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The Manin–Drinfeld theorem and Ramanujan sums

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Abstract. The Manin–Drinfeld theorem asserts the finiteness of the cuspidal divisor class group of a modular curve corresponding to a congruence subgroup. The purpose of the note is to draw attention to the connection between this theorem and Ramanujan sums, and to the question of what happens for non-congruence subgroups.

Keywords. Modular curve; cuspidal class group; Eisenstein series; Ramanujan sums; non-congruence subgroups.

1. Introduction

We consider the upper half plane

$$\mathscr{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$$

on which the group $GL_2^+(\mathbb{R})$ acts by fractional linear transformations. Let Γ denote a discrete subgroup of $SL_2(\mathbb{R})$ of (co-) finite volume, and consider the quotient

$$Y(\Gamma) = \Gamma \backslash \mathscr{H}.$$

We know that $Y(\Gamma)$ is an open Riemann surface which can be compactified by adding the finite set $\Gamma\backslash\mathbb{P}(\mathbb{R})$ of cusps. The resulting compact Riemann surface will be denoted $X = X(\Gamma)$. Viewing X as a smooth algebraic curve, and fixing a base point, we have an embedding $X \to J = J(X)$, the Jacobian of X. We are interested in the subgroup $C = C(\Gamma)$ of J spanned by the image of the cusps. If Γ is a congruence subgroup, the theorem of Manin and Drinfeld asserts that C is finite. The purpose of this paper is to draw attention to the connection between this theorem and Ramanujan sums, and to the question of what happens for general Γ.

The original arguments of Manin [8] and Drinfeld [3] use Hecke operators. For non-congruence subgroups, there does not yet exist a good theory of such operators, (cf. [1] for a discussion of the problems involved) and so this approach does not seem to generalize. Moreover, as observed by Rohrlich (see [6]), $C(\Gamma)$ is infinite for a general non-congruence subgroup Γ.

Another way of studying $C(\Gamma)$ is by observing the connection between cuspidal divisors of degree zero and differentials of the third kind. These differentials are

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given by Eisenstein series of weight 2 for Γ. The finiteness of $C(\Gamma)$ is then equivalent to the rationality of the periods of such (suitably normalized) differentials. This point of view is essentially the one taken by Schoeneberg [11] (see also Stevens [15] for a modern treatment). It was observed by Scholl [12] that the rationality of these periods is, in turn, equivalent to the algebraicity of the Fourier coefficients of the Eisenstein series. We review this in § 2.

When Γ is suitably normalized, the nth Fourier coefficient of the Eisenstein series can be written as a linear combination of sums of the form

$$(*) \qquad \pi^2 n \sum_{c>0} \frac{1}{c^2} \sum_{\substack{0 \leq d < c \\ \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma}} \exp(2\pi i n d/c).$$

The inner sum may be thought of as a generalized Ramanujan sum. If Γ is a congruence subgroup, it can in fact be written in terms of (the usual) Ramanujan sums and evaluated explicitly. (Note that Ramanujan first introduced these sums in the very different context of the circle method.) This explicit evaluation allows us to sum the series in $(*)$ and show that it lies in $\bar{\mathbb{Q}}$. We thus get what seems to be a slightly different proof of the Manin–Drinfeld theorem. The calculations are explained in § 3.

It is a very interesting question whether the classical evaluation of Ramanujan sums can be extended in any way to our generalized sums. We study this in one concrete case, namely that of the Fermat curve F_N given by

$$X^N + Y^N = 1.$$

Following Fricke and Klein [4], F_N can be written as $Y(\Gamma)$ with Γ a non-congruence subgroup of $\Gamma(2)$. We discuss this in § 4. We also raise a question about the Heisenberg curves which are cyclic covers of Fermat curves.

Finally, in § 5, we study the correspondence between the Fermat curve $\bar{F}_N$ and the modular curve $X(2N)$, considered by Kubert and Lang [6]. We view it as an algebraic cycle on the surface $\bar{F}_N \times X(2N)$ and show that, if $N > 7$ is prime, then this cycle is an element of finite order in the Néron–Severi group. It is intriguing to ask whether this has any direct connection with the finiteness of the cuspidal divisor class group on either curve.

2. Eisenstein series

Let X be a compact Riemann surface, and D a divisor of degree zero. Then, by the Riemann–Roch theorem, there is a differential ω of the third kind whose residual divisor $\operatorname{res}\omega$ is D. This ω can be made unique by specifying that its period with respect to any integral 1-cycle disjoint from D is purely imaginary. We shall denote this distinguished ω by ω_D. Then, we see that D is a torsion divisor if and only if the periods of ω_D lie in $2\pi i\mathbb{Q}$. Indeed, if $nD = (f)$, then we may take $\omega_D = \mathrm{d}f/nf$. Conversely, if ω_D has the stated property, then we can define the function

$$f(P) = \exp\left(\int_{\mathcal{O}}^{P} n\omega_D\right),$$

where $\mathcal{O}$ is a base point. It is easily seen that $(f) = nD$.

Now, suppose that $X = X(\Gamma)$ and that Γ is such that $Y(\Gamma)$ has a finite volume. In this case, the set of cusps is finite; let us write it as

$$S = \{\kappa_1, \ldots, \kappa_h\}.$$

We shall *assume* that ∞ is a cusp, say κ_1. To simplify the discussion still further, let us *assume* that $\Gamma \subseteq SL_2(\mathbb{Z})$. For each i, denote by Γ_i the stabilizer of κ_i in Γ. (We shall sometimes write Γ_∞ for Γ_1.) Let us fix an element σ_i in $SL_2(\mathbb{Z})$ so that $\sigma_i \infty = \kappa_i$. Then, there is a positive integer μ so that

$$\sigma_i^{-1}\Gamma_i\sigma_i = \left\{ \pm\begin{pmatrix} 1 & \mu m \\ 0 & 1 \end{pmatrix} : m \in \mathbb{Z} \right\}.$$

For $z = x + iy$, we set (following Hecke)

$$G_i(z) = \lim_{s \to 1} s y^{s-1} \sum_{\substack{\sigma \in \Gamma_i \backslash \Gamma \\ \sigma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)}} (cz + d)^{-2} |cz + d|^{2-2s}.$$

Then, G_i is a real-analytic Eisenstein series of weight 2. Moreover, if $D = \sum m_i \kappa_i$ is a divisor of degree zero, let us set

$$G_D(z) = \sum m_i G_i(z).$$

It is known (see [12], Prop. 2]) that

$$\omega_D = 2\pi i G_D(z) \mathrm{d}z.$$

Moreover, there is the following connection between the order of D and the Fourier expansion of G_D at ∞.

PROPOSITION [12]

The following are equivalent:

(a) *D is of finite order in J*
(b) *All the Fourier coefficients of G_D at ∞ are algebraic numbers.*

Following the treatment in Kubota [7], we find that

$$G_j(z) = \delta_j - \pi y C - 4\pi^2 \sum_{m=1}^{\infty} m A_{j,m} \exp(2\pi i m z)$$

where

$$\delta_j = \begin{cases} 1 & \text{if } j = 1 \\ 0 & \text{if } j \neq 1, \end{cases}$$

C is a constant (independent of j) and

$$A_{j,m} = \lim_{s \to 1+} \sum_{c > 0} \frac{1}{c^{2s}} \left(\sum_{\substack{0 \leq d < \mu c \\ \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \sigma_j^{-1}\Gamma}} \exp(2\pi i m d / c\mu) \right).$$

An immediate consequence of the above Proposition is the following.

PROPOSITION

The following are equivalent:

(a) $C(\Gamma)$ *is finite*

(b) *For each* $1 \leqslant j \leqslant h$, *and* $m \geqslant 1, \pi^2(A_{j,m} - A_{1,m}) \in \bar{\mathbb{Q}}$.

3. Congruence subgroups

If Γ is a congruence subgroup, the series $A_{j,m}$ can be evaluated explicitly. One immediately reduces to the case that $\Gamma = \Gamma(N)\{\pm I\}$ where $\Gamma(N)$ is the principal congruence subgroup of level N. We shall perform the calculation for the cusps $\kappa_1 = \infty$ and $\kappa_2 = 0$, the general case being similar. We have $\sigma_1 = I$ and $\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\mu = N$. We see that

$$\begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma \Leftrightarrow c \equiv 0 \,(\mathrm{mod}\, N), \quad d \equiv \pm 1 \,(\mathrm{mod}\, N), \quad (c,d) = 1$$

$$\begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \sigma_2^{-1} \Gamma \Leftrightarrow c \equiv \pm 1 \,(\mathrm{mod}\, N), \quad d \equiv 0 \,(\mathrm{mod}\, N), \quad (c,d) = 1.$$

Hence,†

$$A_{1,m} = \sum_{\substack{c > 0 \\ c \equiv 0 (\mathrm{mod} N)}} \frac{1}{c^2} \Bigg(\sum_{\substack{0 \leqslant d < Nc \\ d \equiv \pm 1 (\mathrm{mod} N) \\ (d,c) = 1}} \exp(2\pi i m d / cN) \Bigg)$$

and

$$A_{2,m} = \sum_{\substack{c > 0 \\ c \equiv \pm 1 (\mathrm{mod} N)}} \frac{1}{c^2} \Bigg(\sum_{\substack{0 \leqslant d < Nc \\ (d,c) = 1 \\ d \equiv 0 (\mathrm{mod} N)}} \exp(2\pi i m d / cN) \Bigg).$$

We use the Ramanujan sum

$$\sum_{\substack{0 \leqslant h < k \\ (h,k) = 1}} \exp(2\pi i m h / k) = \sum_{j | (m,k)} \mu(k/j)\, j,$$

where $\mu(x)$ denotes the Mobius function. We find that

$$A_{2,m} = \sum_{\substack{j | m \\ (j,N) = 1}} \frac{1}{j} \sum_{c_0 \equiv \pm \bar{j} (\mathrm{mod} N)} \frac{\mu(c_0)}{c_0^2}$$

where $\bar{j}$ denotes the inverse of j (mod N). We can simplify this further:

$$A_{2,m} = \frac{1}{\phi(N)} \sum_{\substack{j | m \\ (j,N) = 1}} \frac{1}{j} \sum_{\chi} (\chi(j) + \chi(-j)) \frac{1}{L(2,\chi)}$$

$$= \frac{2}{\phi(N)} \sum_{\substack{j | m \\ (j,N) = 1}} \frac{1}{j} \sum_{\chi(-1) = 1} \frac{\chi(j)}{L(2,\chi)}$$

†To be precise, we should really write $A_{1,m}$ and $A_{2,m}$ as limits as $s \to 1^+$. However, the discussion in this section justifies setting $s = 1$.

where the inner sum is over even Dirichlet characters (mod N). It follows that $\pi^2 A_{2,m} \in \bar{\mathbb{Q}}$. To evaluate $A_{1,m}$, we observe that if $\varepsilon = \pm 1$, then,

$$\sum_{\substack{0 \leqslant d < Nc \\ d \equiv \varepsilon (\mathrm{mod}\, N) \\ (d,c)=1}} \exp(2\pi i m d/cN) = \sum_{\substack{0 \leqslant d < Nc \\ d \equiv \varepsilon (\mathrm{mod}\, N)}} \exp(2\pi i m d/cN) \sum_{j|(d,c)} \mu(j)$$

$$= \sum_{\substack{j|c \\ (j,N)=1}} \mu(j) \sum_{\substack{0 \leqslant d_0 < Nc/j \\ d_0 \equiv \varepsilon \bar{j} (\mathrm{mod}\, N)}} \exp(2\pi i m j d_0/cN)$$

where, we have written $d = jd_0$ and $\bar{j}$ for the inverse of $j \,(\mathrm{mod}\, N)$. If we further write $d_0 = \varepsilon \bar{j} + d_1 N$, the sum becomes

$$\sum_{\substack{j|c \\ (j,N)=1}} \mu(j) \sum_{0 \leqslant d_1 < (c/j)} \exp[2\pi i m j(\varepsilon \bar{j} + d_1 N)/cN]$$

$$= \sum_{\substack{j|c \\ (j,N)=1}} \mu(j) \exp(2\pi i m \varepsilon j \bar{j}/cN) \sum_{0 \leqslant d_1 < (c/j)} \exp[2\pi i m d_1/(c/j)].$$

Let us write $c = jc_0$. The inner sum is 0 unless $m \equiv 0 (c_0)$ in which case it is c/j. Thus, we find the above equal to

$$c \sum_{\substack{j|c \\ (j,N)=1 \\ m \equiv 0 (\mathrm{mod}\, c/j)}} \frac{\mu(j)}{j} \exp(2\pi i m \varepsilon j \bar{j}/cN).$$

Substituting this into the series for $A_{1,m}$, and rearranging, we find that

$$A_{1,m} = \sum_{\varepsilon = \pm 1} \sum_{\substack{c_0 | m \\ c_0 \equiv 0 (\mathrm{mod}\, N) \\ c_0 > 0}} \frac{1}{c_0} \sum_{\substack{j>0 \\ (j,N)=1}} \frac{\mu(j)}{j^2} \exp(2\pi i \varepsilon m \bar{j}/c_0 N)$$

$$= \sum_{\varepsilon = \pm 1} \sum_{\substack{N|c_0|m \\ c_0 > 0}} \frac{1}{c_0} \sum_{\substack{1 \leqslant a < N \\ (a,N)=1}} \exp(2\pi i \varepsilon m \bar{u}/c_0 N) \sum_{\substack{j \equiv a (\mathrm{mod}\, N) \\ j>0}} \frac{\mu(j)}{j^2}.$$

The innermost sum is written in terms of Dirichlet L-series as before:

$$A_{1,m} = \sum_{\varepsilon} \sum_{c_0} \sum_{u} \frac{1}{c_0} \exp(2\pi i^* \bar{u}) \frac{1}{\phi(N)} \sum_{\chi} \chi(a) \frac{1}{L(2,\chi)}$$

where we have written * for $\varepsilon M/c_0 N$. Rearranging the sums, and observing that

$$\sum_a \exp(-2\pi i^* \bar{a}) \bar{\chi}(a) = \chi(-1) \sum_a \exp(2\pi i^* \bar{a}) \bar{x}(a),$$

we deduce that

$$A_{1,m} = \frac{2}{\phi(N)} \sum_{\substack{N|c_0|m \\ c_0 > 0}} \frac{1}{c_0} \sum_{\substack{1 \leqslant a < N \\ (a,N)=1}} \exp(2\pi i m \bar{a}/c_0 N) \sum_{\chi(-1)=1} \frac{\bar{\chi}(a)}{L(2,\chi)}.$$

Hence, $\pi^2 A_{1,m} \in \bar{\mathbb{Q}}$ also. This, therefore, proves that (∞)–(0) is a point of finite order in the Jacobian of $X(\Gamma)$.

4. Fermat curves and Heisenberg curves

We define the Fermat curve F_N of degree N by

$$X^N + Y^N = 1.$$

There is a natural smooth compactification $\bar{F}_N$ of F_N in $\mathbb{P}^2$. Following Fricke and Klein (cf. Rohrlich [10]), we know that it is possible to write F_N as a modular curve corresponding (in general) to a non-congruence subgroup. We begin by reviewing this parametrization.

For any $\tau \in \mathscr{H}$, let $L_\tau = \mathbb{Z} \oplus \mathbb{Z}\tau$ and

$$p_\tau(z) = \frac{1}{z^2} + \sum_{0 \neq \omega \in L_\tau} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

the corresponding Weierstrass function. Let

$$w_k = \begin{cases} 1 & \text{if } k = 1 \\ \tau + 1 & \text{if } k = 2 \\ \tau & \text{if } k = 3 \end{cases}$$

and set $e_k = p_\tau(\frac{1}{2}w_k)$. Note that $\{0, \frac{1}{2}, \frac{1}{2}\tau, \frac{1}{2}(\tau+1)\}$ represent the points of order 2 on the elliptic curve $\mathbb{C}/L_\tau$. In particular, the e_k are distinct. We define the function

$$\lambda(\tau) = (e_2 - e_3)/(e_1 - e_3).$$

This function is analytic and non-zero on $\mathscr{H}$. We have the following facts regarding λ:

(a) λ is an automorphic function for $\Gamma(2)$
(b) λ maps H onto $\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty]$
(c) λ takes the values $0, 1, \infty$ at $\infty, 0, 1$ respectively.

Because of (a), λ defines a function on the open Riemann surface $Y(2) = \Gamma(2)\backslash\mathscr{H}$. Moreover, we may identify

$$Y(2) \to F_1$$

by $z \to (\lambda(z), 1 - \lambda(z))$. Now, F_N can be described as a finite ramified cover of $F_1 = Y(2)$. In any simply connected open subset U of $\mathbb{P}^1 - \{0, 1, \infty\}$, we can find a well-defined N-th root $\tilde{x}$ of λ and $\tilde{y}$ of $1 - \lambda$. We can extend $(\tilde{x}, \tilde{y})$ as a pair of multi-valued functions on $\mathbb{P}^1 - \{0, 1, \infty\}$. The covering space on which $(\tilde{x}, \tilde{y})$ become single-valued is the affine curve F_N given by

$$\tilde{x}^N + \tilde{y}^N = 1.$$

The monodromy of $\tilde{x}$ and $\tilde{y}$ can be described in a simple way. Take $\gamma \in \Gamma(2) = \pi_1(\mathbb{P}^1 - \{0, 1, \infty\})$. Then, there are integers $s(\gamma)$ and $t(\gamma)$ such that

$${}^{\gamma}\log \lambda = \log \lambda + 2\pi i s(\gamma)$$

$${}^{\gamma}\log(1 - \lambda) = \log(1 - \lambda) + 2\pi i t(\gamma).$$

It follows that $\pi_1(F_N) = \Gamma$ where

$$\Gamma = \{\gamma \in \Gamma(2) \colon s(\gamma) \equiv t(\gamma) \equiv 0 (\operatorname{mod} N)\}.$$

In other words, Γ is the kernel of the map

$$\Gamma(2) \to \mathbb{Z}/N \times \mathbb{Z}/N$$

$$\gamma \to (s(\gamma) \operatorname{mod} N,\ t(\gamma) \operatorname{mod} N).$$

There is yet another way to view Γ. We note that $\Gamma(2)$ is a free group on the generators $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. These two generators correspond to the positively oriented cycles around 0 and 1 in $\mathbb{P}^1 - \{0, 1, \infty\}$. The reduction $\operatorname{mod} N$ of the abelianization map gives:

$$\Gamma(2) \to \mathbb{Z}/N \times \mathbb{Z}/N$$

which coincides with the map of the previous paragraph. It follows easily from this description that in general Γ is a non-congruence subgroup.

It is also possible to describe $s(\gamma)$, $t(\gamma)$ explicitly in terms of Dedekind sums. The key idea is to relate λ to the Dedekind η function and use the well-known transformation formulae for $\log \eta$. We begin by recalling the latter result (see, for example, Siegel [14]).

PROPOSITION

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be in* $SL_2(\mathbb{Z})$. *Then*

$$\log \eta(\gamma z) - \log \eta(z) = \begin{cases} \pi i b/12 & \text{if} \quad c = 0 \\ \tfrac{1}{2}\log(-i(cz + d)) + \pi i \dfrac{a + d}{12c} - \pi i D(d, c) & \text{if} \quad c > 0. \end{cases}$$

Here,

$$D(d, c) = \sum_{0 \leqslant j < c} ((j/c))((jd/c))$$

is the Dedekind sum and

$$((x)) = \begin{cases} x - [x] - \tfrac{1}{2} & \text{if} \quad x \notin \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

Next, we relate λ to the η function.

Lemma. We have

$$\lambda(z) = 16 \exp(\pi i/3)(\eta(2z)/\eta(\tfrac{1}{2}(z + 1)))^8,$$

$$1 - \lambda(z) = \exp(\pi i/3)(\eta(\tfrac{1}{2}z)/\eta(\tfrac{1}{2}(z + 1)))^8.$$

Proof. Let us set $q_2 = \exp(\pi i z)$. From Fricke and Klein [4] we know that

$$\lambda(z) = 16 q_2 \prod_{n=1}^{\infty} (1 + q_2{}^{2n})^8 (1 + q_2{}^{2n-1})^{-8},$$

$$1 - \lambda(z) = \prod_{n=1}^{\infty} (1 - q_2{}^{2n-1})^8 (1 + q_2{}^{2n-1})^{-8}.$$

The result follows easily from these.

Using this lemma, we may define $\log\lambda$ and $\log(1-\lambda)$ in terms of $\log\eta$ and study their transformation properties.

PROPOSITION

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$. *Then*

$$s(\gamma) = \begin{cases} b/2 & \text{if } c = 0 \\ \dfrac{a+d}{2c} - 4\left[D\left(d, \dfrac{c}{2}\right) - D(d-c, 2c)\right] & \text{if } c > 0, \end{cases}$$

$$t(\gamma) = \begin{cases} 0 & \text{if } c = 0 \\ 4[D(d-c, 2c) - D(d, 2c)] & \text{if } c > 0. \end{cases}$$

Proof. From the previous lemma, we have

$$\log\lambda(z) = \log(16\exp(\pi i/3)) + 8\log\eta(2z) - 8\log\eta(\tfrac{1}{2}(z+1)).$$

Suppose that $c > 0$. Since

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 2b \\ c/2 & d \end{pmatrix} \in SL_2(\mathbb{Z}),$$

we have

$$\log\eta\left(\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\gamma z\right) - \log\eta\left(\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} z\right)$$

$$= \frac{1}{2}\log(-i(cz+d)) + \pi i\left(\frac{a+d}{6c}\right) - \pi i D\left(d, \frac{c}{2}\right).$$

Similarly,

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -1/2 \\ 0 & 1/2 \end{pmatrix} = \begin{pmatrix} a+c & \frac{1}{2}(b+d-a-c) \\ 2c & d-c \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Hence,

$$(**) \qquad \log\eta\left(\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\gamma z\right) - \log\eta\left(\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} z\right)$$

$$= \tfrac{1}{2}\log(-i((cz+d))) + \pi i\left(\frac{a+d}{24c}\right) - \pi i D(d-c, 2c).$$

Thus,

$$\frac{1}{8}((\log\lambda)(\gamma z)-(\log\lambda)(z))$$

$$=\left[\pi i\left(\frac{a+d}{6c}\right)-\pi iD\left(d,\frac{c}{2}\right)\right]-\left[\pi i\left(\frac{a+d}{24c}\right)-\pi iD(d-c,2c)\right]$$

and so

$$s(\gamma)=\frac{a+d}{2c}+4\left[D(d-c,2c)-D\left(d,\frac{c}{2}\right)\right].$$

If $c=0$, a similar calculation gives

$$\frac{1}{8}((\log\lambda)(\gamma z)-(\log\lambda)(z))=\pi i\left(\frac{2b}{12}\right)-\frac{\pi i(b+d-a-c)}{24}=\frac{\pi ib}{8},$$

and so $s(\gamma)=b/2$. For $t(\gamma)$, we repeat the above using the relation

$$\log(1-\lambda)(z)=\frac{\pi i}{3}+8(\log\eta(\tfrac{1}{2}z)-\log\eta(\tfrac{1}{2}(z+1)))$$

given by the lemma. Since

$$\begin{pmatrix}1&0\\0&2\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1&0\\0&1/2\end{pmatrix}=\begin{pmatrix}a&b/2\\2c&d\end{pmatrix}$$

we have, for $c>0$,

$$\log\eta(\tfrac{1}{2}\gamma z)-\log\eta(\tfrac{1}{2}z)=\frac{1}{2}\log(-i(cz+d))+\pi i\left(\frac{a+d}{24c}\right)-\pi iD(d,2c).$$

Combining this with (**), we find that

$$2\pi it(\gamma)=8\left[\left(\pi i\frac{a+d}{24c}-\pi iD(d,2c)\right)-\left(\pi i\frac{a+d}{24c}-\pi iD(d-c,2c)\right)\right]$$

and so

$$t(\gamma)=4[D(d-c,2c)-D(d,2c)]$$

if $c>0$. If $c=0$, we have

$$2\pi it(\gamma)=8\left[\frac{\pi ib}{24}-\frac{\pi ib}{24}\right]=0.$$

It is interesting to ask whether the expressions for $s(\gamma)$ and $t(\gamma)$ given by this Proposition can be used to evaluate the generalized Ramanujan sums corresponding to our group Γ, a typical one of which is

$$\sum_{\substack{0\leq d<2Nc\\ \left(\begin{smallmatrix}*&*\\c&d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma}}\exp(2\pi imd/2Nc).$$

We remark that the sum can also be written as

$$\sum_{\substack{0\leqslant d<2Nc \\ s\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right)\equiv 0 \\ r\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right)\equiv 0 \pmod N}} \exp(2\pi imd/2N).$$

On the other hand, Rohrlich [10] has shown that the group $C(\Gamma)$ *is* finite. Therefore, invoking the Proposition at the end of § 2, we get the following curious result.

PROPOSITION

For all $m\geqslant 1$,

$$\lim_{s\to 1+} \pi^2 \sum_{c>0} \frac{1}{c^{2s}} \left\{ \sum_{\substack{0\leqslant d<2Nc \\ \left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma}} \exp(2\pi imd/2Nc) - \sum_{\substack{0\leqslant d<2Nc \\ \left(\begin{smallmatrix} c & d \\ * & *\end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma}} \exp(2\pi imd/2Nc) \right\} \in\bar{\mathbb{Q}}.$$

Indeed this is equivalent to the fact that $(\infty)-(0)$ *is torsion.*

Next, we turn our attention to another class of curves. For any ring R, the R-points of the Heisenberg group are defined to be

$$H_R = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c\in R \right\}.$$

There is a natural homomorphism

$$\phi\colon \Gamma(2)\to H_{\mathbb{Z}}$$

given on the generators by

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

For every integer $N\geqslant 1$, we have a (commutative) diagram:

$$\begin{array}{ccccc} & \overset{\phi}{\nearrow} & H_{\mathbb{Z}} & \xrightarrow[\text{mod } N]{\text{reduction}} & H_{\mathbb{Z}/N\mathbb{Z}} \\ \Gamma(2) & & \big\downarrow \beta & & \big\downarrow \\ & \underset{\alpha}{\searrow} & \mathbb{Z}^2 & \xrightarrow[\text{mod } N]{\text{reduction}} & (\mathbb{Z}/N\mathbb{Z})^2 \end{array}$$

where α and β are the abelianization maps. (In particular, β tends

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

to the pair (a,b)). Let ϕ_N denote the composition of ϕ with reduction mod N. The *Heisenberg curve* H_N is $\Gamma\backslash\mathscr{H}$ where $\Gamma = \ker\phi_N$. The compactification $\bar{H}_N$ of H_N is a cover of $\mathbb{P}^1$ ramified only over $\{0, 1, \infty\}$. It is a fact (due to Deligne) that $\bar{H}_N$ can be defined over $\mathbb{Q}$.

It is known (see [9]) that ϕ can be defined by means of the monodromy of the dilogarithm function

$$\ell_2(z) = -\int_0^z \log(1-t)\frac{\mathrm{d}t}{t}.$$

(More precisely, one considers the monodromy of $\ell_2(z)$, $\log z$ and $\log(1-z)$.) As in the case of the Fermat curves, we can write a formula for the Fourier coefficients of the Eisenstein series of weight 2 for $\ker\phi_N$, in terms of certain generalized Ramanujan sums. It will be a very interesting problem to determine the arithmetic nature of these coefficients and the structure of the cuspidal class group $C(\ker\phi_N)$.

Both in the case of $\bar{F}_N$ and $\bar{H}_N$, we have curves expressed as covers of $\mathbb{P}^1$ ramified only at $\{0, 1, \infty\}$. There is a general result of Belyi [2] that a smooth projective curve has a model over $\bar{\mathbb{Q}}$ if and only if it can be written as a cover of $\mathbb{P}^1$ ramified only at $\{0, 1, \infty\}$. A natural question to ask is the following. Given the cover $X \to \mathbb{P}^1$ ramified only at $\{0, 1, \infty\}$, what further hypotheses are needed to ensure that the inverse image of $\{0, 1, \infty\}$ give points of finite order in the Jacobian of X? (We know that *some* further hypotheses will be necessary since the proof of Belyi shows that *any* finite set of algebraic points can be made to lie over $\{0, 1, \infty\}$.)

5. The Kubert–Lang correspondence

In this section, we discuss a matter related to the parametrization of the Fermat curve which was discussed in the previous section. Using the λ-function, we have seen that it is possible to exhibit $\bar{F}_N$ as a cover of $X(2) = \mathbb{P}^1$ corresponding to a subgroup Γ of $\Gamma(2)$. This cover is ramified only over $\{0, 1, \infty\}$ and is in fact totally ramified at these points. We also have the modular curve $X(2N)$ as a cover of $X(2)$ and, as observed by Kubert and Lang, this cover has the same ramification properties. Therefore we have a diagram

$$\bar{F}_N \longrightarrow X(2) \longleftarrow X(2N)$$

where the arrows are given by rational finite maps. Thus we get an unramified correspondence from $\bar{F}_N$ to $X(2N)$. Kubert and Lang [6, p.196] use this to show that if F_N has infinitely many rational points in a number field k, then $X(2N)$ has infinitely many rational points in a finite extension of k, and conversely.

The correspondence gives, in a natural way, a divisor on the product $\bar{F} \times X(2N)$. It is interesting to consider the algebraic equivalence class of this divisor, D_N say. We have the following result.

PROPOSITION

Let N be a prime larger than 7. Then D_N is torsion in the Neron–Severi group $NS(\bar{F}_N \times X(2N))/(NS(\bar{F}_N) \oplus NS(X(2N)))$.

Proof. This follows easily from known properties of the Jacobians of Fermat and modular curves. Indeed, D_N gives rise to an endomorphism

$$(***)\qquad \mathrm{Jac}(\bar{F}_N) \to \mathrm{Jac}(X(2N)).$$

Let us recall the decomposition of $\mathrm{Jac}(\bar{F}_N)$ into simple factors. We have

$$\mathrm{Jac}(\bar{F}_N) \approx \prod_{i=1}^{N-2} A_i,$$

where each A_i is an abelian variety of dimension $\frac{1}{2}(N-1)$ and having complex multiplication (CM) by $\mathbb{Q}(\zeta_N)$. From the tables of Koblitz [5], we know that if $N > 7$ is prime, then over $\bar{\mathbb{Q}}, A_i$ is isogenous to a power of an absolutely simple abelian variety of dimension > 1. On the other hand, it is a well known result of Shimura that

$$\mathrm{Jac}(X(2N)) \approx \prod A_f,$$

where the product ranges over certain cusp forms of weight 2 and, if A_f is of CM type, there is a splitting over $\bar{\mathbb{Q}}: A_f \approx E^n$ where E is an elliptic curve with CM and $n \geqslant 1$ ([13]). Thus, tensoring the homomorphism (***) with $\mathbb{Q}$ and observing that CM factors must map to CM factors, we conclude the result of the Proposition.

References

[1] Atkin A O L and Swinnerton-Dyer H P F, Modular forms on non-congruence subgroups, *Combinatorics, Proc. Symp. Pure Math. Am Math. Soc.* **19** (1968) 1–25

[2] Belyi G V, Galois extensions of a maximal cyclotomic field, *Izv. Akad. Nauk USSR Ser. Mat. (Russian)* **43** (1979) 267–276

[3] Drinfield V G, Two theorems on modular curves, *Funkc. Anal. i. Prilozen (Russian)* **7** (1973) 83–84

[4] Fricke R and Klein F, *Vorlesungen uber die theorie der elliptischen modulfunktionen* (New York: Johnson Reprint Corpn.) Vol. 2 (1966) (from the 1890 edition)

[5] Koblitz N, Gamma function identities and elliptic differentials on Fermat curves, *Duke Math. J.* **45** (1978) 87–99

[6] Kubert D and Lang S, *Modular units* Grundleheren der Math. Wissenschaften (Springer-Verlag) (1981)

[7] Kubota T, *Elementary theory of Eisenstein series* (Wiley) (1973)

[8] Manin J, Parabolic points and zeta functions of modular curves, *Izv. Akad. Nauk USSR Ser. Math. (Russian)* **36** (1972) 19–64

[9] Ramakrishnan D, A regulator for curves via the Heisenberg group, *Bull. Am. Math. Soc.* **5** (1981) 191–195

[10] Rohrlich D, Points at infinity on the Fermat curves, *Invent. Math.* **39** (1977) 95–127

[11] Schoeneberg B, *Elliptic modular functions, an introduction* (English translation from German) (Grundleheren: Springer-Verlag) (1974)

[12] Scholl A J, Fourier coefficients of Eisenstein series on non-congruence subgroups, *Math. Proc. Cambridge Philos. Soc.* **99** (1986) 11–17

[13] Shimura G, On elliptic curves with complex multiplication as factors of the Jacobians of modular function fields, *Nagoya Math. J.* **43** (1971) 199–208

[14] Siegel C L, *Advanced analytic number theory* (Lecture Notes) (Bombay: Tata Institute of Fundamental Research) 2nd ed. (1980)

[15] Stevens G, *Arithmetic of modular curves, Progress in Mathematics* (Boston: Birkhauser) (1982)