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On Whittaker models and the vanishing of Fourier coefficients of cusp forms†

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Abstract. The purpose of this paper is to construct examples of automorphic cuspidal representations which possess a ψ-Whittaker model even though their ψ-Fourier coefficients vanish identically. This phenomenon was known to be impossible for the group $GL(n)$, but in general remained an open problem. Our examples concern the metaplectic group and rely heavily upon J L Waldspurger's earlier analysis of cusp forms on this group.

Keywords. Whittaker models; cusp forms; automorphic cuspidal representations.

1. Introduction

Consider the following:

Question: Do there exist automorphic cuspidal representations which are *ψ-non-degenerate* and yet not *ψ-generic*? i.e. can there exist an automorphic cuspidal representation which possesses an (abstract) ψ-Whittaker model and yet the Whittaker model consisting of its ψ-th Fourier coefficients vanishes identically?

For $G = GL(n)$, the answer to this question is "no", since any automorphic cuspidal representation is automatically ψ-generic, for any $\psi \not\equiv 1$. Indeed, the ψ-Fourier coefficients of a cusp form on $GL(n)$ are known to completely determine the form; cf. ([9], p. 302, [13]).

The purpose of this paper is to show that the answer to the *question* in general is "yes". We do this by treating the special case of the metaplectic group $\overline{SL}_2(\mathbb{A})$ where we can appeal to Waldspurger's profound analysis of the space of cusp forms. The expert will observe that almost all the ingredients of our proof can be found in (one place or another in) [16], [17]. Our contribution has been to focus attention on the Whittaker model question posed above, and to answer it by reformulating Waldspurger's theory.

To describe our result, let us fix once and for all a nontrivial character ψ of $F\backslash\mathbb{A}$, and regard $\overline{SL_2}$ as part of the dual reductive pair $(\overline{SL_2}, PGL_2)$ in the sense of [6]. The Shimura–Shintani correspondence of [16] then provides a bijection between the set of irreducible automorphic cuspidal representations σ of $\overline{SL_2}(\mathbb{A})$ which have non-

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vanishing ψ-Fourier coefficients and the set of irreducible automorphic cuspidal representations π of $PGL_2(\mathbb{A})$ satisfying $L(\pi,\frac{1}{2}) \neq 0$. In § 2 we describe how the domain of this bijection may be extended (via Howe's correspondence) to include all σ which "merely" possess an abstract ψ-Whittaker functional; the resulting image then consists of all irreducible automorphic cuspidal π with $\varepsilon(\pi,\frac{1}{2}) = 1$. In other words, the cuspidal representations whose existence is questioned in our "problem" above are precisely the Howe lifts of π on $PGL_2(\mathbb{A})$ with

$$L(\pi,\tfrac{1}{2}) = 0 \quad \text{and} \quad \varepsilon(\pi,\tfrac{1}{2}) = 1.$$

In order to show that such representations π exist, we give some examples in § 3. Finally, in § 3, we include some remarks about the possibility of finding similar examples for other groups (and other dual reductive pairs).

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2 The relation between the correspondences of Howe and Waldspurger

2.1 We begin by recalling the key results of [16] needed in this paper. For a detailed analysis of Waldspurger's theory, or assistance with unexplained facts or terminology, the reader is referred to [10] and [4].

Let A_{oo} denote the space of genuine cusp forms on $\overline{SL_2}(\mathbb{A})$ orthogonal to all theta-series in one-variable. For any irreducible automorphic cuspidal representation $\sigma = \otimes \sigma_v$ appearing in A_{oo}, there is attached an irreducible subspace

$$\pi = \theta(\sigma, \psi)$$

of the space of cusp forms on $PGL_2(\mathbb{A})$; this is the "theta-series lift" of σ to PGL_2. Conversely, given any irreducible automorphic cuspidal representation $\pi = \otimes \pi_v$ of $PGL_2(\mathbb{A})$, its theta-series lifting $\theta(\pi, \psi)$, defines an irreducible subspace of A_{oo}. We recall that these maps are adelizations of Shintani and Niwa's treatment of *Shimura's correspondence in* [14]. According to the theory of [16]:

(a) There are local analogues of the correspondences above, such that

$$\theta(\sigma, \psi) = \otimes\, \theta(\sigma_v, \psi_v)$$

and

$$\theta(\pi, \psi) = \otimes\, \theta(\pi_v, \psi_v);$$

moreover, these local correspondences are defined (and non-zero) for arbitrary local σ_v and π_v (provided σ_v is ψ_v-non-degenerate);

(b) $\theta(\sigma, \psi) \equiv \{0\}$ if and only if the ψ-Fourier coefficient of any cusp form in the space of σ is zero;

(c) $\theta(\pi, \psi) \equiv \{0\}$ if and only if $L(\pi, \frac{1}{2}) = 0$.

Remarks

(i) Let $A_{oo}(\psi)$ be the space of cusp forms on $\overline{SL_2}(\mathbb{A})$ with non-vanishing ψ-Fourier coefficients. Then $\sigma \to \theta(\sigma, \psi)$ defines a bijection between the set of irreducible σ in $A_{oo}(\psi)$ and the set of irreducible cuspidal representations π of $PGL_2(\mathbb{A})$ such that $L(\pi, \frac{1}{2}) \neq 0$.

(ii) If ψ' is another non-trivial character of $F\backslash\mathbb{A}$ such that σ belongs to $A_{oo}(\psi')$ as well as $A_{oo}(\psi)$, then there exists b in F^{x} such that $\psi' = \psi^{b}$ and

$$\theta(\sigma, \psi') = \theta(\sigma, \psi) \otimes \chi_b$$

(with χ_b the quadratic character of $F^{x}\backslash\mathbb{A}^{x}$ associated to $F(\sqrt{b})$; cf. Prop. 28 of [16]). Thus one can extend the correspondence $\sigma \to \theta(\sigma, \psi)$ *to all irreducible constituents of* A_{oo} by defining

$$\sigma \to Wd(\sigma, \psi) = \theta(\sigma, \psi^{b}) \otimes \chi_b$$

for any ψ^{b} such that $\theta(\sigma, \psi^{b}) \neq \{0\}$. The image of this extended map consists of all irreducible automorphic cuspidal representations π of $PGL_2(\mathbb{A})$ such that

$$L(\pi \otimes \chi_b, \tfrac{1}{2}) \neq 0$$

for *some* b in F^{x}. (cf. [17], Prop. 19 and Cor. 1; also p. 300 of [10].) The problem with this extended correspondence is that *it is no longer one-to-one.*

2.2 Implicit in our discussion above is the fact that σ appears in A_{oo} with multiplicity one, i.e. in defining the correspondence $Wd(\sigma, \psi)$, we may indeed confuse σ with the space of cusp forms realizing it. Waldspurger derives this multiplicity one result as a Corollary of the assertion that the non-vanishing of the ψ-Fourier coefficients of cusp forms in the space of σ depends only on σ as an abstract representation. More precisely, these coefficients are non-zero if and only if each σ_v is ψ_v-non-degenerate *and* $L(Wd(\sigma, \psi), \frac{1}{2}) \neq 0$. This last assertion then suggests a strategy for finding ψ-non-degenerate σ with trivial ψ-Fourier coefficients: restrict $Wd(\sigma, \psi)$ to the set of ψ-non-degenerate σ in A_{oo}, and try to prove that this restriction is an injection (with image containing π such that $L(\pi, \frac{1}{2}) = 0$).

To this end, let $D(\psi) \supseteq A_{oo}(\psi)$ denote those σ in A_{oo} which support a non-trivial ψ-Whittaker functional. (For the time being, we do not know that $D(\psi)$ properly contains $A_{oo}(\psi)$; it is the purpose of this paper to show this.) Consider also the *local* correspondences

$$\begin{aligned} &\sigma_v \to \theta(\sigma_v, \psi_v) \\ &\pi_v \to \theta(\pi_v, \psi_v) \end{aligned} \tag{1}$$

described in § 2.1(a). These correspondences are precisely *concrete* (non-zero) realizations of the *abstract* "θ-series liftings" defined in [6], i.e. $\mathrm{Hom}_{\overline{SL_2(F_v)} \times PGL_2(F_v)}(r_{\psi}, \sigma_v \otimes \pi_v) \neq 0$ if and only if $\pi_v = \theta(\sigma_v, \psi_v)$ or $\sigma_v = \theta(\pi_v, \psi_v)$, with σ_v non-degenerate with respect to ψ_v. Thus, given *any* $\sigma = \otimes\sigma_v$ in $D(\psi)$, it is natural to consider the abstract representation

$$H(\sigma,\psi)=\otimes\,\theta(\sigma_v,\psi_v) \tag{2}$$

of $PGL_2(\mathbb{A})$.

Note that (2) defines a non-trivial representation of $PGL_2(\mathbb{A})$ independently of the vanishing of $\theta(\sigma,\psi)$; all that matters is that σ_v possess an abstract ψ_v-Whittaker model for every v. We call $H(\sigma,\psi)$ the *Howe lifting* of σ to $PGL_2(\mathbb{A})$. This lifting is clearly one-to-one, but it does not *apriori* preserve "automorphicity". If σ is such that $\theta(\sigma,\psi)$ already defines a non-trivial representation, i.e. σ belongs to $A_{oo}(\psi)$, then $H(\sigma,\psi)=\theta(\sigma,\psi)$; hence $H(\sigma,\psi)$ is obviously automorphic cuspidal in this case. In general, we have the following:

Theorem. *For each σ in $D(\psi)$, $\pi=H(\sigma,\psi)$ is an irreducible automorphic cuspidal representation of $PGL_2(\mathbb{A})$ with the property that*

$$\varepsilon(\pi,\tfrac{1}{2})=1. \tag{3}$$

Conversely, if π is any irreducible automorphic cuspidal representation of $PGL_2(\mathbb{A})$ satisfying (3), *then its Howe lift*

$$H(\pi,\psi)\equiv\otimes\,\theta(\pi_v,\psi_v)$$

is automorphic cuspidal on $\overline{SL_2}(\mathbb{A})$ and belongs to $D(\psi)$; moreover,

$$H(H(\pi,\psi),\psi)=\pi$$

and

$$H(H(\sigma,\psi),\psi)=\sigma.$$

Proof. Consider the cuspidal representation $\pi'=Wd(\sigma,\psi)\equiv\theta(\sigma,\psi^{\xi})\otimes\chi_\xi$ with σ in $D(\psi)$. Locally, for any v,

$$\pi'_v=\theta(\sigma_v,\psi_v^{\xi})\otimes\chi_{\xi,v}=\theta(\sigma_v,\psi_v) \tag{4}$$

since σ_v, by assumption, has a ψ_v-Whittaker model. (This is the local analogue of the result quoted in Remark 2.1(ii) above; it is proved in [17]). Hence

$$\pi=\otimes\,H(\sigma_v,\psi_v)=\otimes\,\theta(\sigma_v,\psi_v)=\pi'=Wd(\sigma,\psi)$$

is indeed automorphic cuspidal. Moreover, since π is in the image of Wd, we know $L(\pi\otimes\chi_\xi,\tfrac{1}{2})\neq 0$ for some $\xi\in F^\times$ (again see Remark (2.1)(ii) above); on the other hand equation (4) implies

$$\theta(\pi_v\otimes\chi_{\xi,v},\psi_v^{\xi})=\sigma_v$$

for each v. Therefore (by Theorem 2.5 and the "Additional note" on p. 301 of [10]) we may conclude $\varepsilon(\pi,\tfrac{1}{2})=1$.

Conversely, suppose π cuspidal on $PGL_2(\mathbb{A})$ satisfies $\varepsilon(\pi,\tfrac{1}{2})=1$. Then directly from Theorem A.1 of [10] (together with the "Additional note" on p. 301) we conclude

$\sigma = H(\pi, \psi) = \otimes \theta(\pi_v, \psi_v)$ is automorphic cuspidal, belongs to $D(\psi)$, and maps back to π under the inverse Howe lifting $\otimes \theta(\sigma_v, \psi_v)$.

Remark.

The fact that $H(\pi, \psi)$ is automorphic cuspidal is not at all obvious. Indeed, if $\varepsilon(\pi, \frac{1}{2}) \neq 1$, then $H(\pi, \psi)$ is *not* automorphic.

COROLLARY 1.

The restriction of $Wd(\sigma, \psi)$ to $D(\psi) \supset A_{oo}(\psi)$ remains a bijection.
Indeed the proof of the Theorem shows that $Wd(\sigma, \psi) = H(\sigma, \psi)$ on $D(\psi)$.

COROLLARY 2.

The irreducible automorphic cuspidal representations of $\overline{SL_2}(\mathbb{A})$ which are ψ-non-degenerate but not ψ-generic are precisely the Howe lifts of cuspidal π on $PGL_2(\mathbb{A})$ satisfying

$$\varepsilon(\pi, \tfrac{1}{2}) = 1 \quad \textit{and} \quad L(\pi, \tfrac{1}{2}) = 0.$$

Indeed, the Howe correspondence and the θ-lifting correspondence $\theta(\cdot, \psi)$ agree on the domains $A_{oo}(\psi)$ and $\{\pi : L(\pi, \frac{1}{2}) \neq 0\}$. Thus our Theorem implies that the Howe correspondence is a bijection between the complements of these domains in $D(\psi)$ and $\{\pi : \varepsilon(\pi, \frac{1}{2}) = 1\}$, i.e. a bijection between the types of representations σ and π described in the Corollary.

Remark.

Recall that $Wd: \sigma \to \pi$ is not 1-1 on all of A_{oo} (see [17], [10] or [4] for a detailed description of $Wd^{-1}(\pi)$). The significance of the Howe lifting $H(\pi, \psi)$ is that for π satisfying

$$\varepsilon(\pi, \tfrac{1}{2}) = 1 \quad \text{and} \quad L(\pi, \tfrac{1}{2}) = 0$$

it picks out exactly the *one* σ in $Wd^{-1}(\pi)$ which possesses a ψ-Whittaker model. It remains therefore to exhibit π on $PGL_2(\mathbb{A})$ satisfying these conditions.

3. Examples of cuspidal π on $PGL_2(\mathbb{A})$ satisfying $L(\pi, \frac{1}{2}) = 0$ and $\varepsilon(\pi, \frac{1}{2}) = 1$

The first example is over $F = \mathbb{Q}$ and requires the theory of elliptic curves.

Example 2.1 Suppose E is an elliptic curve over $\mathbb{Q}$ with complex multiplication. Then there exists an automorphic cuspidal representation π_E of $PGL_2(\mathbb{A})$ such that $L(\sigma, \pi_E)$ is equal to the Hasse-Weil zeta-function $L(s, E)$; moreover, the (weak form of the) Birch-Swinnerton-Dyer conjecture holds for $L(s, E)$ (cf. [1]). Let us suppose also that

(i) the functional equation for E is "even", i.e. $L(s, E) = L(1 - s, E)$, and
(ii) $|E(\mathbb{Q})|$ is infinite, i.e. the rank of the Mordell-Weil group of E is positive.

Condition (i) implies that $\varepsilon(\pi, \frac{1}{2}) = 1$, whereas condition (ii) implies (by Coates-Wiles) that $L(\pi, \frac{1}{2}) = 0$.

Our second example is more elementary, but not defined over $\mathbb{Q}$.

Example 2.2. Let $\tau = \otimes \tau_v$ be an automorphic cuspidal representation $PGL_2(\mathbb{A}_F)$ satisfying the conditions:

(i) τ_v belongs to the principal series for all v; and
(ii) $\varepsilon(\tau, \frac{1}{2}) = -1$ (and hence $L(\tau, \frac{1}{2}) = 0$, by the functional equation $L(\tau, s) = \varepsilon(\tau, s)L(\tau, 1-s)$).

Such a representation τ is constructed in [10] (bottom of p. 300) by assuming F is an imaginary quadratic extension of $\mathbb{Q}$ and base change lifting a classical holomorphic cusp form with respect to $SL_2(\mathbb{Z})$. (The significance of τ is that its Howe lifting to $\overline{SL_2}(\mathbb{A})$ is *not* automorphic, let alone cuspidal, since $\varepsilon(\tau, \frac{1}{2}) = -1$.)

PROPOSITION.

Let K be a quadratic extension of F, and π the base change lift of τ to $PGL_2(\mathbb{A}_K)$. Then

$$L(\pi, \tfrac{1}{2}) = 0 \quad \textit{and} \quad \varepsilon(\pi, \tfrac{1}{2}) = 1.$$

Proof. Let $\chi = \otimes \chi_v$ be the quadratic character of $\mathbb{A}^\times$ associated to K over F. Then we know (cf. [8]) that

$$\varepsilon(\pi, s) = \varepsilon(\tau, s)\varepsilon(\tau \otimes \chi, s)$$

and

$$L(\pi, s) = L(\tau, s)L(\tau \otimes \chi, s).$$

Thus $L(\pi, \frac{1}{2}) = 0$, since $L(\tau, \frac{1}{2}) = 0$. On the other hand, since τ_v belongs to the principal series for all v, there are quasicharacters μ_v of $F_v^\times$ such that $\tau_v = \pi(\mu_v, \mu_v^{-1})$ for each v; hence

$$\varepsilon(\tau_v, \tfrac{1}{2}) = \varepsilon(\mu_v, \tfrac{1}{2})\varepsilon(\mu_v^{-1}, \tfrac{1}{2}) = \mu_v(-1),$$

$$\varepsilon(\tau_v \otimes \chi_v, \tfrac{1}{2}) = \varepsilon(\mu_v\chi_v, \tfrac{1}{2})\varepsilon(\mu_v^{-1}\chi_v, \tfrac{1}{2}) = \mu_v\chi_v(-1),$$

and

$$\begin{aligned}\varepsilon(\pi, \tfrac{1}{2}) &= \prod_v \varepsilon(\tau_v, \tfrac{1}{2})\varepsilon(\tau_v \otimes \chi_v, \tfrac{1}{2}) \prod_v \chi_v(-1) \\ &= \chi(-1) = 1\end{aligned}$$

as claimed.

Remarks.

(i) We recall that the base change lift of an automorphic cuspidal representation τ of $PGL_2(\mathbb{A})$ fails to be cuspidal only if τ is of the form $\pi(\eta)$ with η a character of

$K^x \backslash \mathbb{A}_K^x$ such that $\bar{\eta} \neq \eta$. Thus the π in our Proposition is indeed cuspidal, since "τ_v principal series for all v" precludes the possibility that τ be such a $\pi(\eta)$.

(ii) The idea of executing a sequence of *consecutive* (quadratic) base changes in order to produce interesting examples of automorphic forms comes from [2].

4. Concluding remarks

How general is the phenomenon described by the examples in this paper? i.e. for which quasi-split reductive groups G are there automorphic cuspidal representations σ such that, for some "non-degenerate" character ψ of the maximal unipotent subgroup U of G, σ admits a ψ-Whittaker model, and yet the ψ-Fourier coefficients of cusp forms in the space of σ vanish identically?

4.1 Note that for the group $\overline{SL_2}$, there is more than one orbit of non-trivial characters ψ with respect to the natural action of the maximal torus. In cases where the maximal torus *does* act transitively on non-degenerate ψ (i.e. ψ such that the restriction to any simple root subgroup in U is non-trivial), Piatetski-Shapiro has formalized the notion of an automorphic cuspidal representation σ being *hypercuspidal* (cf. [11]). This means precisely that all the Fourier coefficients of σ along U (with respect to any ψ) vanish identically. By contrast, σ is called *generic* if it is orthogonal to all hypercusp forms. Clearly σ generic implies σ admits a non-trivial (abstract) Whittaker model with respect to one (and hence all) ψ. However, it is also clearly suggested now that the converse need not be true. It would be desirable, therefore, to construct examples of hypercusp forms which are non-degenerate. For the group $U_{2,1}$, these notions are discussed in [3]. It is shown there that hypercuspidal σ do exist, but the examples (constructed by way of Weil's representation) have no Whittaker models at all. This is also the case for the "Saito-Kurokawa" cusp forms constructed in [12] by way of the dual pair $(PGSp_4, \overline{SL_2})$; cf. the last section of [17].

4.2 It seems natural that more dual reductive pair liftings be studied in the spirit of Waldspurger's work. Two natural candidates are $(Sp_4, O_{2,2})$ and $(GSp_4, GSO(6))$, examples already examined carefully in [5] (resp. [7], [18] and [15]). Note that GSp_4 satisfies the transitivity condition alluded to in §4.1. Only for the pair (SL_2, PGL_2), however, is the θ-series lifting sufficiently well understood to be able to resolve the Whittaker model question discussed in this paper.

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