Z(G)$ is arbitrary, we can write $\xi = \xi^1 - \xi^2$ where ξ^1, ξ^2 are characters of actual representations ρ^1, ρ^2 with no common irreducible constituent; the trivial representation does not occur in $\rho^1 \otimes (\rho^2)^{\text{conj}}$ and $(\rho^1)^{\text{conj}} \otimes \rho^2$, so that, using (a), we have

$$\nu(\xi\xi^*) = \nu(\xi^1\xi^{1*}) + \nu(\xi^2\xi^{2*}) - \nu(\xi^1\xi^{2*}) - \nu(\xi^2\xi^{1*}) \geqslant 0.$$

For finite G the proof is as follows. Taking ξ to be the character of the regular representation and using τ to denote irreducible characters, we get

$$0 \leqslant 1 + \Sigma_{\tau \neq 1} \deg(\tau)\nu(\tau) = 1 - \Sigma_{\tau \neq 1} \deg(\tau)|\nu(\tau)|.$$

So either $\nu(\tau) = 0$ for all irreducible $\tau \neq 1$ or else $|\nu(\tau_0)| = 1$ for exactly one $\tau_0 \neq 1$ with $\nu(\tau_0) = -1$, $\deg(\tau_0) = 1$; τ_0 then has to be real so that $(\tau_0)^2 = 1$. If G is infinite, the regular representation is infinite dimensional and one will have to modify this argument. We extend ν linearly to $R(G)$, write ν again for this extension, and note that ν is a *state* of the $*$-algebra $R(G)$:

$$\nu(\xi\xi^*) \geqslant 0, \quad \xi \in R(G).$$

One would expect that the state ν is defined by an invariant probability measure $p = p\nu$ on G:

$$\nu(\xi) = \int_G \xi \, \mathrm{d}p, \quad \xi \in R(G).$$

If we assume this, Deligne's argument may be understood as follows. For any invariant continuous function g let $g^\wedge$ denote its Fourier transform defined on the set of irreducible characters by

$$g^\wedge(\tau) = (g, \tau) = \int_G g\tau^* \, \mathrm{d}x.$$

One has $(g_1 * g_2)^\wedge(\tau) = (g_1)^\wedge(\tau)(g_2)^\wedge(\tau)\deg(\tau)^{-1}$. Let k_1 be any invariant continuous function on G which is $\geqslant 0$, symmetric, i.e., $k_1(x) = k_1(x^{-1})$, and normalized by the condition $\int_G k_1 \, \mathrm{d}x = 1$, and let $k = k_1 * k_1$. Then $(k_1)^\wedge$ is real and $k^\wedge = ((k_1)^\wedge)^2 \deg^{-1}$ is $\geqslant 0$. By the Peter–Weyl theorem, $\Sigma_\tau k^\wedge(\tau)\tau$ converges to k in $L^2(G)$. But,

$$\Sigma_\tau |k^\wedge(\tau)| \sup|\tau| \leqslant \Sigma_\tau k^\wedge(\tau)\deg(\tau) = \Sigma_\tau (k_1)^\wedge(\tau)^2 = \|k_1\|_2^2 < \infty$$

so that we actually have

$$\Sigma_\tau k^\wedge(\tau)\tau = k$$

in the topology of uniform convergence. Integrating this relation with respect to p we get

$$\Sigma_\tau k^\wedge(\tau)\nu(\tau) = \int_G k \, \mathrm{d}p \geqslant 0.$$

Since $\int_G k\,dx = 1$ and $\nu(\tau) \leqslant 0$ for $\tau \neq 1$, we get, for any finite set F of such τ's,

$$1 - \Sigma_{\tau\in F} k^{\wedge}(\tau) |\nu(\tau)| \geqslant 0.$$

Let us now take $k_1 = k_{1n}$ where the supports of the k_{1n} decrease to (e). Then, as $n \to \infty$, $k_n{}^{\wedge}(t) \to \deg(\tau)$ for each τ, and so we get

$$1 - \Sigma_{\tau\in F} |\nu(\tau)| \deg(\tau) \geqslant 0.$$

Since F is arbitrary, we get

$$\Sigma_{\tau\neq 1} |\nu(\tau)| \deg(\tau) \leqslant 1,$$

and the rest of the argument is the same as in the finite case.

This leads us to ask if states of a commutative $*$-algebra, especially $R(G)$, have integral representations. We discuss briefly the notion of a state and its representability in § 2. In § 3 we shall determine which states of $R(G)$ are representable by invariant measures on G and complete the above argument by proving the representability of ν. We shall also give an example of a state of $R(\mathrm{SU}(2))$ which is not representable by an invariant measure on SU(2), but which is nevertheless representable by a measure in $\mathrm{SL}(2, \mathbf{C})$, and we shall formulate a general result for all compact Lie groups that subsumes this example. We also discuss briefly the case when one deals with compact groups that are not Lie.

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2. Commutative $*$-algebras and their states

Let $\mathbf{A}$ be a commutative complex $*$-algebra, i.e. a commutative algebra with unit over the field $\mathbf{C}$ of complex numbers with an involution $*(a^{**} = a$, $1^* = 1$, $(ab)^* = a^*b^*$, and $a \to a^*$ is conjugate linear). A linear map $\lambda(\mathbf{A} \to \mathbf{C})$ is a *state* if $\lambda(1) = 1$ and $\lambda(aa^*) \geqslant 0$ for all $a \in \mathbf{A}$. The notion of a state of course makes sense even if $\mathbf{A}$ is non-commutative; the terminology and motivation come from quantum mechanics. It is standard that $\lambda(a^*) = \lambda(a)^{\mathrm{conj}}$ and that $(a, b)_\lambda := \lambda(ab^*)$ is a non-negative definite Hermitian scalar product for $\mathbf{A}$. If $\mathbf{A}$ is a $\mathbf{C}*$-algebra, i.e. a Banach algebra such that $\|aa^*\| = \|a\|^2$ for all $a \in \mathbf{A}$, then $\mathbf{A}$ is isomorphic to the $\mathbf{C}*$-algebra of all complex valued continuous functions on the Gel'fand spectrum of $\mathbf{A}$; it is then easy to show that there is a natural one-one correspondence $\lambda \leftrightarrow p\lambda$ between states λ and probability measures $p\lambda$ on the Gel'fand spectrum given by $\lambda(a) = \int a\, dp\lambda (a \in \mathbf{A})$ [3]. However, as the example of $R(G)$ shows, one often has to work with $*$-algebras $\mathbf{A}$ that are neither complete nor even normed; and it is natural to ask how much of the above integral representation remains valid in the more general contexts.

Given $\mathbf{A}$ and a state λ on $\mathbf{A}$ let $N_\lambda = \{a | (a, b)\lambda = 0$ for all $a \in \mathbf{A}\}$; then N_λ is an ideal in $\mathbf{A}$ and $(.,.)\lambda$ defines the structure of a pre-Hilbert space on $\mathbf{A}/N_\lambda$; let $\mathscr{H}_\lambda$ be the completion of $\mathbf{A}/N_\lambda = \mathscr{H}_\lambda{}^0$. For each $a \in \mathbf{A}$ multiplication by a defines a linear operator $\pi|(a)$ on $\mathscr{H}_\lambda{}^0$; the map $a \to \pi\lambda(a)$ is a $*$-representation and $\pi\lambda(a^*)$ is the adjoint of $\pi\lambda(a)$ in the sense that $(\pi\lambda(a)b, c)\lambda = (b, \pi\lambda(a^*)c)\lambda$ for all $b, c \in \mathbf{A}$. Guided by the theory of $*$-representations we now make the following definition ([3] p. 99).

DEFINITION 1.

A state λ of **A** is called *continuous* if the $\pi\lambda(a)$ extend to bounded operators on $\mathscr{H}_\lambda$ for each $a \in \mathbf{A}$.

Obviously, if λ is continuous, the $\pi\lambda(a)$ for self-adjoint a ($a^* = a$) form a set of commuting self-adjoint operators. This suggests making a second definition.

DEFINITION 2.

A state λ of **A** is called *regular* if for each self-adjoint $a, \pi\lambda$ is essentially self-adjoint on $\mathscr{H}_\lambda{}^0$, and the self-adjoint operators corresponding to different elements of **A** commute strongly.

Strong commutativity means the commutativity of the spectral projections or equivalently, the commutativity of the unitary groups generated by the self-adjoint extensions.

Let Spec(**A**) be the spectrum of **A**, namely the set of homomorphisms of **A** into **C**; it has an involution $h \to h^*$ given by $h^*(a) = h(a^*)^{\text{conj}}$; the fixed points under this involution form the set $\text{Spec}(\mathbf{A})_{\mathbf{R}}$ of $*$-homomorphisms of **A**. We equip $\text{Spec}(\mathbf{A})_{\mathbf{R}}$ (and Spec(**A**)) with the smallest σ-algebra of sets with respect to which the evaluation maps $h \to h(a)$ ($a \in \mathbf{A}$) are all Borel. It is then natural to make the following definition.

DEFINITION 3.

A state λ of **A** is called *representable* if there exists a probability measure p on $\text{Spec}(\mathbf{A})_{\mathbf{R}}$ such that

$$\lambda(a) = \int_{\text{Spec}(\mathbf{A})_{\mathbf{R}}} a \, \mathrm{d}p, \quad (a \in \mathbf{A}).$$

If p is unique (resp. has compact support), λ is called *uniquely representable* (resp. *compactly representable*).

The mutual relationships among these definitions are generally known and are summarized below. Let λ be a state of **A**.

PROPOSITION 4.

(a) *If λ is regular, it is uniquely representable. If it is continuous it is regular.*
(b) *Suppose for each $a \in \mathbf{A}$ there is a number $t(a) > 0$ such that*

$$\Sigma_{n \geq 0} t(a)^n |\lambda(a^n)|/n! < \infty.$$

Then λ is regular.
(c) *λ is continuous $\Leftrightarrow \lambda$ is compactly representable $\Leftrightarrow$ for each $a \in \mathbf{A}$ there is a constant $L(a) \geq 1$ such that $|\lambda(a^n)| \leq L(a)^n$ for all $n \geq 0$.*
(d) *If **A** has a single self-adjoint generator, all of its states are representable.*

I do not prove this result since the proof involves only standard aspects of the spectral theory of self-adjoint operators. If **A** is the polynomial algebra the representability problem is just the classical problem of moments. It is classical that there is always a solution when the number of variables is 1 (this leads to (d)). I do not know whether the states of an arbitrary countably generated algebra are always representable; if this is so, it would be of interest to examine the nature of the set of extreme points of the convex set of representing measures for a given state.

3. States and spectrum of $R(G)$

We have

Theorem 1. *Let G be a compact group and λ a state of $R(G)$. The following are equivalent:*
(a) *There are constants $C>0, r\geqslant 0$ such that*

$$|\lambda(\tau)|\leqslant C\deg(\tau)^r$$

for all irreducible characters τ of G.
(b) *There is an invariant probability measure p on G such that*

$$\lambda(a)=\int_G a\,dp,\quad a\in R(G).$$

If these conditions are satisfied, p is unique and we have the estimate

$$|\lambda(\tau)|\leqslant \deg(\tau),\quad \tau \text{ irreducible}.$$

COROLLARY 2.

The state ν of Lemma 1.1 satisfies the above estimate and is hence represented by a unique invariant probability measure on G.

Proof. We begin with the Corollary. If $\tau\neq 1$ is an irreducible character,

$$\tau\tau^* = 1+\Sigma_j m_j\tau_j.$$

where the m_j are integers >0 and $\tau_j\neq 1$ are irreducible characters; and we have

$$0\leqslant \nu(\tau\tau^*) = 1-\Sigma_j m_j|\nu(\tau_j)|\leqslant 1.$$

Consequently,

$$|\nu(\tau)|^2 = |(\tau,1)_\nu|^2\leqslant(\tau,\tau)_\nu=\nu(\tau\tau^*)\leqslant 1$$

showing that the estimate in question is trivially true for ν.

We now take up the proof of the theorem. The non-trivial part is the implication (a)⇒(b). The uniqueness of p follows from the fact that $R(G)$ is dense in the Banach space of invariant continuous functions on G and $a\rightarrow\int_G a\,dp$ is a continuous linear functional.

It is enough to prove (a)⇒(b) when G is a Lie group. For, if this is done, we have unique invariant measures pH on the Lie quotients H of G representing the restriction of λ to $R(H)\subset R(G)$; and the uniqueness of pH shows that the system $\{(H,pH)\}$ is coherent, leading in a standard fashion to the required pG as $\lim_H pH$.

For a Lie group G let $\mathfrak{g} = \mathrm{Lie}(G)$ and let $X_1, \ldots, X_n$ be a basis for $\mathfrak{g}$, orthonormal with respect to the form $(X, Y) \to \mathrm{tr}(\mathrm{ad}X\,\mathrm{ad}Y)$ $(X, Y \in \mathfrak{g})$. Let the element $\square$ of the enveloping algebra of $\mathfrak{g}$ be given by

$$\square = 1 - (X_1^2 + \cdots + X_n^2).$$

Then $\square$ is independent of the choice of the basis and is invariant under the action of G. We shall now view $\square$ in the usual manner as a differential operator on G. If π is an irreducible representation of G, $\pi(\square)$ is a scalar depending only on the character τ of π, say $c(\tau)$. Clearly $c(\tau) \geqslant 1$. We then have the following ([5], pp. 316–318; (ii) may also be proved by Sobolev estimates for elliptic operators on compact manifolds). (i) There are constants $C > 0, r, s \geqslant 0$ such that

$$\deg(\tau) \leqslant Cc(\tau)^r, \Sigma_\tau c(\tau)^{-s} < \infty,$$

where τ varies over the set of irreducible characters of G.
(ii) The collection of seminorms

$$C^\infty(G) \ni f \to \| \square^r f \|_2 = \left(\int_G | \square^r f|^2 \, dx \right)^{1/2}$$

for $r = 0, 1, 2, \ldots$, already induces the topology of $C^\infty(G)$.
It follows from this that for f in $C^\infty(G)$,

$$\Sigma_\tau f^\wedge(\tau)\tau = f^\#:= \int_G \mathrm{Ad}(u) f \, du,$$

the series converging absolutely in $C^\infty(G)$.

The key to the proof is to show that the estimates for λ permit us to extend λ uniquely to $C^\infty(G)$ as a *distribution* in the sense of L. Schwartz on G. Indeed, the estimates (a) shows that for some $m \geqslant 0$, $\lambda(\tau) = O(c(\tau)^m)$. Since $f^\wedge(\tau) = O(c(\tau)^{-q})$ for any $q \geqslant 0$, for any $f \in C^\infty(G)$, we can define

$$T(f) = \Sigma_\tau \lambda(\tau) f^\wedge(\tau) \quad (f \in C^\infty(G)).$$

Obviously $T = \lambda$ on $R(G)$.
It follows from the estimate

$$|f^\wedge(\tau)| \leqslant c(\tau)^{-q} \| \square^q f \|_2$$

that

$$|T(f)| \leqslant C \| \square^m f \|_2$$

showing that T is an invariant distribution on G. Since T is an extension of the state λ, it is clear that $T(ff^*) \geqslant 0$ for all invariant $f \in C^\infty(G)$, hence $T(g) \geqslant 0$ for all non-negative $g \in C^\infty(G)$, as T is invariant. But then T is a measure, by a classical result.

Example 1. Let $G = \mathrm{SU}(2)$, $G_{\mathbb{C}} = \mathrm{SL}(2, \mathbb{C})$ and let T (resp. $T_{\mathbb{C}}$) be the diagonal subgroup of G (resp. $G_{\mathbb{C}}$). The irreducible characters τ of G satisfy $\tau = \tau^*$ and so a homomorphism $\lambda(R(G) \to \mathbb{C})$ is a state as soon as $\lambda(\tau)$ is real for all τ. The irreducible characters of G

are the τ_n $(n \geqslant 0)$ where

$$\tau_n(\operatorname{diag}(z, z^{-1})) = z^n + z^{n-2} + \cdots + z^{-n}.$$

Note that the elements of $R(G)$ are (holomorphic) functions on $G_{\mathbf{C}}$ and so evaluations at points of $G_{\mathbf{C}}$ define homomorphisms of $R(G)$. Since $\deg(\tau_n) = n + 1$, the estimate in Theorem 1 becomes $\lambda(\tau_n) = O(n^k)$ for some $k \geqslant 0$. Let r be a real number > 1 and let $g(r) = \operatorname{diag}(r, r^{-1})$. Then $\lambda(r)(\xi \to \xi(g(r)))$ is a state; but

$$|\lambda(r)(\tau_n)| = r^n + \cdots + r^{-n} \geqslant r^n + r^{-n}$$

and so $\lambda(r)$ cannot be represented by an invariant measure on G.

The point of this example is that $\operatorname{Spec} R(G)_{\mathbf{R}}$ is *not* exhausted by the conjugacy classes of G. Indeed, if we identify $R(G)$ with $\mathbf{C}[z, z^{-1}]^W$ where W is the reflexion $z \to z^{-1}$, we see that $\operatorname{Spec}(R(G)) = W\backslash\mathbf{C}$ and so $\operatorname{Spec}(R(G))$ gets identified with the set $\{r | r > 1\} \amalg (W\backslash\mathbf{T})$ ($\mathbf{T}$ = unit circle); only the points of $\mathbf{T}$ correspond to conjugacy classes in G.

The well-known representation theory of semi-simple groups can be used to show that essentially all aspects of this example generalize to the case where G is compact, semi-simple and simply connected. We may then view G as a real form of the complex group $C_{\mathbf{C}}$ whose Lie algebra $\mathfrak{g}_{\mathbf{C}}$ is the complexification of $\mathfrak{g} = \operatorname{Lie}(G)$. Let T be a maximal (topological) torus of G and $T_{\mathbf{C}}$ the maximal (algebraic) torus of $G_{\mathbf{C}}$ containing T. The compact group G defines the conjugation $\xi \to \xi^*$ in $R(G)$. Let

$$T_{\mathbf{R}} = \{t \in T_{\mathbf{C}} | \xi^*(t) = \xi(t)^* \text{ for all } \xi \in R(G)\}.$$

If W is the Weyl group of $(G_{\mathbf{C}}, T_{\mathbf{C}})$, it is obvious that T, $T_{\mathbf{C}}$ and $T_{\mathbf{R}}$ are all invariant under W. We have $T \subset T_{\mathbf{R}}$, but in general, $T \neq T_{\mathbf{R}}$. We select a positive chamber in $T_{\mathbf{C}}$ and parametrize the irreducible characters of G by τ_μ, μ a highest weight. We write $o(\mu)$ for the order of μ. The following theorem is then not difficult to establish.

Theorem 3. *A state of λ of $R(G)$ is continuous if and only if for some constant $D \geqslant 1$,*

$$|\lambda(\tau_\mu)| \leqslant D^{o(\mu)}$$

for all highest weights μ. The continuous states are precisely those uniquely represented as integrals with respect to compactly supported W-invariant probability measures over $T_{\mathbf{R}}$.

From the point of view of conjugacy classes, we know that $W\backslash T_{\mathbf{C}}$ is the space of all semi-simple conjugacy classes of $G_{\mathbf{C}}$. In the previous theorem we may actually assume that G is any compact Lie group, connected or not. Then $G_{\mathbf{C}}$ is defined as the spectrum of the algebra which is the (Hopf) algebra $H(G)$ of matrix coefficients of G. If we drop the assumption that G is Lie, $H(G)$ and $G_{\mathbf{C}}$ can still be defined and $G_{\mathbf{C}}$ becomes a proalgebraic group. It will be interesting to describe $\operatorname{Spec}(R(G))$ and $\operatorname{Spec}(R(G))_{\mathbf{R}}$ in terms of semi-simple conjugacy classes in this situation.

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