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# Poincaré series for $SO(n, 1)$

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**Abstract.** A theory of Poincaré series is developed for Lobachevsky space of arbitrary dimension. For a general non-uniform lattice a Selberg-Kloosterman zeta function is introduced. It has meromorphic continuation to the plane with poles at the corresponding automorphic spectrum. When the lattice is a unit group of a rational quadratic form, the Selberg-Kloosterman zeta function is computed explicitly in terms of exponential sums. In this way a non-trivial Ramanujan-like bound analogous to "Selberg's 3/16 bound" is proved in general.

**Keywords.** Poincaré series; Lobachevsky space; Selberg-Kloosterman zeta function; non-uniform lattices.

## 1. Introduction

The theory of holomorphic Poincaré series has been developed and studied quite generally [11]. It allows one to construct explicitly holomorphic cusp forms and to study their Fourier coefficients. Non-holomorphic Poincaré series were introduced by Selberg [10] for $SL(2, \mathbb{R})$. By expanding these series once spectrally in $L^2(\Gamma\backslash SL(2, \mathbb{R})/K)$ and once directly in a Fourier series in a cusp, one obtains a relation between the $L^2$ spectrum and sums of Kloosterman sums. In this way using bounds on Kloosterman sums due to Weil, Selberg established the well-known estimate

$$\lambda_1 \geqslant 3/16 \tag{1}$$

for the second smallest eigenvalue of the Laplacian for any congruence subgroup of $SL(2, \mathbb{R})$. The bound (1) goes part of the way towards the 'Ramanujan Conjecture', $\lambda_1 \geqq 1/4$, which remains a basic outstanding problem; (see Iwaniec [5] for some recent progress).

The above relation and set-up has numerous striking applications [4] which we do not enter into here. Suffice it to say that it is desirable to develop such a theory for more general groups. For $GL(n)$, $n \geqq 3$, computations have been done by Bump–Friedberg–Goldfeld [1] and Stevens [12]. However, it appears that direct estimation of the resulting exponential sums does not produce results better than what follows from property $T$.

In this note we describe results for $G = SO(n, 1)$. We will obtain the meromorphic continuation to $\mathbb{C}$ of the 'Kloosterman-Selberg' zeta function for arbitrary $\Gamma \subseteq G$ as well as the analogue of (1) for congruence subgroups of unit groups of rational quadratic forms. Detailed proofs will appear elsewhere.

## 2. General non-uniform lattices

Let $G$ be the real orthogonal group of a form of signature $(r+1,1)$, $r \geqq 2$, i.e. $G = SO(r+1,1)$. We realize $G$ as

$$G = \left\{ g \in SL_{r+2} \,\middle|\, {}^t g \begin{pmatrix} & & 1 \\ & I_r & \\ 1 & & \end{pmatrix} g = \begin{pmatrix} & & 1 \\ & I_r & \\ 1 & & \end{pmatrix} \right\}, \tag{2}$$

where $I_r$ is the $r \times r$ identity matrix. Denote by $A, H$ and $U$ the subgroups

$$A = \left\{ \begin{pmatrix} a & & \\ & I_r & \\ & & a^{-1} \end{pmatrix} \,\middle|\, a \in \mathbb{R}^* \right\} \tag{3}$$

$H = SO_r$, the standard orthogonal group which we embedded in $G$ via

$$h \to \begin{pmatrix} 1 & & \\ & h & \\ & & 1 \end{pmatrix},$$

$$U = \left\{ \begin{pmatrix} 1 & -{}^t u & -\frac{1}{2}\langle u, u\rangle \\ & I_r & {}^t u \\ & & 1 \end{pmatrix} : u \in \mathbb{R}^r \right\} \tag{5}$$

where $u$ is thought of as a column vector and $\langle\ ,\ \rangle$ is the usual inner product on $\mathbb{R}^r$,

Then $P = UAH$ is a parabolic subgroup of $G$ with unipotent radical $U \cong \mathbb{R}^r$ and Levi component $M = AH$.

Let $\Gamma \subseteq G$ be a discrete subgroup of finite covolume but not compact. We may take one of its cusps to correspond to $U$ i.e. $\Gamma \cap U = \Gamma_\infty$ is a lattice of full rank in $U$. For simplicity assume that in fact $\Gamma \cap P = \Gamma_\infty$ and that this is the only cusp subgroup of $\Gamma$. We fix non-trivial additive characters $\psi$, $\eta$ of $\Gamma_\infty$ (i.e. $\psi, \eta$ are members of the lattice dual to $\Gamma_\infty$ in $\mathbb{R}^r$). The Bruhat decomposition of $G$ asserts that

$$G = P \cup PwU \tag{6}$$

where $w = \begin{pmatrix} & & -1 \\ & I_r & \\ 1 & & \end{pmatrix}$. In this way we may write every $\gamma \in \Gamma$, $\gamma \notin P$ uniquely as

$$\begin{aligned} \gamma &= u(\gamma)a(\gamma)h(\gamma)wv(\gamma), \quad \text{or} \\ &= u(\gamma)m(\gamma)wv(\gamma) \end{aligned} \tag{7}$$

with the obvious notation concerning membership.

Correspondingly let

$$M(\Gamma) = \{ m(\gamma) \mid \gamma \in \Gamma - \Gamma_\infty \}.$$

One may then choose a set of representatives for $\gamma \in \Gamma_\infty \backslash \Gamma / \Gamma_\infty$, $\gamma \notin \Gamma_\infty$ in the form

$$umwv, \tag{8}$$

where for each $m\in M(\Gamma)$, $u$ and $v$ run over a finite set in $U\times U$. The 'Kloosterman sum' $\mathrm{Kl}(\psi,\eta,m)$ for $m\in M(\Gamma)$ is defined by

$$\mathrm{Kl}(\psi,\eta,m)=\sum\psi(u)\eta(v), \tag{9}$$

where the sum ranges over the $u$ and $v$ above. Let $\tau$ be an irreducible (finite dimensional) unitary representation of $H$, we define the 'Kloosterman–Selberg' zeta function by

$$Z(\psi,\eta,\tau,s)=\sum_{m\in M(\Gamma)}\mathrm{Kl}(\psi,\eta,m)\tau(h_m)|a_m|^{s+r/2}. \tag{10}$$

By comparing this series with the Eisenstein series (corresponding to $\Gamma_\infty$) it follows that the series converges absolutely for $\mathrm{Re}(s)>r/2$.

**Theorem 1:** *$Z(\psi,\eta,\tau,s)$ extends to a meromorphic function in $s$ in the entire plane. Furthermore the poles of $Z$ may be explicitly described in terms of the automorphic spectrum (representations) of $L^2(\Gamma\backslash G)$. In particular it is analytic in $\mathrm{Re}(s)\geqq r/2$ and has only finitely many poles in $\mathrm{Re}(s)>0$—these latter will be referred to as exceptional poles (or spectrum).*

In order to describe briefly the precise location of poles as well as the method of proof we consider the case of $r=2$. In this case $G\cong SL(2,\mathbb{C})$. This case contains the essential features. Changing our notation slightly we have $P=UAH$ with

$$U=\left\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\middle| x\in\mathbb{C}\right\},\quad A=\left\{\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}\middle| a\in\mathbb{R}^*\right\},\quad H\left\{\begin{pmatrix}e^{i\theta} & 0\\ 0 & e^{-i\theta}\end{pmatrix}\right\}.$$

Hence $M=AH=\left\{\begin{pmatrix}z & 0\\ 0 & z^{-1}\end{pmatrix}\middle| z\in\mathbb{C}^*\right\}$ and we identify $M$ with $\mathbb{C}^*$ by $\begin{pmatrix}z & \\ & z^{-1}\end{pmatrix}\leftrightarrow z$. The irreducible representations of $H$ correspond to integers $r\in\mathbb{Z}$. Let $b_1, b_2$ be smooth radial functions of compact support on $\mathbb{C}$ and $\mathbb{C}^*$ respectively. Define the function $f$ on $G$ by

$$\begin{aligned} f(u_1wu_2z)&=\psi(u_1)b_1(|u_2|)b_2(|z|)\left(\frac{z}{|z|}\right)^r\\ &=0 \quad \text{if } g \text{ is not of the above form.} \end{aligned} \tag{11}$$

Here $u_1,u_2\in U$, $w=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, $z\in M$ and $\psi$ (and later $\eta$) are characters as before. The Poincaré series $P_f(g)$ on $G$ is defined by

$$P_f(g)=\sum_{\Gamma_\infty\backslash\Gamma}f(\gamma g). \tag{12}$$

The series (12) consists of only a finite number of non-zero terms and in fact $P_f(g)$ is $\Gamma$-automorphic and in $C_0^\infty(\Gamma\backslash G)$ (i.e. smooth of compact support). For $\mathrm{Re}(s)$ large consider

$$I=\int_{\mathbb{C}^*}\int_{\Gamma_\infty\backslash U}P_f(uz)\bar{\eta}(u)\,\mathrm{d}u\left(\frac{(z)}{|z|}\right)^r|z|^s\,\mathrm{d}^*z. \tag{13}$$

A calculation shows that

$$I = \left(\int_{\mathbb{C}^*} b_2(|z|)|z|^s \mathrm{d}^* z\right)$$

$$\times \sum_{n=0}^{\infty} \frac{(i\pi)^{2n}|\xi|^{2n}}{(n!)^2} \int_{\mathbb{C}^*} b_1(|u|)|u|^{2n} \mathrm{d}u.\, Z(\psi, \eta, 4+s+2n, r) \tag{14}$$

where $\eta(u) = \exp[-\pi i(\xi u + \bar{\xi} u)]$ and for $v \in \mathbb{Z} = \hat{H}$

$$Z(\psi, \eta, v, s) = \sum_{\alpha \in M(\Gamma)} K_I(\psi, \eta, \alpha)\left(\frac{\alpha}{|\alpha|}\right)^v |\alpha|^s. \tag{15}$$

Note that due to the slightly different parametrization of $G, Z(\psi, \eta, v, s)$ converges absolutely for $\mathrm{Re}(s) > 4$. Theorem 1 in this case, that is the meromorphic continuation and location of poles of $Z$ follows immediately from (13), (14) and

*Lemma 2*: *Let* $\Theta \in C_0^\infty(\Gamma \backslash G)$. *Then*

$$\int_{\mathbb{C}^*} \int_{\Gamma_\infty \backslash U} \Theta(uz)\psi(u)\,\mathrm{d}u \left(\frac{z}{|z|}\right)^{-r} |z|^s \mathrm{d}^* z$$

*is meromorphic in* $\mathbb{C}$ *with possible poles at the poles of* $L_{\pi \times \chi_r}((s/4) + (1/2))$, *where* $\pi$ *runs through the discrete irreducible constituents of* $L^2(\Gamma \backslash G)$, *as well as the poles of the Eisenstein series* $E(g, (s/4) + (1/2))$. *Here* $\chi$ *is the character* $(z/|z|)^r$ *of* $\mathbb{C}^*$ *and* $L_\pi(s)$ *is the local L-function corresponding to* $\pi$ *(see Jacquet-Langlands* [6]) *which is a product of two-gamma functions.*

In particular the first infinite set of poles of $Z(\psi, \eta, v, s)$ is along the line $\mathrm{Re}(s) = 2$ corresponding to the discrete occurrences of weight $r$ representations of $G$ appearing in $L^2(\Gamma \backslash G)$. Lemma 2 is proved by expanding $\Theta$ spectrally and using the theory of Whittaker functions.

## 3. Lower bounds for $\lambda_1$

In order to obtain a lower bound for exceptional spectrum (i.e. analogue of (1)) we restrict our attention to $\Gamma$'s which are congruence subgroups of unit groups of rational quadratic forms. This is necessary since in fact the bound we obtain, see Corollary 4, is false for general $\Gamma$ (see the comments at the end of this section). For simplicity of exposition we stick to the algebraic group $G$ defined in (2). Let $G$ be the subgroup of $G$ whose entries are integers (i.e. a unit group) and for $D \geqslant 1$ an integer we let $\Gamma(D)$ be the congruence subgroup $\Gamma(D) = \{g \in G \mid g \equiv 1 \pmod{D}\}$. $\Gamma(D)$ is a non-uniform lattice in $G(r \geqslant 3)$. At this point it is both convenient and essential in our approach to work adelically. So we let $A$ be the adele ring over $Q$ and let $G(A)$, $G(Q)$, $P(A)$, $P(Q)$, $U(A)$, $U(Q)\ldots$, etc, be the adelic, respectively $Q$, points of the corresponding subgroups. Let $\psi = \prod_p \psi_p$ be a standard character of $Q \backslash A$. For $\xi \in (1/D)\mathbb{Z}^r$ we define a character $\psi^\xi$ of $U(Q) \backslash U(A)$ by

$$\psi^\xi(u) = \psi(\langle u, \xi \rangle), \quad u \in U(A). \tag{16}$$

Clearly we may decompose $\psi^\xi$ as $\psi^\xi = \prod_p \psi_p^\xi$. To introduce Poincaré series on

$G(Q)\backslash G(A)$ we define functions $f^{\xi}(g)$, $g\in G(A)$ as follows:

(a) For $p=\infty$ we set

$$f^{\xi}_{\infty}(uyk)=\psi^{\xi}_{\infty}(u)y^{s+r/2}\exp(-2\pi|\xi|y),$$

where $u\in U(R)$, $y\in A(R)$, $k\in K_{\infty}$ and $uyk$ is the Iwasawa decomposition. Note here that this is a somewhat different choice to (11) above and that it corresponds to $\tau$ the identity representation of $H$.

(b) For $p<\infty$, let

$$K_p=\{g\in G(\mathbb{Z}_p)|g\equiv 1 \bmod p^l \quad \text{where} \quad p^l\,\|\, D\}.$$

Set $f^{\xi}_p(uk)=\psi^{\xi}_p(u), u\in U(Q_p), k\in K_p$ and $f^{\xi}_p(g)=0$ for $g$ outside the open set $U(Q_p)K_p$. Define

$$f^{\xi}(g)=\prod_p f^{\xi}_p(g_p), \quad g=(g_{\infty},g_2,\ldots)\in G(A).$$

Now $f^{\xi}$ satisfies

$$f^{\xi}(ug)=\psi^{\xi}(u)f^{\xi}(g).$$

Finally the Poincaré series $P_{\xi}(g,s)$ is defined by

$$P_{\xi}(g,s)=\sum_{\gamma\in U(Q)\backslash G(Q)} f^{\xi}(\gamma g). \tag{17}$$

Again by comparison with the Eisenstein series we find that the series (17) converges absolutely for $\mathrm{Re}(s)>r/2$. An inspection shows that the function $P_{\xi}(g,s)$ really lives on $\Gamma(D)\backslash G(R)/K_{\infty}$. As before we consider

$$I(\xi,s)=\int_{U(Q)\backslash U(A)} P_{\xi}(ug,s)\psi^{\xi}(-u)\,\mathrm{d}u. \tag{18}$$

A calculation as before relates $I(\xi,s)$ to the following Kloosterman-Selberg zeta function: For $p<\infty$, $\mathrm{Kl}_p(\xi,\xi,\gamma)$ for $\gamma\in M(Q)$, is the local "Kloosterman integral" (it is a sum in fact)

$$\mathrm{Kl}_p(\xi,\xi,\gamma)=\int_{U(Q_p)} f^{\xi}_p(\xi wu)\psi^{\xi}_p(-u)\,\mathrm{d}u. \tag{19}$$

For example if $\gamma=ah\in M(Q)$ with $|h|_p=|a|_p>1$, where $|h|_p=\max_{ij}(|h_{ij}|_p)$, then

$$\mathrm{Kl}_p(\gamma)=\sum_{\lambda\in(\mathbb{Z}_p/a^{-1}\mathbb{Z}_p)^*}\psi^{\xi}_p\left(\lambda u_0+\frac{1}{\lambda}v_0\right) \tag{19'}$$

for suitable $u_0,v_0\in U(Q_p)$, $|u_0|_p=|v_0|_p=|a|_p$. (19′) is a classical one-dimensional Kloosterman sum over the field with $p$ elements. In the other cases $\mathrm{Kl}_p(\gamma)$ is more complicated being an exponential sum in more than one variable.

The Kloosterman sum corresponding to $\gamma \in M(A)$ is

$$\mathrm{Kl}(\xi, \xi, \gamma) = \prod_{p<\infty} \mathrm{Kl}_p(\xi, \xi, \gamma). \tag{20}$$

The Kloosterman zeta function is

$$Z(\xi, \xi, s) = \sum_{\gamma \in M(Q)} \mathrm{Kl}(\xi, \xi, \gamma) |a_\gamma|^{s+r/2}. \tag{21}$$

This zeta function is of course a special case of those introduced in § 2 (corresponding to $\Gamma = \Gamma(D)$) but as it appears here the Kl's are factored naturally over the primes. This is the reason the adelic approach was taken since with this factorization nontrivial estimates on $\mathrm{Kl}(\xi, \xi, \gamma)$ can be made via ones for $\mathrm{Kl}_p(\xi, \xi, \gamma)$.

The precise relation between $I(\xi, s)$ and $Z(\xi, \xi, s)$ is that in $\mathrm{Re}(s) > \frac{1}{2}r - 1$ they have the same poles. On the other hand by spectrally decomposing $P_\xi(g)$ one can determine the precise location of poles of $I(\xi, s)$. To state the result we let $\Theta_j(g), j = 1, \ldots, R$ be a basis of eigenfunctions of $\Delta$ (the Laplacian) on $\Gamma(D)\backslash G(R)/K_\infty$ which are exceptional, i.e. their eigenvalues $\lambda_j = (\frac{1}{2}r - v_j)(\frac{1}{2}r + v_j)$ satisfy $0 < \lambda_j < (\frac{1}{2}r)^2$, or what is the same $v_j$ is real and $v_j > 0$.

PROPOSITION 3:

*$I(\xi, s)$ has simple poles at $s = v_j$ and positive residue if the $\xi$th Fourier coefficient of $\theta_j$ is non-zero.*

We now come to the main point.

**Theorem 4:** *$Z(\xi, \xi, s)$ is holomorphic in* $\mathrm{Re}(s) > \frac{1}{2}(r-1)$.

This is proved by appealing to the definition of $Z(\xi, \xi, s)$ in (21) and estimating the terms $\mathrm{Kl}(\xi, \xi, \gamma)$. The $\mathrm{Kl}_p(\xi, \xi, \gamma)$ of the form (19′) are estimated (when they are non-trivial) by Weil's bound [13] while all other $\mathrm{Kl}_p$'s are estimated by a trivial volume bound.

To obtain the result one needs to carefully examine those $\gamma$'s for which $f^\xi(\gamma w u) \neq 0$ for some $u$. In this way one is able to control the contribution from those terms which are estimated trivially.

From the above considerations we obtain the following analogue of (1) (see also [9] for the case $SL_2(\sigma) \subset SL_2(C)$ where $\sigma$ is the ring of integers in an imaginary quadratic field).

COROLLARY 5:

*If $\lambda_1$ is the second smallest ($\lambda_0 = 0$) eigenvalue of $\Delta$ on $L^2(\Gamma(D)\backslash G(R)/K_\infty)$ then*

$$\lambda_1 \geqq \tfrac{1}{2}(r - \tfrac{1}{2}).$$

In this context the analogue of the "Ramanujan conjecture" would be the assertion that $\lambda_1 \geqq (\frac{1}{2}r)^2$ (for $\Gamma(D)$), that is there is no exceptional spectrum. Corollary 4 says that to some extent this conjecture is true, however, we note that for $r \geqq 3$ it is in fact false. One can construct discrete exceptional spectrum on $\Gamma(D)\backslash G(R)/K_\infty$ (even cuspidal) with $\lambda = \frac{1}{2}r - 1, \frac{1}{2}r - 3 \ldots$. This phenomenon is similar to what happens for half integral weight for $SL(2)^{\sim}$. In fact the above exceptional forms are constructed by lifting holomorphic cusp forms for $SL(2)^{\sim}$ (or $SL(2)$ depending on the parity of $r$) to $\Gamma(D)\backslash G(R)$. These 'counter examples' are similar to those constructed by Howe and Piatetski-Shapiro [3].

To conclude we note that the above analysis and results hold equally well for any unit group of a rational quadratic form. Also similar non-trivial estimates for the representation parameters occurring in $L_0^2(G(Q)\backslash G(A))$ at places $p$ other than $\infty$ may be derived in a similar way as long as $G_p$ is still rank 1. (The other cases represent no problem in view of property $T$). For a treatment of the theory at finite places over $GL(2)$ and for a general introduction to the $GL(2)$ adelic Poincaré series, see Piatetski-Shapiro [7].

Finally, note that Theorem 3 does not hold for all $\Gamma$. Using deformation argument as in [10, 8] one may construct $\Gamma$'s with $\lambda_1$ as small as we please. Of course such $\Gamma$'s cannot be congruence groups.

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**Note:** Some results similar to those obtained here were announced recently by Elstrodt Grunewald and Mennicke [2]. Their methods however seem different to those presented here.

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