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Multiplicative properties of the partition function

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Abstract. A lower bound for the number of multiplicatively independent values of $p(n)$ for $N \leqslant n < N + R$ is given. The proof depends on the Hardy-Ramanujan formula and is of an elementary nature.

Keywords. Partition function; Hardy-Ramanujan formula; multiplicative independence of integers.

1. Introduction

P Erdös and A Ivić [1] in their study of the number of non-isomorphic Abelian groups of a given order needed a lower estimate for the number $m(N)$ of multiplicatively independent values of the partition function $p(n)$ in $1 \leqslant n \leqslant N$. The first named author has proved (see [1], lemma 2) that, denoting the number of prime devisors of n by $\omega(n)$, one has

$$\omega\left(\prod_{n=1}^{N} p(n)\right) \to \infty,$$

whence $m(N) \to \infty$, as $N \to \infty$.

In the present paper we give an explicit estimate for $m(N)$, at the same time eliminating from the proof the appeal to "linear forms in logarithms". What we actually treat is the number $m(N, N + R)$ of multiplicatively independent values of $p(n)$ in $N \leqslant n < N + R$, where R is relatively small compared to N.

2. Theorem

There is an N_0 *such that*

$$m(N, N + R) \geqslant R \frac{\log N - \log R}{\frac{3}{2}\log N + R \log 2}$$

for $N \geqslant N_0$ *and all* $R \in \mathbb{N}$.

The same lower bound applies to

$$\omega\left(\prod_{N\leqslant n<N+R} p(n)\right)$$

which is, of course, $\geqslant m(N, N+R)$.

It will be seen that, given N, our bound decreases as a function of R for $R \geqslant \log^2 N$, more precisely if $R \log 2 > 2/3 \log N(\log N - 1 - \log R)$. Thus in the proof and in applications like corollary 2 below there is no need to consider larger R.

COROLLARY 1

$$m(N, N+R) \geqslant (\tfrac{2}{3} - o(1))R \quad \textit{if } R = o(\log N) \textit{ as } N \to \infty.$$

COROLLARY 2

$$m(N, N+R) \geqslant \left(\frac{1}{\log 2} - o(1)\right)\log N \quad \textit{if } \frac{R}{\log N} \to \infty \textit{ as } N \to \infty.$$

In particular we may state that

$$\omega\left(\prod_{1}^{N} p(n)\right) \geqslant m(N) \geqslant (1-\varepsilon)\frac{\log N}{\log 2} \quad \textit{if } N \geqslant N_0(\varepsilon).$$

Concerning the paper of Erdös and Ivić we have

COROLLARY 3

Let $a(n)$ be the number of non-isomorphic Abelian groups of order n and $C(x)$ the number of distinct values of $a(n)$ for $n \leqslant x$. Then for every $\varepsilon > 0$ and $x \geqslant x_1(\varepsilon)$

$$\log C(x) \geqslant \frac{(\log\log x)^2}{\log 16 + \varepsilon}.$$

Similarly for the number $D(x)$ of distinct values $a(n) \leqslant x$ with any n one has, if $x \geqslant x_2(\varepsilon)$,

$$\log D(x) \geqslant \frac{(\log\log x)^2}{\log 4 + \varepsilon}.$$

Proof of corollary 3. Notation as in [1]. Our corollary 2 allows to pick $p(k_i)$ multiplicatively independent, $k_1, \ldots, k_t \leqslant \sqrt{\log x}$, with $t(\log 4 + \varepsilon) \sim \log\log x$. The construction gives $C(x) \geqslant r^t$ where $2rt \log(rt) \leqslant \sqrt{\log x}$. Here $r = [(\log x)^{1/2-\varepsilon}]$ is admissible if $x \geqslant x_1$, which gives our proposition.

For the estimate of $D(x)$ take $k_1, \ldots, k_t \leqslant \log x$ with $t(\log 2 + \varepsilon) \sim \log\log x$. Now $D(x) \geqslant r^t$ upon the condition that $\sum_i r \log p(k_i) \leqslant \log x$, which because of $\log p(k) \ll \sqrt{k} \leqslant \sqrt{\log x}$ is again satisfied by $r \sim (\log x)^{1/2-\varepsilon}$.

3. The proof uses the Hardy-Ramanujan formula for $p(n)$ (see (1) below) to construct a large number of distinct linear combinations of the numbers $\log p(n)$ with bounded integral coefficients. If, on the other hand, the dimension of this $\mathbb{Z}$-module were too small it would contain too few elements with bounded height. This mechanism is rather unspecific. Thus the theorem as it stands applies to any function $q(n) = p(n) + O[\exp(c\sqrt{n})]$ where $c < \pi\sqrt{2/3}$, and similar results can be proved whenever an arithmetic function $q(n)$ allows an expansion

$$\log q(n) = a_0 n^{\alpha_0} + a_1 n^{\alpha_1} + \cdots + b \log n + \text{remainder},$$

where $a_0 > 0, \alpha_0 > 0, \alpha_0 \notin \mathbb{N}, \alpha_0 > \alpha_1 > \cdots, \alpha_i - \alpha_{i+1} \gg 1$, and where the remainder term is small enough. The factor 2/3 in corollary 2 would become $(\alpha_0 + 1)^{-1}$.

Lemma 1. Write Δ *for the forward difference and put*

$$\Delta_r := |\Delta^r \log p(N)|.$$

Then, as $N \to \infty$, *we have*

$$\Delta_r = c_2 r! \left| \binom{1/2}{r} \right| N^{1/2 - r}(1 + O(N^{-1/3}))$$

with some constant c_2, *uniformly in* $0 \leqslant r \leqslant N^{1/6}$.

Proof. The Hardy–Ramanujan formula (see [2]) gives

$$p(n) = c_1 f(n - \tfrac{1}{24}) + O[\exp(c_3\sqrt{n})], \tag{1}$$

where

$$f(x) := \left(\frac{1}{\sqrt{x}} \exp(c_2\sqrt{x}) \right)', \quad c_2 > c_3 > 0.$$

The actual values

$$\left(c_1 = \frac{1}{\pi\sqrt{8}}, c_2 = \pi\sqrt{2/3}, c_3 = \frac{c_2}{2} \right)$$

are mostly irrelevant for our purpose. Keeping the abbreviation $x = n - 1/24$ we find

$$p(n) = \frac{c_1}{2x}\left(c_2 - \frac{1}{\sqrt{x}} \right) \exp(c_2\sqrt{x}) + O[\exp(c_3\sqrt{x})]$$

$$= \frac{c_1}{2x}\left(c_2 - \frac{1}{\sqrt{x}} \right) \exp(c_2\sqrt{x})(1 + O[\exp(-c_4\sqrt{x})])$$

with some $c_4 > 0$. Therefore

$$\log p(n) = g(x) + O[\exp(-c_4\sqrt{n})],$$

where

$$g(x) = c_2\sqrt{x} + \log\frac{c_1 c_2}{2} + \log\left(1 - \frac{1}{c_2\sqrt{x}} \right) - \log x$$

—apart from the $\log x$ term—is a power series that converges for all large x,

$$g(x) = \sum_{i=-1}^{\infty} a_i x^{-i/2} - \log x.$$

Actually

$$a_{-1} = c_2, \quad |a_i| \leqslant c_2^{-i} \quad \text{for } i \geqslant 1.$$

The case $r = 0$ of the lemma is now obvious.

For $r \geqslant 1$ the generalized mean-value theorem gives

$$\Delta^r \log p(N) = g^{(r)}(\xi) + O(2^r \exp(-c_4\sqrt{N})),$$

where $N - \frac{1}{24} < \xi < N + r$. Hence for $r \geqslant 1$

$$\Delta^r \log p(N) = r! \sum_{\substack{i=-1 \\ i \neq 0}}^{\infty} a_i \binom{-i/2}{r} \xi^{-i/2-r} + (-1)^r (r-1)! \xi^{-r}$$
$$+ O(2^r \exp(-c_4\sqrt{N})).$$

If $i \geqslant 1$, $r \geqslant 1$, then

$$\binom{-(i+1)/2}{r}\binom{-i/2}{r}^{-1} = \prod_{j=0}^{r-1} \frac{\frac{i+1}{2}+j}{\frac{i}{2}+j}$$

$$< \prod_{0}^{r-1} \frac{\frac{i}{2}+1+j}{\frac{i}{2}+j} = 1 + \frac{2r}{i} \leqslant 3r.$$

Similarly

$$r!\left|\binom{-1/2}{r}\right|(r-1)!^{-1} \leqslant r, \quad (r-1)!\left(r!\left|\binom{1/2}{r}\right|\right)^{-1} \leqslant 4r.$$

Therefore

$$\Delta_r = c_2 r!\left|\binom{1/2}{r}\right|\xi^{1/2-r}\left(1 + \sum_{i=1}^{\infty} O\left(\left(\frac{4r}{c_2\sqrt{\xi}}\right)^i\right)\right) + O(2^r \exp(-c_4\sqrt{N}))$$
$$= c_2 r!\left|\binom{1/2}{r}\right|\xi^{1/2-r}\left(1 + O\left(\frac{r}{\sqrt{N}}\right)\right) + O(2^r \exp(-c_4\sqrt{N})).$$

We also have

$$\xi^{1/2-r} = (N + O(r))^{1/2-r} = N^{1/2-r}\left(1 + O\left(\frac{r}{N}\right)\right)^{1/2-r}$$
$$= N^{1/2-r}\left(1 + O\left(\frac{r^2}{N}\right)\right)$$
$$= N^{1/2-r}(1 + O(N^{-2/3}))$$

by our bound for r. Finally

$$2^r\left(r!\left|\binom{1/2}{r}\right|N^{1/2-r}\right)^{-1} \ll (2N)^r = \exp[o(\sqrt{N})],$$

whence altogether

$$\Delta_r = c_2 r!\left|\binom{1/2}{r}\right|N^{1/2-r}(1+O(N^{-1/3})+O[\exp(-c_4\sqrt{N}/2)])$$

for all $r \leqslant N^{1/6}$, which is the lemma.

Lemma 2. Let $R \leqslant N^{1/6}$ and $1 \leqslant r \leqslant R$. Then

$$\Delta_r/\Delta_{r-1} \leqslant R/N$$

provided that $N \geqslant N_0$. The N_0 does not depend on R.

Proof. Lemma 1 for $2 \leqslant r \leqslant N^{1/6}$ supplies

$$\frac{\Delta_r}{\Delta_{r-1}} = \frac{r-\frac{3}{2}}{N}(1+O(N^{-1/3}))$$
$$= \frac{r}{N} - \frac{3}{2N} + O\left(\frac{r}{N^{4/3}}\right) < \frac{r}{N} \leqslant \frac{R}{N}$$

if $N \geqslant N_0$, and similarly for $r = 1$.

Proof of the theorem

We assume, somewhat arbitrarily, $R \leqslant N^{1/6}$. This includes for large N the range $R \leqslant \log^2 N$ that by an earlier remark is relevant.

Consider now the numbers

$$\omega = \sum_{r=0}^{R-1} x_r\Delta_r, \quad \text{where } x_r \in \mathbb{N}_0, \quad x_r \leqslant \frac{N}{R} - 1.$$

They all are distinct because of

$$\sum_{r=s+1}^{R-1} x_r\Delta_r \leqslant \sum_{s+1}^{R-1}\left(\frac{N}{R}-1\right)\Delta_r \leqslant \sum_{s+1}^{R-1}(\Delta_{r-1}-\Delta_r) < \Delta_s.$$

The number A of the ω's is therefore

$$A = \left[\frac{N}{R}\right]^R > \left(\frac{N}{R}-1\right)^R = \left(\frac{N}{R}\right)^R\left(1-\frac{R}{N}\right)^R$$
$$\geqslant \left(\frac{N}{R}\right)^R\left(1+O\left(\frac{1}{\sqrt{N}}\right)\right)$$

as $N \to \infty$.

On the other hand

$$\Delta_r = \varepsilon_r \sum_{s=0}^{r} (-1)^s \binom{r}{s} \log p(N+s), \quad \varepsilon_r = \pm 1$$

implies that each ω is a linear combination over $\mathbb{Z}$ of the $\log p(n)$, $N \leqslant n < N+R$. If $q_1, \ldots, q_k$ denote the primes that make up the $p(N), \ldots, p(N+R-1)$,

$$p(n) = \prod_{j=1}^{k} q_j^{a_j(n)}, \quad a_j(n) \in \mathbb{N}_0,$$

say, then we obtain the representation

$$\omega = \sum_{j=1}^{k} y_j \log q_j$$

with

$$y_j := \sum_{0 \leqslant s \leqslant r < R} \varepsilon_r (-1)^s \binom{r}{s} x_r a_j(N+s).$$

Trivially $a_j(n) \leqslant (\log p(n))/\log 2$. Therefore, if N is large,

$$a_j(n) \leqslant c_2 \frac{\sqrt{n}}{\log 2},$$

$$\begin{aligned} |y_j| &\leqslant \frac{c_2}{\log 2} \sqrt{N+R} \sum_{r=0}^{R-1} 2^r x_r \\ &\leqslant \frac{c_2}{\log 2} \sqrt{N+R} \cdot 2^R \left(\frac{N}{R} - 1 \right) \\ &\leqslant \frac{c_2}{\log 2} N^{3/2} \frac{2^R}{R} \end{aligned}$$

for $1 \leqslant j \leqslant k$.

If now $m(N, N+R) =: l$ then the ω's are elements of an l-dimensional $\mathbb{Z}$-module. A suitable choice of l of the coordinates $y_j (j = 1, \ldots, k)$ will then determine the $k - l$ others. The number A of the ω's can therefore not exceed

$$\left(\frac{2c_2}{\log 2} N^{3/2} \frac{2^R}{R} + 1 \right)^l.$$

Consequently

$$\left(\frac{3c_2}{\log 2} N^{3/2} \frac{2^R}{R} \right)^l \geqslant \left(\frac{N}{R} \right)^R \left(1 + O\left(\frac{1}{\sqrt{N}} \right) \right). \tag{2}$$

Thus, if any $R_0 > 3c_2/\log 2$ and a suitable N_0 are chosen then for all $R \geqslant R_0$ and $N \geqslant N_0$

$$(2^R N^{3/2})^l \geqslant \left(\frac{N}{R} \right)^R \frac{R_0 \log 2}{3c_2} \left(1 + O\left(\frac{1}{\sqrt{N}} \right) \right) \geqslant \left(\frac{N}{R} \right)^R,$$

as claimed. For each of the remaining $R < R_0$, formula (2) implies $3l \geqslant 2R - \varepsilon$, and therefore $3l \geqslant 2R$, if N is large enough, hence again $(2^R N^{3/2})^l \geqslant (N/R)^R$.

References

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[2] Hardy G H and Ramanujan S, Asymptotic formulae in combinatory analysis, *Proc. London Math. Soc.* **17** (1918) 75–115