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Approximation of exponential sums by shorter ones

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Abstract. A new theorem on approximation of exponential sum by shorter one is proved.

Keyword. Exponential sums.

1. Introduction

This paper is dedicated to the centenary of S Ramanujan, whose works form a fundamental part of additive number theory—one of the largest areas in analytic number theory. A number of problems in the theory involve formulae of the kind

$$S = S_1 + R, \tag{1}$$

where

$$S = \sum_{a < x \leqslant b} \varphi(x)\exp[2\pi i f(x)],$$

$$S_1 = \sum_{\alpha < n \leqslant \beta} G(n)\exp[2\pi i F(n)],$$

$\varphi(x)$, $f(x)$, $G(n)$, $F(n)$ being some real functions, $b-a>\beta-\alpha>0$, and R—the "remainder" term. It seems that the first relations of type (1) with concrete functions φ, f, G, F were obtained by Vinogradov [5] and Hardy and Littlewood [1]. In a general form the theorem was proved by van der Corput [4] (c.f. also [3]). One of the most exact versions of (1) from the point of view of the estimate of R was proved by Vinogradov in [6]. In the conditions of Vinogradov's theorem algebraicity of the functions $\varphi(x)$ and $f''(x)$ was required. In [2] the present author proved (1) under weaker assumptions concerning $\varphi(x)$ and $f(x)$. A disadvantage of the theorem in [2] is the monotonicity condition, of a certain auxiliary function which, being controllable in every given case of applications seems nevertheless superfluous to a certain extent. Besides the estimate of the quantity R is rough. The aim of this paper is to present a new version of the proof of (1), which preserves the accuracy of Vinogradov's theorem and is close in the sense of the assumptions to those of van der Corput.

We use the following notations. For positive A the relations $B \ll A$, $A \gg B$ mean that $|B| \leqslant cA$, where $c>0$ is an absolute constant, $\|\alpha\| = \min(\{\alpha\}, 1-\{\alpha\})$, where $\{\alpha\}$ is the fractional part of α and for positive integer k, $f^{(k)}(x) = \mathrm{d}^k f(x)/\mathrm{d}x^k$.

2. Statement of the result

Theorem. *Let the real functions $f(x)$ and $\varphi(x)$ satisfy the following conditions on the segment $[a, b]$:*

a. $f^{(4)}(x)$ *and* $\varphi^{(2)}(x)$ *are continuous;*
b. There exist numbers H, U, A, $0<H$, $1 \ll A \ll U$, $0<b-a\leqslant U$, *such that*

$$A^{-1} \ll f^{(2)}(x) \ll A^{-1}, \quad f^{(3)}(x) \ll A^{-1}U^{-1}, \quad f^{(4)}(x) \ll A^{-1}U^{-2},$$

$$\varphi(x) \ll H, \quad \varphi^{(1)}(x) \ll HU^{-1}, \quad \varphi^{(2)}(x) \ll HU^{-2}.$$

Then if we define the numbers x_n *from the equation*

$$f^{(1)}(x_n)=n,$$

we have

$$\sum_{a<x\leqslant b} \varphi(x)\exp[2\pi i f(x)] = \sum_{f^{(1)}(a)\leqslant n\leqslant f^{(1)}(b)} c(n)Z(n)+R, \tag{2}$$

where

$$R = O(HA(b-a)^{-1}) + O(HT_a) + O(HT_b)$$
$$+\, O(H\ln(f^{(1)}(b)-f^{(1)}(a)+2)), \tag{3}$$

$$T_\mu = \begin{cases} 0, & \textit{if } f^{(1)}(\mu) \textit{ is an integer} \\ \min\left(\dfrac{1}{\| f^{(1)}(\mu)\|}, \sqrt{A}\right), & \textit{if } \| f^{(1)}(\mu)\| \neq 0, \end{cases}$$

$$c(n) = \begin{cases} 1, & \textit{if } f^{(1)}(a)<n<f^{(1)}(b), \\ \frac{1}{2}, & \textit{if } n=f^{(1)}(a) \textit{ or } n=f^{(1)}(b), \end{cases}$$

$$Z(n) = \frac{1+i}{\sqrt{2}}\frac{\varphi(x_n)}{[f^{(2)}(x_n)]^{1/2}}\exp[2\pi i(f(x_n)-nx_n)]. \tag{4}$$

3. Lemmas

We prove two auxiliary statements.

Lemma 1. Let $f(x)$ *and* $\varphi(x)$ *be real functions, satisfying the following requirements on the segment* $[a,b]$:

(*a*) $f^{(2)}(x)$ *and* $\varphi^{(1)}(x)$*—are continuous:*
(*b*) $0<f^{(2)}(x)\ll 1$;
(*c*) *there exist numbers* $0<H$, $0<b-a\leqslant U$, *such that*

$$\varphi(x) \ll H, \quad \varphi^{(1)}(x) \ll HU^{-1}.$$

Then for each $\Delta\in(0,1)$ *the equality*

$$\sum_{a<x\leqslant b}\varphi(x)\exp[2\pi i f(x)] = \sum_{\alpha-\Delta\leqslant n\leqslant \beta+\Delta}\int_a^b \varphi(x)\exp[2\pi i(f(x)-nx)]\,dx$$
$$+\, O(H\ln(\beta-\alpha+2)), \tag{5}$$

is valid, where $\alpha=f^{(1)}(a)$, $\beta=f^{(1)}(b)$ *and the factor in the sign* O *depends only on* Δ.

Proof. We assume that $b - a > 10$, since in the contrary case the assertion of the lemma is trivial. Let $\varphi(x) = 1$. Take a positive integer $m = [10U^2K]$, $K = 1 + |\alpha| + |\beta|$, and for $[a] + 2 \leqslant M \leqslant [b] - 1$ consider the integral W_M,

$$W_M = \int_{-0\cdot 5}^{0\cdot 5} \frac{\sin(2m+1)\pi x}{\sin \pi x} \exp[2\pi i f(M+x)]\,dx.$$

Since

$$\frac{\sin(2m+1)\pi x}{\sin \pi x} = \sum_{n=-m}^{m} \exp[2\pi i n x], \tag{6}$$

we have

$$\int_{-0\cdot 5}^{0\cdot 5} \frac{\sin(2m+1)\pi x}{\sin \pi x}\,dx = 1,$$

and consequently

$$W_M = \exp[2\pi i f(M)] + V_M,$$

where

$$V_M = \int_{-0\cdot 5}^{0\cdot 5} \frac{\sin(2m+1)\pi x}{\sin \pi x}\{\exp[2\pi i f(M+x)] - \exp[2\pi i f(M)]\}\,dx.$$

To estimate $|V_M|$ from above we represent V_M as a sum of the following three integrals:

$$V_M = \int_{-1/m}^{1/m} + \int_{-1/2}^{-1/m} + \int_{1/m}^{1/2}.$$

We estimate the first integral on the right hand side, using the finite differences formula to the expression in paranthesis:

$$\int_{-1/m}^{1/m} \ll \int_{-1/m}^{1/m} \frac{|f^{(1)}(x)||x|\,dx}{|x|} \ll \frac{K}{m} \ll \frac{1}{U}.$$

The second and the third integrals are treated in the same way. For example, the latter may be estimated by first integrating by parts as:

$$\int_{1/m}^{1/2} \frac{\sin(2m+1)\pi x}{\sin \pi x}(\exp[2\pi i f(M+x)] - \exp[2\pi i f(M)])\,dx$$

$$= -\frac{\exp[2\pi i f(M+x)] - \exp[2\pi i f(M)]}{\sin \pi x}\cdot\frac{\cos(2m+1)\pi x}{(2m+1)\pi}\Bigg|_{1/m}^{1/2}$$

$$+ \int_{1/m}^{1/2} \frac{\cos(2m+1)\pi x}{(2m+1)\pi} Y_x\,dx,$$

where

$$Y_x = \frac{\exp[2\pi i f(M+x)]2\pi i f^{(1)}(M+x)}{\sin \pi x}$$

$$-\frac{(\exp[2\pi i f(M+x)] - \exp[2\pi i f(M)])\pi\cos \pi x}{\sin^2 \pi x}.$$

The first summand is of the order $O(Km^{-1}) = O(U^{-1})$. Furthermore

$$Y_x \ll \frac{K}{|x|}, \quad \int_{1/m}^{1/2} \frac{\cos(2m+1)\pi x}{(2m+1)\pi} Y_x \, dx \ll \frac{K}{m} \ln m \ll \frac{1}{U}.$$

Thus

$$V_M \ll U^{-1}; \quad W_M = \exp[2\pi i f(M)] + O(U^{-1}).$$

If we sum up the latter expression with respect to M and use the definition of W_M and formula (6), we obtain:

$$\sum_{a<x\leqslant b} \exp[2\pi i f(x)] = \sum_{M=[a]+2}^{[b]-1} W_M + O(1)$$

$$= \sum_{M=[a]+2}^{[b]-1} \int_{-0\cdot5}^{0\cdot5} \sum_{n=-m}^{m} \exp[2\pi i(f(M+x) - nx)] \, dx + O(1)$$

$$= \sum_{n=-m}^{m} \sum_{M=[a]+2}^{[b]-1} \int_{M-0\cdot5}^{M+0\cdot5} \exp[2\pi i(f(x) - nx)] \, dx + O(1)$$

$$= \sum_{n=-m}^{m} I_n + O(1), \tag{7}$$

where

$$I_n = \int_{[a]+1\cdot5}^{[b]-0\cdot5} \exp[2\pi i(f(x) - nx)] \, dx.$$

Next we estimate the sum of the summands from (7) with $n < f^{(1)}(a) - \Delta$ and $n > f^{(1)}(b) + \Delta$, i.e. the sum Σ,

$$\sum = \sum_{-m\leqslant n<f^{(1)}(a)-\Delta} I_n + \sum_{f^{(1)}(b)+\Delta<n\leqslant m} I_n.$$

Integrating by parts we find:

$$I_n = \frac{\exp[2\pi i(f(x) - nx)]}{2\pi i(f^{(1)}(x) - n)}\bigg|_{a_1}^{b_1} + O\left(\int_{a_1}^{b_1} \frac{f^{(2)}(x)\,dx}{(f^{(1)}(x) - n)^2}\right)$$

$$= \frac{\exp(\pi i n)}{2\pi i}\left(\frac{\exp[2\pi i f(b_1)]}{f^{(1)}(b_1) - n} - \frac{\exp[2\pi i f(a_1)]}{f^{(1)}(a_1) - n}\right)$$

$$+ O\left(\frac{f^{(1)}(b_1) - f^{(1)}(a_1)}{(f^{(1)}(a_1) - n)(f^{(1)}(b_1) - n)}\right),$$

where $a_1 = [a] + 1\cdot5$, $b_1 = [b] - 0\cdot5$. Since the fractions

$$(f^{(1)}(b_1) - n)^{-1}, \quad (f^{(1)}(a_1) - n)^{-1}$$

for $-m \leqslant n < f^{(1)}(a) - \Delta$ and $f^{(1)}(b) + \Delta < n \leqslant m$ are monotonically increasing, $\exp(\pi i n) = (-1)^n$, we have

$$\sum \frac{(-1)^n}{f^{(1)}(b_1) - n} = O(1), \quad \sum \frac{(-1)^n}{f^{(1)}(a_1) - n} = O(1).$$

Furthermore, for $-m \leqslant n < f^{(1)}(a) - \Delta$ the fraction

$$\frac{f^{(1)}(b_1) - f^{(1)}(a_1)}{(f^{(1)}(a_1) - n)(f^{(1)}(b_1) - n)}$$

increases monotonically and thus

$$\sum_{-m \leqslant n < f^{(1)}(a) - \Delta} \frac{f^{(1)}(b_1) - f^{(1)}(a_1)}{(f^{(1)}(a_1) - n)(f^{(1)}(b_1) - n)} \ll \frac{f^{(1)}(b_1) - f^{(1)}(a_1)}{\Delta(f^{(1)}(b_1) - f^{(1)}(a_1) + \Delta)}$$
$$+ \int_{-m}^{f^{(1)}(a) - \Delta} \frac{f^{(1)}(b_1) - f^{(1)}(a_1)}{(f^{(1)}(a_1) - x)(f^{(1)}(b_1) - x)} \,\mathrm{d}x = O(\ln(\beta - \alpha + 2)).$$

The sum of the fractions for $f^{(1)}(b) + \Delta < n \leqslant m$ is estimated analogously. Thus,

$$\sum = O(\ln(\beta - \alpha + 2));$$
$$\sum_{a < x \leqslant b} \exp[2\pi i f(x)] = \sum_{\alpha - \Delta \leqslant n \leqslant \beta + \Delta} I_n + O(\ln(\beta - \alpha + 2)).$$

Let us substitute I_n for the integral over the interval (a, b):

$$I_n = \int_a^b \exp[2\pi i(f(x) - nx)]\,\mathrm{d}x - I_{n,1} - I_{n,2},$$

where

$$I_{n,1} = \int_a^{a_1} \exp[2\pi i(f(x) - nx)]\,\mathrm{d}x, \quad I_{n,2} = \int_{b_1}^b \exp[2\pi i(f(x) - nx)]\,\mathrm{d}x.$$

Each segment $[\alpha - \Delta, f^{(1)}(a_1) + \Delta]$ and $[f^{(1)}(b_1) - \Delta, \beta + \Delta]$ can contain not more than $O(1)$ of integers n. Excluding those numbers for the remaining ones n we have the estimates

$$I_{n,1} = O(|f^{(1)}(a_1) - n|^{-1} + |\alpha - n|^{-1}),$$
$$I_{n,2} = O(|f^{(1)}(b_1) - n|^{-1} + |\beta - n|^{-1}).$$

Consequently

$$\sum_{\alpha - \Delta \leqslant n \leqslant \beta + \Delta} (I_{n,1} + I_{n,2}) = O(\ln(\beta - \alpha + 2));$$
$$\sum_{a < x \leqslant b} \exp[2\pi i f(x)] = \sum_{\alpha - \Delta \leqslant n \leqslant \beta + \Delta} \int_a^b \exp[2\pi i(f(x) - nx)]\,\mathrm{d}x$$
$$+ O(\ln(\beta - \alpha + 2)).$$

Thus, our lemma is proved for $\varphi(x) = 1$. Now let $\varphi(x)$ be an arbitrary function. If we apply the formula of summation by parts, we see that

$$\sum_{a < x \leqslant b} \varphi(x) \exp[2\pi i f(x)] = -\int_a^b \mathbb{C}(u)\varphi'(u)\,\mathrm{d}u + \mathbb{C}(b)\varphi(b),$$

where

$$\mathbb{C}(u) = \sum_{a < x \leqslant u} \exp[2\pi i f(x)].$$

We apply the already proved statement to $\mathbb{C}(u)$:

$$\mathbb{C}(u) = \sum_{\alpha-\Delta\leqslant n\leqslant\gamma+\Delta} \int_a^u \exp[2\pi i(f(x) - nx)]\,dx + O(\ln(\beta-\alpha+2)),$$

$$\gamma = f^{(1)}(u) \leqslant \beta;$$

as for $\gamma + \Delta < n \leqslant \beta + \Delta$

$$\int_a^u \exp[2\pi i(f(x) - nx)]\,dx = O\left(\frac{1}{n-\gamma}\right).$$

We have

$$\mathbb{C}(u) = \sum_{\alpha-\Delta\leqslant n\leqslant\beta+\Delta} \int_a^u \exp[2\pi i(f(x) - nx)]\,dx + O(\ln(\beta-\alpha+2));$$

$$\sum_{a<x\leqslant b} \varphi(x)\exp[2\pi i f(x)] = -\sum_{\alpha-\Delta\leqslant n\leqslant\beta+\Delta} \int_a^b \left(\int_a^u \exp[2\pi i(f(x) - nx)]\,dx\right)$$

$$\times\, \varphi^{(1)}(u)\,du + \sum_{\alpha-\Delta\leqslant n\leqslant\beta+\Delta} \left(\int_a^b \exp[2\pi i(f(x) - nx)]\,dx\right)\varphi(b)$$

$$+\, O(H\ln(\beta-\alpha+2)) = \sum_{\alpha-\Delta\leqslant n\leqslant\beta+\Delta} \int_a^b \varphi(x)\exp[2\pi i(f(x) - nx)]\,dx$$

$$+\, O(H\ln(\beta-\alpha+2)),$$

which completes the proof of the lemma.

Lemma 2. Let $f(x)$ and $\varphi(x)$ be real functions satisfying the following conditions on the segment $[a, b]$:

(a) $f^{(4)}(x)$ and $\varphi^{(2)}(x)$ are continuous;
(b) there exist numbers H, U, A, $0 < H$, $1 \ll A \ll U$, $0 < b - a \leqslant U$, such that

$$A^{-1} \ll f^{(2)}(x) \ll A^{-1}, \quad f^{(3)}(x) \ll A^{-1}U^{-1}, \quad f^{(4)}(x) \ll A^{-1}U^{-2},$$

$$\varphi \ll H, \quad \varphi^{(1)}(x) \ll HU^{-1}, \quad \varphi^{(2)}(x) \ll HU^{-2};$$

(c) for a certain c, $a \leqslant c \leqslant b$, $f'(c) = 0$.
Then the following formula is valid:

$$\int_a^b \varphi(x)\exp[2\pi i f(x)]\,dx = \frac{1+i}{\sqrt{2}}\frac{\varphi(c)\exp[2\pi i f(c)]}{[f^{(2)}(c)]^{1/2}} + O(HAU^{-1})$$

$$+\, O\left(H_{\min}\left(\frac{1}{|f^{(1)}(a)|}, \sqrt{A}\right)\right) + O\left(H_{\min}\left(\frac{1}{|f^{(1)}(b)|}, \sqrt{A}\right)\right).$$

Proof. If we divide the interval of integration by the point c into two parts, we obtain:

$$\int_a^b \varphi(x)\exp[2\pi i f(x)]\,dx = \int_a^c + \int_c^b. \tag{8}$$

Compute the second integral. It is obvious that

$$\int_c^b \varphi(x)\exp[2\pi i f(x)]\,dx = \varphi(c)\int_c^b \exp[2\pi i f(x)]\,dx$$
$$+\int_0^{b-c}(\varphi(x+c)-\varphi(c))\exp[2\pi i f(x+c)]\,dx.$$

Estimate the second integral of this formula. Integrating by parts, we get:

$$\int_0^{b-c}(\varphi(x+c)-\varphi(c))\exp[2\pi i f(x+c)]\,dx = \frac{\varphi(x+c)-\varphi(c)}{2\pi i f^{(1)}(x+c)}\exp[2\pi i f(x+c)]\Big|_0^{b-c}$$
$$-\frac{1}{2\pi i}\int_0^{b-c}\frac{\varphi^{(1)}(x+c)f^{(1)}(x+c)-f^{(2)}(x+c)(\varphi(x+c)-\varphi(c))}{(f^{(1)}(x+c))^2}\exp[2\pi i f(x+c)]\,dx.$$

Since

$$\varphi(x+c)-\varphi(c)=\varphi^{(1)}(c)x+O(HU^{-1}x^2),$$
$$f^{(1)}(x+c)=f^{(2)}(c)x+O(A^{-1}U^{-1}x^2),\quad f^{(1)}(x+c)\gg xA^{-1}\text{ and}$$
$$f^{(2)}(x+c)=f^{(2)}(c)+O(A^{-1}U^{-1}x),$$

the first summand is of the order $O(HAU^{-1})$, and the absolute value of the integrand does not exceed

$$|\varphi^{(1)}(c)+O(HU^{-1}x))(f^{(2)}(c)x+O(A^{-1}U^{-1}x^2))$$
$$-(f^{(2)}(c)+O(A^{-1}U^{-1}x))(\varphi^{(1)}(c)x+O(HU^{-1}x^2))|$$
$$\ll HA^{-1}U^{-2}x^2.$$

Thus the integral as a whole is $O(HAU^{-1})$. Now compare J and J_1 defined by

$$J=\int_0^{b-c}\exp[2\pi i f(x+c)]\,dx,\quad J_1=\int_0^{b-c}\frac{f^{(1)}(x+c)\exp[2\pi i f(x+c)]\,dx}{[2f^{(2)}(c)(f(x+c)-f(c))]^{1/2}}.$$

Observing the difference $J-J_1$, let us integrate by parts. Then we find:

$$J-J_1=\frac{1}{2\pi i}\left(\frac{1}{f^{(1)}(x+c)}-\frac{1}{[2f^{(2)}(c)(f(x+c)-f(c))]^{1/2}}\right)\exp[2\pi i f(x+c)]\Big|_0^{b-c}$$
$$+O\left(\int_0^{b-c}\left|-\frac{f^{(2)}(x+c)}{(f^{(1)}(x+c))^2}+\frac{f^{(1)}(x+c)}{[8f^{(2)}(c)](f(x+c)-f(c))^{3/2}}\right|dx\right).$$

Using the relations

$$f(x+c)-f(c)=\tfrac{1}{2}f^{(2)}(c)x^2+\tfrac{1}{6}f^{(3)}(c)x^3+O(A^{-1}U^{-2}x^4),$$
$$f^{(1)}(x+c)=f^{(2)}(c)x+\tfrac{1}{2}f^{(3)}(c)x^2+O(A^{-1}U^{-2}x^3)\ \text{ and}$$
$$f^{(2)}(x+c)=f^{(2)}(c)+f^{(3)}(c)x+O(A^{-1}U^{-2}x^2),$$

we easily find that the first summand is of the order $O(AU^{-1})$ while the integrand of the second one is $O(AU^{-2})$. Consequently,

$$J = J_1 + O(AU^{-1}).$$

We now compute J_1 by introducing the new variable of integration $f(x+c) - f(c) = u$. Denoting by λ the difference $f(b) - f(c)$, we obtain

$$J_1 = \frac{\exp[2\pi i f(c)]}{[2f^{(2)}(c)]^{1/2}} \int_0^{\lambda} \frac{\exp(2\pi iu)\,du}{\sqrt{u}} = \frac{\exp[2\pi i f(c)]}{[2f^{(2)}(c)]^{1/2}} \int_0^{\infty} \frac{\exp(2\pi iu)\,du}{\sqrt{u}}$$

$$- \frac{\exp[2\pi i f(c)]}{[2f^{(2)}(c)]^{1/2}} \int_{\lambda}^{\infty} \frac{\exp(2\pi iu)\,du}{\sqrt{u}}.$$

We estimate the latter integral in two ways. First of all

$$\left| \int_{\lambda}^{\infty} \frac{\exp(2\pi iu)\,du}{\sqrt{u}} \right| \leqslant \int_{\lambda}^{\lambda+1} \frac{du}{\sqrt{u}} + \left| \int_{\lambda+1}^{\infty} \frac{\exp(2\pi iu)\,du}{\sqrt{u}} \right| \ll 1;$$

and besides, for $0 < \lambda$

$$\left| \int_{\lambda}^{\infty} \frac{\exp(2\pi iu)\,du}{\sqrt{u}} \right| \ll \frac{1}{\sqrt{\lambda}} = \frac{1}{[f(b) - f(c)]^{1/2}}.$$

Since

$$f(b) - f(c) = \tfrac{1}{2} f^{(2)}(\xi)(b-c)^2 \gg (b-c)^2 A^{-1},$$

$$|f^{(1)}(b)| = |f^{(1)}(b - c + c)| = f^{(2)}(\xi_1)(b-c) \ll (b-c)A^{-1},$$

we have

$$\frac{1}{[f(b) - f(c)]^{1/2}} \ll \frac{\sqrt{A}}{b-c} \ll \frac{1}{|f^{(1)}(b)|\sqrt{A}}.$$

Thus we obtain that

$$J_1 = \frac{\exp[2\pi i f(c)]}{[2f^{(2)}(c)]^{1/2}} \int_0^{\infty} \frac{\exp(2\pi iu)\,du}{\sqrt{u}} + O\left(\min\left(\frac{1}{|f^{(1)}(b)|}, \sqrt{A} \right) \right);$$

$$\int_c^b = \frac{\varphi(c) \exp[2\pi i f(c)]}{[2f^{(2)}(c)]^{1/2}} \int_0^{\infty} \frac{\exp(2\pi iu)\,du}{\sqrt{u}} + O(HAU^{-1})$$

$$+ O\left(H_{\min}\left(\frac{1}{|f^{(1)}(b)|}, \sqrt{A} \right) \right).$$

The first integral in (8) is computed in an analogous manner:

$$\int_a^c = \frac{\varphi(c) \exp[2\pi i f(c)]}{[2f^{(2)}(c)]^{1/2}} \int_0^{\infty} \frac{\exp(2\pi iu)\,du}{\sqrt{u}} + O(HAU^{-1})$$

$$+ O\left(H_{\min}\left(\frac{1}{|f^{(1)}(a)|}, \sqrt{A} \right) \right).$$

Since

$$\int_0^\infty \frac{\cos 2\pi u \, du}{\sqrt{u}} = \int_0^\infty \frac{\sin 2\pi u \, du}{\sqrt{u}} = \frac{1}{2},$$

the proved relations imply the statement of the lemma.

Remark. If we estimate the right hand side of the formula of the lemma trivially, we obtain

$$\left| \int_a^b \varphi(x) \exp[2\pi i f(x)] \, dx \right| \ll H\sqrt{A}.$$

The above estimate is valid also under weaker assumptions concerning the functions $f(x)$ and $\varphi(x)$, namely, if $f^{(2)}(x)$ and $\varphi^{(1)}(x)$ are continuous on $[a, b]$ and, besides,

$$A^{-1} \ll f^{(2)}(x) \ll A^{-1}, \quad \varphi(x) \ll H, \quad \varphi^{(1)}(x) \ll HU^{-1}, \quad a \leqslant x \leqslant b.$$

Let us prove it. We can assume, with no loss of generality, that $b - a \geqslant 4\sqrt{A}$. If the root, say c, of the equation $f^{(1)}(x) = 0$ belongs to the segment $[a, b]$, then we use the equality

$$\int_a^b = \int_a^{c-\sqrt{A}} + \int_{c-\sqrt{A}}^{c+\sqrt{A}} + \int_{c+\sqrt{A}}^b.$$

and estimate each of the three appearing integrals. The integral in the middle is estimated trivially:

$$\int_{c-\sqrt{A}}^{c+\sqrt{A}} \ll H\sqrt{A}.$$

The integrals on the extremes are estimated in the same way. E.g., consider the first one. We can assume that $a < c - \sqrt{A}$. Integrating by parts we find:

$$\int_a^{c-\sqrt{A}} \varphi(x) \exp[2\pi i f(x)] \, dx = \frac{1}{2\pi i} \frac{\varphi(x)}{f^{(1)}(x)} \exp[2\pi i f(x)] \Big|_a^{c-\sqrt{A}}$$

$$- \frac{1}{2\pi i} \int_a^{c-\sqrt{A}} \frac{\varphi^{(1)}(x) f^{(1)}(x) - \varphi(x) f^{(2)}(x)}{(f^{(1)}(x))^2} \exp[2\pi i f(x)] \, dx.$$

Since $f^{(1)}(x)$ is monotonically increasing,

$$f^{(1)}(c) = 0, \quad a \leqslant c \leqslant b, \quad |f^{(1)}(a)| \geqslant |f^{(1)}(c - \sqrt{A})| \gg A^{-0 \cdot 5}$$

and the first summand of the latter formula are of the order $H\sqrt{A}$. The second summand, i.e. the integral, is estimated trivially, using that

$$f^{(1)}(x) = f^{(1)}(c + x - c) = f^{(2)}(\xi)(x - c),$$

we find

$$\int_a^{c-\sqrt{A}} \ll \int_a^{c-\sqrt{A}} \frac{HU^{-1}AU + HA^{-1}}{A^{-2}(x-c)^2} \, dx \ll H\sqrt{A}.$$

On the other hand, if c does not belong to $[a, b]$, we represent the integral in the form

$$\int_a^b = \int_a^{a+\sqrt{A}} + \int_{a+\sqrt{A}}^{b-\sqrt{A}} + \int_{b-\sqrt{A}}^b$$

The integrals on the extremes are trivially estimated by the quantity $\ll H\sqrt{A}$. The integral in the middle is estimated by means of partial integration:

$$\int_{a+\sqrt{A}}^{b-\sqrt{A}} \varphi(x)\exp[2\pi i f(x)]\,\mathrm{d}x = \frac{\varphi(x)}{2\pi i f^{(1)}(x)}\exp[2\pi i f(x)]\Bigg|_{a+\sqrt{A}}^{b-\sqrt{A}}$$

$$-\frac{1}{2\pi i}\int_{a+\sqrt{A}}^{b-\sqrt{A}} \frac{\varphi^{(1)}(x)f^{(1)}(x)-\varphi(x)f^{(2)}(x)}{(f^{(1)}(x))^2}\exp[2\pi i f(x)]\,\mathrm{d}x.$$

If $f^{(1)}(x) > 0$ for $a \leqslant x \leqslant b$, then

$$f^{(1)}(b-\sqrt{A}) > f^{(1)}(a+\sqrt{A}) = f^{(1)}(a) + f^{(2)}(\xi)\sqrt{A} \gg A^{-0\cdot 5},$$

$$f^{(1)}(x) = f^{(1)}(a + x - a) \gg A^{-1}(x-a).$$

Thus the first summand is of the order $O(H\sqrt{A})$.

The second summand, i.e. the integral, is estimated trivially:

$$\int_{a+\sqrt{A}}^{b-\sqrt{A}} \ll \int_{a+\sqrt{A}}^{b-\sqrt{A}} \left(\frac{HU^{-1}A}{x-a} + \frac{HA}{(x-a)^2}\right)\mathrm{d}x$$

$$\ll HU^{-1}A\ln(b-a) + H\sqrt{A} \ll H\sqrt{A}.$$

On the other hand, if $f^{(1)}(x) < 0$, then the estimation runs in an analogous way. Thus, the statement of the remark is proved.

4. Proof of the theorem

At first, let $\|f^{(1)}(a)\| \neq 0$, $\|f^{(1)}(b)\| \neq 0$. Take $\Delta = \frac{1}{2}$ in lemma 1; then we find:

$$\sum_{a\leqslant x\leqslant b} \varphi(x)\exp[2\pi i f(x)]$$

$$= \sum_{f^{(1)}(a)-\frac{1}{2}\leqslant n\leqslant f^{(1)}(b)+\frac{1}{2}} \int_a^b \varphi(x)\exp[2\pi i(f(x)-nx)]\,\mathrm{d}x$$

$$+ O(H\ln(f^{(1)}(b) - f^{(1)}(a) + 2)). \tag{9}$$

Obviously,

$$f^{(1)}(a) < [f^{(1)}(a)] + 1, \quad [f^{(1)}(b)] < f^{(1)}(b).$$

If $[f^{(1)}(a)] + 1 \leqslant n \leqslant [f^{(1)}(b)]$ we apply lemma 2 to the integrals

$$\int_a^b \varphi(x)\exp[2\pi i(f(x)-nx)]\,\mathrm{d}x,$$

taking $f(x)$ as $f(x) - nx$ and c as x_n. Then we have:

$$\int_a^b \varphi(x)\exp[2\pi i(f(x)-nx)]\,\mathrm{d}x = \frac{1+i}{\sqrt{2}}\frac{\varphi(x_n)\exp[2\pi i(f(x_n)-nx_n)]}{[f^{(2)}(x_n)]^{1/2}}$$
$$+O(HAU^{-1}) + O\left(H_{\min}\left(\frac{1}{|f^{(1)}(a)-n|},\sqrt{A}\right)\right)$$
$$+O\left(H_{\min}\left(\frac{1}{|f^{(1)}(b)-n|},\sqrt{A}\right)\right).$$

Summing the above relation with respect to n, $[f^{(1)}(a)] < n \leqslant [f^{(1)}(b)]$ we get:

$$\sum_{[f^{(1)}(a)]<n\leqslant[f^{(1)}(b)]}\int_a^b \varphi(x)\exp[2\pi i(f(x)-nx)]\,\mathrm{d}x$$
$$=\frac{1+i}{\sqrt{2}}\sum_{f^{(1)}(a)\leqslant n\leqslant f^{(1)}(b)}\frac{\varphi(x_n)\exp[2\pi i(f(x_n)-nx_n)]}{[f^{(2)}(x_n)]^{1/2}}$$
$$+O\left(H_{\min}\left(\frac{1}{\|f^{(1)}(a)\|},\sqrt{A}\right)\right)+O\left(H_{\min}\left(\frac{1}{\|f^{(1)}(b)\|},\sqrt{A}\right)\right)$$
$$+O(H\ln(f^{(1)}(b)-f^{(1)}(a)+2)).$$

The right hand side of (9) may contain additional summands of the form

$$\int_a^b \varphi(x)\exp[2\pi i(f(x)-nx)]\,\mathrm{d}x,$$

where $n=[f^{(1)}(a)]$ or $n=[f^{(1)}(b)]+1$. In those cases $(\mathrm{d}/\mathrm{d}x)(f(x)-nx)\neq 0$ for $a\leqslant x\leqslant b$. Thus integrating by parts and estimating the right hand side trivially, after this we obtain:

$$\int_a^b \varphi(x)\exp[2\pi i(f(x)-nx)]\,\mathrm{d}x = \frac{\varphi(x)}{2\pi i(f^{(1)}(x)-n)}\exp[2\pi i(f(x)-nx)]\Big|_a^b$$
$$-\frac{1}{2\pi i}\int_a^b \frac{\varphi^{(1)}(x)(f^{(1)}(x)-n)-\varphi(x)f^{(2)}(x)}{(f^{(1)}(x)-n)^2}\exp[2\pi i(f(x)-nx)]\,\mathrm{d}x$$
$$\ll \frac{H}{\|f^{(1)}(\mu)\|},$$

where $\mu = a$, if $n=[f^{(1)}(a)]$ and $\mu=b$, if $n=[f^{(1)}(b)]+1$. According to the remark following lemma 2 we have also the estimate

$$\int_a^b \varphi(x)\exp[2\pi i(f(x)-nx)]\,\mathrm{d}x \ll H\sqrt{A}.$$

Thus, the statement of the theorem is proved in the case $\|f^{(1)}(a)\|\neq 0$, $\|f^{(1)}(b)\|\neq 0$.

Now, let $\| f^{(1)}(a) \| = 0$, i.e. $f^{(1)}(a)$ is an integer, $\| f^{(1)}(b) \| \neq 0$. Again according to lemma 1, the relation is valid. Extracting the summand corresponding to $n = f^{(1)}(a)$ from the right hand side we apply the above observations to the remaining sum. We get:

$$\sum_{a<x\leq b} \varphi(x)\exp[2\pi i f(x)] = \int_a^b \varphi(x)\exp[2\pi i(f(x) - f^{(1)}(a)x)]\,dx$$

$$+ \frac{1+i}{\sqrt{2}} \sum_{f^{(1)}(a)<n\leq f^{(1)}(b)} \frac{\varphi(x_n)\exp[2\pi i(f(x_n) - nx_n)]}{[f^{(2)}(x_n)]^{1/2}}$$

$$+ O\left(H_{\min}\left(\frac{1}{\| f^{(1)}(b)\|}, \sqrt{A}\right)\right) + O(H \ln(f^{(1)}(b) - f^{(1)}(a) + 2)).$$

Here, the first integral was computed in lemma 2 (with $c = a$) and it equals

$$\frac{1+i}{2\sqrt{2}} \frac{\varphi(a)\exp[2\pi i(f(a) - f^{(1)}(a)a)}{[f^{(2)}(a)]^{1/2}} + O(HA(b-a)^{-1}).$$

This implies that the assertion of the theorem is valid for $\| f^{(1)}(a) \| = 0$, $\| f^{(1)}(b) \| \neq 0$ as well. The assertion of the theorem in the other two remaining cases, namely, for $\| f^{(1)}(a) \| \neq 0$, $\| f^{(1)}(b) \| = 0$ and for $\| f^{(1)}(a) \| = \| f^{(1)}(b) \| = 0$ is proved analogously. The theorem is completely proved.

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