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Fluctuations in the mean of Euler's phi function

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Abstract. We consider the error term in the mean value estimate of Euler's phi function $\varphi(n)$, and show that it is $\Omega_{\pm}(x(\log\log x)^{\frac{1}{2}})$. This improves on the earlier results of Pillai and Chowla, and of Erdös and Shapiro.

Keywords. Euler's phi function; fluctuations; mean value estimate.

1. Statement of results

Let $\varphi(n)$ denote Euler's totient function, which counts the number of reduced residue classes (mod n). We study the remainder term in the asymptotic function

$$\sum_{n\leqslant x} \varphi(n) = (3/\pi^2)x^2 + R(x). \tag{1}$$

Dirichlet showed that $R(x) \ll x^{1+\varepsilon}$ for any $\varepsilon > 0$, and Mertens [5] refined this to $R(x) \ll x\log x$. More recently, Walfisz [12] improved this to $R(x) \ll x(\log x)^{2/3}(\log\log x)^{4/3}$. For most x, $R(x)$ is not much larger than $O(x)$, since Pillai and Chowla [7] have shown that

$$\sum_{n\leqslant x} R(n) \sim \frac{3}{2\pi^2}x^2, \tag{2}$$

while Chowla [1] found that

$$\int_0^x R(u)^2\,du \sim (6\pi^2)^{-1}x^3. \tag{3}$$

Instead of $R(x)$, some authors have considered the remainder $R_0(x)$ defined by the relation

$$\sum_{n\leqslant x} \varphi(n)/n = (6/\pi^2)x + R_0(x). \tag{4}$$

Pillai and Chowla [7] showed that $R(x)$ and $R_0(x)$ are closely linked. We formulate this in the following refined form.

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Theorem 1. *For* $x \geqslant 2$,

$$R_0(x) = R(x)/x + O(\exp(-c\sqrt{\log x})). \tag{5}$$

Here c is a positive absolute constant.

Sylvester [9], [10] conjectured that $R(x) > 0$ whenever x is a positive integer, but this was disproved by Sarma [8], who showed that $R(820) < 0$. In fact $R(n) > 0$ and $R_0(n) > 0$ for $n < 820$, while $R(820) = -9{\cdot}092\ldots$ and $R_0(820) = -0{\cdot}012\ldots$. Karl K Norton [6, pp. 86–87] made the interesting observation that Sylvester [10] tabulated $\varphi(n)$, $\sum_{m \leqslant n} \varphi(m)$, and $(3/\pi^2)n$ for $n = 1, 2, \ldots, 1000$, but failed to note that $R(820) < 0$. Pillai and Chowla [7] showed that

$$R(x) = \Omega(x \log\log\log x). \tag{6}$$

That $R(x)$ changes sign infinitely often was finally proved by Erdös and Shapiro [3], who demonstrated that

$$R(x) = \Omega_{\pm}(x \log\log\log\log x). \tag{7}$$

We now improve on these latter estimates.

Theorem 2. *Let* $R(x)$ *be determined by* (1). *Then*

$$R(x) = \Omega_{\pm}(x\sqrt{\log\log x}).$$

Concerning the maximum order of magnitude of $R(x)$, we conjecture that

$$R(x) \ll x \log\log x \tag{8}$$

and that

$$R(x) = \Omega_{\pm}(x \log\log x). \tag{9}$$

From the construction of Pillai and Chowla [7] one sees that $\varphi(n)$ can be small, say $\varphi(n) < n/4$, for all n in an interval $x < n < x + c \log\log\log x$. However, in intervals longer than this $\varphi(n)$ has the expected mean value, since it may be shown that if $2 \leqslant h \leqslant x$ then

$$(6/\pi^2)h + O(\log h) + O(\log\log\log x) \leqslant \sum_{x < n \leqslant x+h} \varphi(n)/n$$
$$\leqslant (6/\pi^2)h + O(\log h).$$

This is a slight refinement of Theorem 6 of Erdős [2].

2. Basic lemmas

In the discussion below we take $s(x)$ to be the 'saw tooth' function. That is, $s(x)$ has period 1, $s(0) = 0$, and $s(x) = 1/2 - x$ for $0 < x < 1$.

Lemma 1. We have

$$R_0(x) = \tfrac{1}{2}\varphi(x)/x + \sum_{d \leqslant y} \mu(d)s(x/d)/d + O(\exp(-c_1\sqrt{\log x})) \tag{10}$$

uniformly for $x \geqslant 2$, $y \geqslant x\exp(-c_1\sqrt{\log x})$.

Here we set $\varphi(x) = 0$ if x is not an integer. The constants c_i are positive and absolute. The method of Walfisz [12, Chapter IV] can be used to show that (10) holds in the larger range $y \geqslant x^\eta$, for any given $\eta > 0$, but such a strengthening of the lemma would have no impact on our main results.

Proof. We first establish (10) in the case $y = x$. We recall that $\varphi(n)/n = \sum_{d|n}\mu(d)/d$. But if $f(n) = \sum_{d|n} g(d)$, then

$$\sum_{n \leqslant x} f(n) = x\sum_{d \leqslant x} g(d)/d - \sum_{d \leqslant x} g(d)\{x/d\}. \tag{11}$$

Thus in particular,

$$\sum_{n \leqslant x} \varphi(n)/n = x\sum_{d \leqslant x} \mu(d)/d^2 - \sum_{d \leqslant x} \mu(d)\{x/d\}/d.$$

Here the first sum on the right, if extended over all $d \geqslant 1$, would be $1/\zeta(2) = 6/\pi^2$. Hence we see that

$$R_0(x) = x\sum_{d \geqslant x} \mu(d)/d^2 - \sum_{d \leqslant x} \mu(d)\{x/d\}/d.$$

Put $L(x) = \sum_{d \leqslant x}\mu(d)/d$. By a variant of the quantitative form of the prime number theorem we know that $L(x) \ll \exp(-c_0\sqrt{\log x})$ for $x \geqslant 2$. Thus the first sum above is

$$-L(x)/x + \int_x^\infty L(u)/u^2\,du \ll \frac{1}{x}\exp(-c_0\sqrt{\log x}).$$

Using this, and also adding $L(x)/2$ to the expression, we see that

$$R_0(x) = \sum_{d \leqslant x} \mu(d)(1/2 - \{x/d\})/d + O(\exp(-c_0\sqrt{\log x})).$$

But $1/2 - \{x\} = s(x)$ if x is not an integer. We note that if x/d is an integer then x is an integer and $d|x$. In the case that x is an integer, these d combine to contribute an amount $\tfrac{1}{2}\varphi(x)/x$. Thus we have (10) with $c_1 = c_0$ when $y = x$.

To establish (10) when $y > x$ it suffices to note that

$$\sum_{x < d \leqslant y} \mu(d)s(x/d)/d = \left(\frac{1}{2} - \frac{x}{y}\right)L(y) + \tfrac{1}{2}L(x) - \int_x^y L(u)/u^2\,du$$

$$\ll \exp(-c_0\sqrt{\log x}).$$

We now consider y in the range $x\exp(-c_1\sqrt{\log x}) \leqslant y \leqslant x$. Suppose that k is a positive integer, $1 \leqslant k \leqslant x/y$. Then $s(x/d)$ is monotonic for $x/(k+1) < d < x/k$, so that

$$\sum_{x/(k+1) < d < x/k} \mu(d)s(x/d)/d \ll \exp(-\tfrac{1}{2}c_0\sqrt{\log y}).$$

On summing this over $1 \leqslant k \leqslant x/y$ we find that

$$\sum_{y < d \leqslant x} \mu(d)s(x/d)/d \ll xy^{-1}\exp(-\tfrac{1}{2}c_0\sqrt{\log y}).$$

This suffices to give (10) with $c_1 = c_0/4$.

Lemma 2. Let b and $r > 0$ be relatively prime integers, and let β be a real number. Then

$$\sum_{n=1}^{r} s(bn/r + \beta) = s(r\beta). \tag{12}$$

Proof. Both sides are periodic with period $1/r$, so we may assume that $0 \leqslant \beta < 1/r$. The numbers bn run through a complete residue system (mod r), so we may assume that $b = 1$. If $\beta = 0$ then the left hand side is

$$\sum_{n=1}^{r-1} s(n/r) = \sum_{n=1}^{r-1} (1/2 - n/r) = 0.$$

If $0 < \beta < 1$ then the sum is

$$\sum_{n=0}^{r-1} s(n/r + \beta) = \sum_{n=0}^{r-1} (1/2 - n/r - \beta) = 1/2 - r\beta.$$

Thus (12) holds in all cases.

We now consider a sum as in (12), but extended over an arbitrary interval $1 \leqslant n \leqslant N$. The size of such a sum depends in a complicated way on the continued fraction of b/r, but the following crude estimate is sufficient for our present purpose.

Lemma 3. Let b and $r > 0$ be relatively prime integers, and let β be a real number. Then for any positive N,

$$\sum_{n=1}^{N} s(nb/r + \beta) \ll \frac{N}{r}s(r\beta) + O(r). \tag{13}$$

Proof. Write $N = Qr + R$, $0 \leqslant R < r$. By Lemma 2 the above is

$$Qs(r\beta) + \sum_{n=1}^{R} s(nb/r + \beta).$$

Here the first term is $(N/r)s(r\beta) + O(1)$, and the second term is $\ll R \ll r$.

Lemma 4. If q is a positive integer, $q \leqslant \exp(c_2\sqrt{\log N})$, α is a non-integral real number, $0 < \alpha < q$, then

$$\sum_{n=1}^{N} R_0(nq + \alpha) = C(q, \alpha)N + O(N\exp(-c_2\sqrt{\log N})). \tag{14}$$

where

$$C(q, \alpha) = \frac{6}{\pi^2}\left(\prod_{p|q}(1 - p^{-2})\right)^{-1} \sum_{d|q} \mu(d)s(\alpha/d)/d. \tag{15}$$

This estimate is substantially equivalent to Theorem 3.1 of Erdős and Shapiro [3], although both the formulation of the result and the proof are quite different. With more work we could establish an estimate of this sort for larger q, but it would not affect our main results. By taking $q = 1$, $\alpha = 0^+$, we find that

$$\sum_{n=1}^{N} R_0(n) = (3/\pi^2)N + O(N \exp(-c\sqrt{\log N})),$$

a result similar to (2).

Proof. We apply Lemma 1 with $y = (N+1)q \exp(-c_1\sqrt{\log N})$. The sum of the error term in (10) contributes an amount $\ll N \exp(-c_1\sqrt{\log N})$, so it suffices to consider the double sum

$$\sum_{n=1}^{N} \sum_{d \leqslant y} \mu(d)s((nq + \alpha)/d)/d. \tag{16}$$

We invert the order of summation and apply Lemma 3 with $r = d/(d, q)$, $\beta = \alpha/d$. Thus the inner sum is

$$\sum_{n=1}^{N} s((qn + \alpha)/d) = (d, q)s(\alpha/(d, q))N/d + O(d).$$

The error term here, when inserted in (16), contributes an amount $\ll y$, which is acceptable in (14) if we take $c_2 = c_1/3$. We may suppose that d is square-free, and then write $d = ef$ with $e|q$, $(f, q) = 1$. Thus the main term above contributes to (16) the amount

$$N \sum_{e|q} \mu(e)s(\alpha/e)/e \sum_{\substack{f \leqslant y/e \\ (f,q)=1}} \mu(f)/f^2.$$

If we delete the constraint $f \leqslant y/e$ in the inner sum then the above becomes $C(q, \alpha)N$. In making this change the inner sum is altered by an amount $\ll e/y$, which introduces an error term

$$\ll 2^{\omega(q)}N/y \ll (2^{\omega(q)}/q)\exp(c_1\sqrt{\log N}) \ll N \exp(-c_1\sqrt{\log N}).$$

3. Proof of theorem 1

Proof. We recall that $\sum \varphi(n)/n^s = \zeta(s-1)/\zeta(s)$ for $\sigma > 1$. Thus by a familiar Mellin transform,

$$\sum_{n \leqslant x} \left(1 - \frac{n}{x}\right)\varphi(n)/n = \frac{1}{2\pi i}\int_{a - i\infty}^{a + i\infty} \frac{\zeta(s)x^s}{\zeta(s+1)s(s+1)}\, \mathrm{d}s \tag{17}$$

for $a > 1$. Let $\mathscr{C}$ denote the contour $\sigma = -c/\log\tau$, $-\infty < t < +\infty$, where $\tau = |t| + 4$. Then $1/\zeta(s+1) \ll \log\tau$ and $\zeta(s) \ll \tau^{1/2}\log\tau$ on this contour, so that

$$\int_{\mathscr{C}} \cdots \ll \exp(-c\sqrt{\log x}).$$

In the region between $\mathscr{C}$ and the original line of integration, the integrand is regular, apart from a simple pole at $s = 1$ with residue $(3/\pi^2)x$. Thus the expression (17) is $(3/\pi^2)x + O(\exp(-c\sqrt{\log x}))$. But from (1) and (4) we see easily that the left hand side is $(3/\pi^2)x + R_0(x) - R(x)/x$, so we have the stated result.

By treating the identity

$$\sum_{n\leqslant x} (x-n)\varphi(n) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\zeta(s-1)x^{s+1}}{\zeta(s)s(s+1)}\,ds$$

similarly, we find that

$$\int_0^x R(u)\,du \ll x^2\exp(-c\sqrt{\log x}).$$

Thus we see that the main term in (2) is due to the fact that the remainder term is summed, rather than integrated. If in (2) we were to replace $R(n)$ by $R(n^-)$, then the sign of the main term would be reversed on the right hand side.

4. Proof of theorem 2

In view of Theorem 1, it suffices to show that $R_0(x) = \Omega_\pm(\sqrt{\log\log x})$. We apply Lemma 4 with q taken to be the product of all primes $p \equiv 3 \pmod 4$ for which $p \leqslant z$. Here $z \approx \sqrt{\log N}$ is chosen so that $\omega(q)$ is even, which is to say $q \equiv 1 \pmod 4$, and so that $q \leqslant \exp(c_2\sqrt{\log x})$. If $d|q$, then $q/d \equiv 1$ or $3 \pmod 4$ according as $\omega(d)$ is even or odd. That is, $\mu(d)s(q/(4d)) = 1/4$ for all $d|q$, so that

$$C(q, q/4) \approx \sum_{d|q}\mu^2(d)/d \approx \sqrt{\log z} \approx \sqrt{\log\log N}.$$

Thus by taking $\alpha = q/4$ in Lemma 4 we see that there are x for which $R_0(x)$ is large and positive. Since $C(q, 3q/4) = -C(q, q/4)$, we take $\alpha = 3q/4$ to obtain the result with the negative sign.

The construction above, of q for which $\mu(d)$ and $s(q/(4d))$ are correlated, is modelled after the work of Lehmer [4], who showed that there exist q for which

$$\max_N \left| \sum_{\substack{n=1\\(n,q)=1}}^{N} 1 - \varphi(q)N/q \right| \geqslant 2^{\omega(q)-2}.$$

Vijayaraghavan [11] showed that there exist q for which the above quantity is $> 2^{\omega(q)-1} - \delta$, for any $\delta > 0$, but in his constructions $\varphi(q)/q$ is very near 1. In the present context we want $\varphi(q)/q$ to be as small as possible.

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