Proc. Indian Acad. Sci. (Math. Sci.), Vol. 97, Nos 1–3, December 1987, pp. 157–166.

On exponential sums involving the Ramanujan function

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Abstract. Let $\tau(n)$ be the arithmetical function of Ramanujan, α any real number, and $x \geqslant 2$. The uniform estimate

$$\sum_{n \leqslant x} \tau(n) e(n\alpha) \ll x^6 \log x$$

is a classical result of J R Wilton. It is well known that the best possible bound would be $\ll x^6$. The validity of this hypothesis is proved.

Keywords. Exponential sums; Ramanujan τ function.

1. Introduction

The Ramanujan function $\tau(n)$, usually defined by the identity

$$x \prod_{m=1}^{\infty} (1 - x^m)^{24} = \sum_{n=1}^{\infty} \tau(n) x^n \quad (|x| < 1),$$

also appears in the Fourier series

$$\Delta(z) = (2\pi)^{12} \sum_{n=1}^{\infty} \tau(n) e(nz) \quad (\operatorname{Im} z > 0)$$

of the discriminant $\Delta(z)$, which is a cusp form of weight 12 for the full modular group ([1], Theorems 1.19, 3.2 and 3.3). The latter property implies, after Hecke, an analytic continuation and a functional equation for the generating Dirichlet series of $\tau(n)$. The functional equation, in turn, implies a formula of the Voronoi type for the sum function of $\tau(n)$. More generally, a formula for the exponential sum

$$A(x, \alpha) = \sum_{n \leqslant x}{}' \tau(n) e(n\alpha),$$

where $\alpha = h/k$ is a rational number and the summation convention $\sum'$ is as in [4], can be proved in the same way. The result is as follows ([4], Theorem 1.6):

$$A(x, h/k) = x^6 \sum_{n=1}^{\infty} \tau(n) e_k(-n\bar{h}) n^{-6} J_{12}(4\pi\sqrt{nx}/k), \tag{1}$$

where $e_k(x) = e(x/k) = \exp(2\pi i x/k)$, $h\bar{h} \equiv 1 \pmod{k}$, and $J_{12}(x)$ is a Bessel function in the standard notation.

Two approximate variants of (1) may be noted. First, on applying the well-known asymptotic formula for $J_{12}(x)$ and the mean value theorem

$$\sum_{n\leqslant x}\tau^2(n)=Ax^{12}+O(x^{12-2/5}) \tag{2}$$

due to R A Rankin [7], we obtain

$$A(x,h/k)=(\pi\sqrt{2})^{-1}k^{1/2}x^{23/4}\sum_{n=1}^{\infty}\tilde{\tau}(n)e_k(-n\bar{h})n^{-3/4}$$
$$\times\cos(4\pi\sqrt{nx}/k-\pi/4)+O(k^{3/2}x^{21/4}), \tag{3}$$

where $\tilde{\tau}(n)=\tau(n)n^{-11/2}$ is the "normalized" Ramanujan function. By (2), $\tilde{\tau}(n)$ is bounded on the average. It was conjectured by S Ramanujan and proved by P Deligne [2] that $|\tilde{\tau}(n)|\leqslant d(n)$, where $d(n)$ is the usual divisor function. As a truncated version of (3), we have ([4], Theorem 1.1)

$$A(x,h/k)=(\pi\sqrt{2})^{-1}k^{1/2}x^{23/4}\sum_{n\leqslant N}\tilde{\tau}(n)e_k(-n\bar{h})n^{-3/4}$$
$$\times\cos(4\pi\sqrt{nx}/k-\pi/4)+O(kx^{6+\varepsilon}N^{-1/2}), \tag{4}$$

where $k\leqslant x$, $1\leqslant N\ll x$, and ε is an arbitrarily fixed small positive number. If N is appropriately chosen and the sum on the right is estimated by absolute values, then it follows that (see (1.5.21) in [4])

$$A(x,h/k)\ll k^{2/3}x^{6-1/6+\varepsilon}. \tag{5}$$

But sometimes it is possible to make use of cancellation in the sum to prove more refined estimates; such a result is given in Lemma 2 below.

The problem of the order of $A(x,\alpha)$ is more difficult if α is not a rational number with a small denominator (in comparison with x). It was proved by J R Wilton [8] that

$$A(x,\alpha)\ll x^6\log x \tag{6}$$

uniformly in α. This is not far from being best possible, for the mean square of $A(x,\alpha)$ with respect to α over the interval [0, 1] is of the order Ax^{12} by (2), so that the right hand side of (6) cannot be replaced by anything of a smaller order than x^6. It was conjectured by L A Parson and M Sheingorn [6] that the factor $\log x$ can in fact be removed from (6). It is our object in this paper to prove this conjecture.

Theorem *We have*

$$A(x,\alpha)\ll x^6, \tag{7}$$

uniformly for all real α and $x\geqslant 1$.

It will be more convenient to deal with a segment

$$A(x_1,x;\alpha)=A(x,\alpha)-A(x_1,\alpha)$$

of the sum $A(x,\alpha)$ instead of the complete sum itself. We denote by $\tilde{A}(x,\alpha)$ and

$\tilde{A}(x_1, x; \alpha)$ similar sums in which $\tau(n)$ is replaced by $\tilde{\tau}(n)$. Obviously it suffices to prove that

$$A(x_1, x; \alpha) \ll x^6 \tag{8}$$

for $x/2 \leqslant x_1 < x$, or equivalently, that

$$\tilde{A}(x_1, x; \alpha) \ll x^{1/2}. \tag{9}$$

Our arguments will be similar to those in [3], with some refinements. In [3] we studied sums involving $d(n)$, but from the point of view of methods this does not make an essential difference, for the respective summation formulae involving $d(n)$ or $\tau(n)$ are analogous (see [4], Theorem 1.7). Moreover, in place of $\tau(n)$ we might have the Fourier coefficients of any cusp form; if κ is the weight of the form, then $x^{\kappa/2}$ should stand on the right of (7). But for simplicity we work with the Ramanujan function.

A natural approach to the assertion (9) is to write a rational approximation

$$\alpha = h/k + \eta \tag{10}$$

to our number α and to make use of our knowledge about $A(x, h/k)$. Given a number $K \geqslant 1$, we take h/k from the Farey sequence of the order K, in other words

$$(h, k) = 1, \quad k \leqslant K, \quad |\eta| \leqslant (kK)^{-1}. \tag{11}$$

We shall choose $K = x^{1/4}$, and consider separately the cases

$$k^2\eta^2 x \leqslant 1/2 \tag{12}$$

and

$$k^2\eta^2 x \gg 1. \tag{13}$$

The case (12) is easier, for then η is relatively small, and therefore it is possible to make effective use of partial summation and estimates for $A(x, h/k)$.

The case (13) is more delicate. The crucial relation in our proof of (9) is the approximate functional equation

$$\frac{\tilde{A}(x_1, x; \alpha)}{x^{1/2}} = \frac{\tilde{A}(y_1, y; \beta)}{y^{1/2}} + O(y^{-a}), \tag{14}$$

where

$$y_1 = k^2\eta^2 x_1, \quad y = k^2\eta^2 x, \tag{15}$$

$$\beta = -\bar{h}/k - (k^2\eta)^{-1}, \tag{16}$$

and a is a positive constant. An analogous relation for exponential sums involving $d(n)$ was obtained in [3], with the weaker error term $O(\log x)$ however.

Now it is easy to deduce (9) from (14) by an iterative argument. Indeed, we have $y \leqslant K^{-2}x = x^{1/2}$, so that the length of the sum is considerably reduced at each step of the iteration. We stop if we come to a sum for which either (12) holds *mutatis mutandis*, or whose length is $O(1)$. In any case, the last quotient, say $\tilde{A}(z_1, z; \gamma)/z^{1/2}$, is bounded. Then, since the sum of the error terms appearing in the course of the iteration procedure is bounded as well, we find that the quotient on the left of (14) is bounded. Hence (9) holds.

2. Lemmas on exponential sums and integrals

We first quote from [5], § 1.6, a general estimate for exponential sums of the type

$$S = \sum_{M}^{M'} \tilde{\tau}(m) g(m) e(f(m)).$$

(Actually the result in [5] is given for a similar sum in which $d(m)$ stands in place of $\tilde{\tau}(m)$, but the same method applies in both cases.) To formulate the lemma, define for a real interval $[a,b]$ and for $\mu > 0$ the complex domain

$$D(a,b;\mu) = \{z : |z - x| < \mu \text{ for some } x \in [a,b]\}.$$

We write $A \asymp B$ to mean that $A \ll B \ll A$.

Lemma 1. Let $2 \leqslant M < M' \leqslant 2M$, *and let* f *be a holomorphic function in the domain* $D = D(M,M';\mu)$, *where* $\mu \asymp M$. *Suppose that* $f(x)$ *is real for* $M \leqslant x \leqslant M'$, *and that for a certain positive number* $F \gg M$ *we have*

$$f'(z) \ll FM^{-1} \quad \text{for} \quad z \in D,$$

$$|f''(x)| \asymp FM^{-2} \quad \text{for} \quad M \leqslant x \leqslant M'.$$

Let g *be a continuously differentiable function in the interval* $[M,M']$, *where* $g(x) \ll G$ *and* $g'(x) \ll G'$. *Then*

$$S \ll (G + (M' - M)G')M^{1/2}F^{1/3+\varepsilon}.$$

Remark. An exponential sum of the type

$$S' = \sum_{M}^{M'} \tilde{\tau}(m) g(m) e(f(m) + \theta m),$$

where f and g satisfy the above conditions, occurs in the proof of the next lemma. We may suppose that $|\theta| \leqslant 1/2$. Hence if $M \ll F$, then Lemma 1 can be applied with $f(z) + \theta z$ in place of $f(z)$. In addition to this, we need an estimate for S' in the case $F \ll M \ll F^{1+d}$, where d is a small positive constant. The method in [5], §4.2, works for S' as well and yields an estimate which is at most by a small power of F (depending on d) weaker than the corresponding estimate for S in Lemma 1.

Lemma 2. For $x \geqslant 1$ *and* $k \ll x^{1/4}$ *we have*

$$\tilde{A}(x, h/k) \ll k^{1/4} x^{3/8+\varepsilon} + \min(k^2 x^{\varepsilon}, x^{1/2-c}), \tag{17}$$

where c *is a certain positive constant.*

Proof. Obviously (17) is equivalent to the same estimate for $x^{-11/2}A(x, h/k)$. To study this, we divide (4) for $A(x, h/k)$ by $x^{11/2}$, and consider the right hand side choosing

$$N = \max(k^{-2}x, x^{1/2+\gamma}),$$

where γ is a positive constant to be chosen suitably. Then the error term is

$$\ll \min(k^2 x^{\varepsilon}, x^{1/2-\gamma/2+\varepsilon}). \tag{18}$$

In the sum of length N, we estimate the initial terms with $n \leqslant k^{-1}x^{1/2}$ by absolute values, and apply Lemma 1 and the succeeding remark to the rest. The first mentioned terms contribute $\ll k^{1/4}x^{3/8}$.

Consider next those terms with $M \leqslant n \leqslant 2M$ for $k^{-1}x^{1/2} \leqslant M < N$. The parameters in Lemma 1 and in the remark for this sum are

$$F = k^{-1}(Mx)^{1/2}, \quad G = k^{1/2}M^{-3/4}x^{1/4}, \quad G' = G/M, \quad \theta = -\bar{h}/k.$$

Then we have $M \ll F$ if $M \leqslant k^{-2}x$, and otherwise $M \ll F^{1+d}$, where d depends on γ. Then the contribution to be estimated is

$$\ll k^{1/6}M^{-1/12}x^{5/12+\varepsilon} \ll k^{1/4}x^{3/8+\varepsilon}$$

if $M \ll k^{-2}x$, and otherwise this must perhaps be multiplied by a certain factor $F^{d'}$, where d' depends on γ. But in the latter case we have $k \gg x^{1/4-\gamma/2}$, and if we suppose that γ is small enough, then $k^{1/4}x^{3/8+\varepsilon}F^{d'}$ is dominated by the quantity in (18). Then (17) holds with $c = \gamma/3$.

For technical reasons, it is sometimes useful to provide sums or integrals with suitable smooth weights. In [4] we defined the weight function $\eta_J(x)$ in the interval $[a, b]$ as follows. Given a positive integer J and a positive number $U < (b-a)/2J$, we write for any integrable function h

$$U^{-J}\int_0^U \mathrm{d}u_1 \cdots \int_0^U \mathrm{d}u_J \int_{a+u}^{b-u} h(x)\,\mathrm{d}x = \int_a^b \eta_J(x)h(x)\,\mathrm{d}x, \tag{19}$$

where $u = u_1 + \cdots + u_J$. In addition, we define $\eta_0(x)$ to be the characteristic function of the interval $[a, b]$.

Exponential integrals can be approximately evaluated by the "saddle point method". The following saddle point lemma for smoothed exponential integrals is a simplified version of Theorem 2.2 from [4].

Lemma 3. Suppose that the function f satisfies the conditions of Lemma 1, and that $f''(x) > 0$ in the interval $[M, M']$. Suppose also that g is holomorphic and $g(z) \ll G$ in the domain D. Let $U > 0$, J a fixed non-negative integer, and $JU < (M' - M)/2$. Denote the characteristic function of the union of the intervals $(M, M + JU)$ and $(M' - JU, M')$ by $\delta(x)$, let α be a real number, x_0 the (possibly existing) zero of $f'(x) + \alpha$ in the interval (M, M'), and suppose that $U \gg \delta(x_0)F^{-1/2}M$. Write

$$E_J(x) = G(|f'(x) + \alpha| + F^{1/2}M^{-1})^{-J-1}.$$

Then

$$\int_M^{M'} \eta_J(x)g(x)e(f(x) + \alpha x)\,\mathrm{d}x = \xi_J(x_0)g(x_0)f''(x_0)^{-1/2}e(f(x_0) + \alpha x_0 + 1/8)$$

$$+ O((1+\delta(x_0)F^{1/2})GMF^{-3/2}) + O\left(U^{-J}\sum_{j=0}^{J}(E_J(M+jU)+E_J(M'-jU))\right),$$

where $\xi_J(x)$ *is bounded in the interval* (M, M'), $\xi_J(x)=1$ *for* $M+JU<x<M'-JU$, *and furthermore* $\xi_J'(x)$ *is continuous and* $\xi_J'(x)\ll U^{-1}$ *in the intervals* $(M+jU, M+(j+1)U)$ *and* $(M'-(j+1)U, M'-jU)$ *for* $j=0,1,\ldots,J-1$. *If* x_0 *does not exist, then the terms and conditions involving* x_0 *are to be omitted.*

Remark. In Theorem 2.2 in [4], there is one more error term, but under the present assumptions it is absorbed by the others. The bound for $\xi_J'(x)$ given above is not included in Theorem 2.2, but it follows easily from the explicit expression for $\xi_J(x)$ in the theorem (which also shows that $\xi_1(x)$ is piecewise *linear*).

3. An estimate for a short exponential sum

As a preliminary for the proof of the theorem in the case (13), we show that the assertion (8) holds in a more precise form for sufficiently short sums.

Lemma 4. Suppose that α *is of the form (10)–(11), where* $K=x^{1/4}$, *and that (13) holds. Then there are positive constants a and b such that*

$$A(x_1, x; \alpha) \ll x^6 y^{-a} \tag{20}$$

for

$$x - x_1 \ll |\eta|^{-1/2} x^{1/2+b}. \tag{21}$$

Proof. Suppose, for simplicity, that $\eta>0$. By partial summation, we have

$$A(x_1, x; \alpha) = A(x, h/k)e(x\eta) - A(x_1, h/k)e(x_1\eta) - 2\pi i\eta\int_{x_1}^{x} e(t\eta)A(t, h/k)\,\mathrm{d}t. \tag{22}$$

The first two terms are of the desired order by Lemma 2.

We replace $A(t,h/k)$ in the integral in (22) by its expression (3). The contribution of the error term is

$$\ll \eta^{1/2}k^{3/2}x^{23/4+b} \leqslant K^{1/2}x^{23/4+b} \leqslant x^{6-1/8+b},$$

which is admissible at least if $b<1/16$. The function $\cos(\ldots)$ in (3) is written in terms of $e(\pm 2\sqrt{nx}/k)$. The critical terms are those involving the negative sign, and the others can be treated similarly but more trivially. So the actual problem is to prove that

$$\eta k^{1/2}\sum_{n=1}^{\infty}\tilde{\tau}(n)e_k(-n\bar{h})n^{-3/4}I_n \ll x^{1/4}y^{-a}, \tag{23}$$

where

$$I_n = x^{-23/4}\int_{x_1}^{x} t^{23/4}e(\psi_n(t))\,\mathrm{d}t \tag{24}$$

and

$$\psi_n(t) = \eta t - 2\sqrt{nt}/k.$$

We integrated here a series term by term, which is justified by bounded convergence.

The order of I_n depends on the size of $|\psi_n'(t)|$ and $|\psi_n''(t)|$ in the interval of integration. Suppose that $|\psi_n^{(j)}(t)| \gg \lambda_j > 0$ for $j = 1$ or 2. Then it is well known that $I_n \ll \lambda_j^{-1/j}$. The derivative $\psi_n'(t)$ has a zero in $[x_1, x]$ if and only if $y_1 \leqslant n \leqslant y$, in which case we have to use $\lambda_2 \asymp \eta x^{-1}$. As to λ_1 for $n \notin [y_1, y]$, we have

$$\lambda_1 \gg \begin{cases} \eta y^{-1}(y_1 - n) & \text{for } n < y_1, \\ (kn^{1/2}x^{1/2})^{-1}(n - y) & \text{for } n > y. \end{cases} \tag{25}$$

Hence

$$I_n \ll \eta^{-1/2}x^{1/2} \qquad \text{for } n \asymp y, \tag{26}$$

$$I_n \ll \eta^{-1}y(y_1 - n)^{-1} \qquad \text{for } n < y_1, \tag{27}$$

$$I_n \ll kn^{1/2}x^{1/2}(n - y)^{-1} \qquad \text{for } n > y. \tag{28}$$

Now the left hand side of (23) can be estimated by using (26) to (28). We obtain

$$\begin{aligned} &\ll (k^{1/2}y^{1/4} + \eta^{1/2}k^{1/2}x^{1/2}y^{-3/4}(y - y_1 + 1))y^{\varepsilon} \\ &\ll (k\eta^{1/2} + y^{-1/2} + k\eta x^{-1/2}(x - x_1))x^{1/4}y^{\varepsilon} \\ &\ll (k\eta^{1/2}x^{b} + y^{-1/2})x^{1/4}y^{\varepsilon}. \end{aligned} \tag{29}$$

Thus the assertion (23) holds if $k\eta^{1/2} \ll x^{-a_0}$ for some positive constant a_0 and $b < a_0/2$. Hence we may assume in the sequel that

$$k\eta^{1/2} \gg x^{-a_0}. \tag{30}$$

The constant a_0 will be chosen sufficiently small, in any case $a_0 < 1/16$. Then $y \gg k^{-2}x^{3/4} \gg x^{1/4}$, and thus the right hand side of (23) can be replaced by $x^{1/4-a}$ (on changing the meaning of a).

The argument leading to (29) shows that if those terms with

$$y^{1-\delta} \leqslant n \leqslant y^{1+\delta} \tag{31}$$

are excluded, then the estimate is improved by a factor $y^{-\delta/4}$. This saving is sufficient if $b < \delta/16$, so it remains to estimate the sum in (23) over the interval (31) in the case (30).

At this stage, we have to make a digression to the case $x - x_1 \ll x^{1/2}$. We claim there is a numerical constant a_1 such that

$$\tilde{A}(x_1, x; \xi) \ll x^{1/2-a_1}((x - x_1)x^{-1/2} + 1), \tag{32}$$

uniformly for all $x/2 \leqslant x_1 < x$ and all real ξ. The interesting case here is $x - x_1 \ll x^{1/2}$, for if this is settled, then the general result follows immediately.

We write a rational approximation to ξ as in (10) and (11), where $K = X^{1/4}$. Then either (12) or (13) holds for ξ. By (22) (with A replaced by $\tilde{A}$) and Lemma 2, we have

$$\tilde{A}(x_1, x; \xi) \ll (1 + \eta x^{1/2})x^{1/2-c}. \tag{33}$$

This implies (32) in the case (12).

Assuming now (13), we sharpen the argument leading to (29) on applying the trivial estimate $I_n \ll x^{1/2}$ instead of (26). In this way, we obtain an estimate for $A(x_1, x; \xi)$, and then

$$\tilde{A}(x_1, x; \xi) \ll k\eta^{1/2}x^{1/2+\varepsilon} + x^{1/2-c} \tag{34}$$

by partial summation.

There are two alternatives: either $k \leqslant Kx^{-c/2}$, or $Kx^{-c/2} < k \leqslant K$. In the former case,

$$k\eta^{1/2} \leqslant (k/K)^{1/2} \leqslant x^{-c/4},$$

and (32) follows from (34). In the latter case,

$$\eta x^{1/2} \leqslant (kK)^{-1}x^{1/2} \leqslant K^{-2}x^{1/2+c/2} = x^{c/2},$$

and (32) follows from (33).

We now return to (23). So far we have estimated terms on the left individually, but in the rest of the proof it is essential that some cancellation takes place in the sum over the interval $[y^{1-\delta}, y^{1+\delta}]$. The estimate (32) was proved for this purpose. It implies that if (30) holds and a_0 is sufficiently small, then there is an absolute constant a_2 such that

$$\sum_u^v \tilde{\tau}(n)e_k(-n\bar{h})e(-2\sqrt{nt}/k) \ll (1 + (v-u)y^{-1/2})y^{1/2-a_2} \tag{35}$$

for all $t \in [x/2, x]$ and $y^{1-\delta} \leqslant u < v \leqslant y^{1+\delta}$. It suffices to prove this for $v - u \ll y^{1/2}$. Write the sum on the left as

$$\sum_u^v \tilde{\tau}(n)e(n\xi)e(-n(\xi + \bar{h}/k) - 2\sqrt{nt}/k),$$

where $\xi = -\bar{h}/k - t^{1/2}u^{-1/2}k^{-1}$. By partial summation (32) and (30), this sum is

$$\ll (k^2\eta)^{-1}y^{1/2-a_1+3\delta/2} \ll x^{2a_0}y^{1/2-a_1+3\delta/2},$$

which is $\ll y^{1/2-a_1/2}$ if a_0 and δ are sufficiently small. Then we may choose $a_2 = a_1/2$.

Let now

$$Y \asymp \eta^{3/2}k^2x^{1/2+b},$$

be a number exceeding $2(y - y_1)$ (this is possible, for $y - y_1 \ll Y$). Consider first the sum over the interval $|n - y| \leqslant Y$ in (23). We invert the order of the summation and integration, apply (35), and estimate the integral trivially, to obtain

$$\ll (k\eta)^{1/2}y^{-1/4-a_2}x^{1/2+2b} \ll x^{1/4+2b-a_2/4},$$

which is $\ll x^{1/4-a}$ if a and b are chosen sufficiently small.

The sums over the intervals $[y^{1-\delta}, y - Y]$ and $[y + Y, y^{1+\delta}]$ can be estimated in the same way; consider the latter as an example. We subdivide the range of summation into segments $[y + \Delta, y + \Delta']$, where $Y \leqslant \Delta < \Delta' \leqslant 2\Delta \ll y^{1+\delta}$. The number of these is $\ll \log x$. We estimate the sum over one of the segments. By integration by parts, the integral in (24) can be written as

$$\Big|_{x_1}^{x} t^{23/4}e(\psi_n(t))(\psi_n'(t))^{-1}$$

$$+(1/2)k^{-1}n^{1/2}\int_{x_1}^{x} t^{17/4}e(\psi_n(t))(\psi_n'(t))^{-2}\,dt$$

$$-(23/4)\int_{x_1}^{x} t^{19/4}e(\psi_n(t))(\psi_n'(t))^{-1}\,dt. \tag{36}$$

The contribution of the integrated terms is estimated as above, and the result is of the order $O(x^{1/4-a})$ again. The integrands in the two integrals in (36) are at least by a factor x^{-2b} smaller in absolute value than the original integrand. If the integration by parts is repeated, then the integrands decrease at each step in the same way, until we end up with integrals which can be estimated trivially. The integrated terms can be dealt with like those in (36), and the proof of (23) is complete.

In retrospect, the crucial fact in the proof of Lemma 4 is seen to be the saving x^{-c} in (17) over the estimate $\ll x^{1/2}$.

4. Proof of the theorem

The case (12) can be settled by arguments similar to those in the proof of Lemma 4, or in [3], §4.1. It is to be shown that the last term on the right of (22) is $\ll x^6$. In fact, the estimate will be $\ll x^{23/4}k^{1/2} \ll x^{6-1/8}$. The point is that the exponential integrals occurring in the proof have no saddle point, because the bound 1/2 on the right of (12) is less than 1.

In the case (13), we have to prove the relation (14). To begin with, we replace $\tilde{A}(x_1, x; \alpha)$ by the smoothed sum

$$\tilde{A}_1(x_1, x; \alpha) = \sum_{x_1}^{x} \eta_1(n)\tilde{\tau}(n)e(n\alpha), \tag{37}$$

where the weight function η_1 is constructed as in (19) on choosing $J=1$ and $U=\eta^{-1/2}x^{1/2}y^b$ (the constant b is as in Lemma 4). The smoothing error is $\ll x^{1/2}y^{-a}$ by Lemma 4. Hence, as far as the proof of (14) is concerned, we may replace $\tilde{A}(x_1, x; \alpha)$ by the sum (37). It suffices to show that

$$\tilde{A}_1(x_1, x; \alpha) = (x/y)^{1/2}\tilde{A}(y_1, y; \beta) + O(x^{1/2}y^{-a}), \tag{38}$$

where the constant a is not necessarily the same as in Lemma 4.

We follow the argument in the proof of Theorem 5 (i) in [3], where an analogous formula with the error term $O(x^{1/2}\log x)$ was obtained for exponential sums involving $d(n)$. Without repeating the details of the proof, we indicate how the saving in the error term is achieved.

The sum (37) is first expanded by an identity (equation (1.9.2) in [4]), analogous to Voronoi's summation formula, as a series of exponential integrals. When these are evaluated by Lemma 3, then the estimate for the contribution of the error terms turns out to be by a power of y smaller than that in [3]. The reason is that in [3] we had $U = \eta^{-1/2}x^{1/2}$, while the presence of the extra factor y^b in our parameter U improves the effect of the smoothing by a comparable saving factor. Also, Lemma 3 is more suitable for dealing with smoothed integrals than the saddle-point lemma applied in [3].

As to the main terms, Lemma 3 gives in the first place the sum

$$(x/y)^{1/2} \sum_{y_1 \leqslant n \leqslant y} w(n)\tilde{\tau}(n)e(n\beta), \tag{39}$$

where β is as in (16), and $w(n) = \xi_1(n/(k\eta)^2)$ in the notation of Lemma 3; here $n/(k\eta)^2$ is the saddle point x_0 in the nth term. Since $\xi_1(u) = 1$ for $x_1 + U < u < x - U$, we have $w(n) = 1$ for $y_1 + k^2\eta^2 U < n < y - k^2\eta^2 U$. We make $w(n) = 1$ for all n, and estimate the error. Because $w(n)$ varies smoothly as a function of n, by Lemma 3, the error is by partial summation at most

$$\ll (x/y)^{1/2} \max_{u,v} |\tilde{A}(u, v; \beta)|, \tag{40}$$

where u and v lie either in the interval $[y_1, y_1 + k^2\eta^2 U]$ or in $[y - k^2\eta^2 U, y]$. The length of these intervals is $\asymp k^2\eta^{3/2}x^{1/2}y^b \ll y^{1/2+b}$. Hence, by (32), the maximum in (40) is $\ll y^{1/2-a}$ for some constant a if b is supposed to be smaller than the constant a_1 in (32). This means that the expression (39) can be written as $(x/y)^{1/2}\tilde{A}(y_1, y; \beta)$ with an error $\ll x^{1/2}y^{-a}$, and the assertion (38) follows.

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