Proc. Indian Acad. Sci. (Math. Sci.), Vol. 97, Nos 1–3, December 1987, pp. 61–66.

Almost poised basic hypergeometric series

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Abstract. Given a basic hypergeometric series with numerator parameters $a_1, a_2, \ldots, a_r$ and denominator parameters $b_2, \ldots, b_r$, we say it is *almost poised* if $b_i = a_1 q^{\delta_i}/a_i$, $\delta_i = 0, 1$ or 2, for $2 \leqslant i \leqslant r$. Identities are given for almost poised series with $r = 3$ and $r = 5$ when $a_1 = q^{-2n}$.

Keywords. Hypergeometric series; poised series.

A hypergeometric series is a power series, say

$$1 + \sum a_n x^n, \quad n \geqslant 1,$$

for which the ratio of successive coefficients, a_n/a_{n-1}, is a rational function of n. A basic hypergeometric series is a power series in which the ratio of successive coefficients is a rational function in q^n. Traditionally,

$$q = \exp(2\pi i \tau),$$

where τ is a complex variable in the upper half plane. Many theta function identities are most easily understood as identities on basic hypergeometric series. Ramanujan contributed appreciably to our knowledge and understanding of these series. They permeate all of his work.

We shall restrict our attention to series of the form

$${}_r\varphi_{r-1}\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_2, \ldots, b_r \end{matrix}; x\right) = 1 + \sum \frac{(a_1)_n (a_2)_n \cdots (a_r)_n}{(q)_n (b_2)_n \cdots (b_r)_n} x^n, \quad n \geqslant 1,$$

where

$$(a)_n = \prod \frac{(1 - aq^i)}{(1 - aq^{n+i})}, \quad i \geqslant 0,$$

$$= (1 - a)(1 - aq) \cdots (1 - aq^{n-1}).$$

Observe that the first equality defines $(a)_n$ for all real n.

Following W N Bailey [3], we call the series *well-poised* if $b_i = a_1 q/a_i$ for $2 \leqslant i \leqslant r$. F H Jackson [5] has proved the following identities for well-poised basic hyper-

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* Partially supported by N.S.F. Grant No. DMS-8521580.

geometric series:

$$
{}_5\varphi_4\left(\begin{matrix} q^{-2n}, b, c, d, e \\ b^{-1}q^{1-2n}, c^{-1}q^{1-2n}, d^{-1}q^{1-2n}, e^{-1}q^{1-2n} \end{matrix}; \frac{q^{2-3n}}{bcde}\right)
$$

$$
= \frac{(q^{-2n})_n (b^{-1}c^{-1}q^{1-2n})_n (b^{-1}d^{-1}q^{1-2n})_n (c^{-1}d^{-1}q^{1-2n})_n}{(b^{-1}q^{1-2n})_n (c^{-1}q^{1-2n})_n (d^{-1}q^{1-2n})_n (b^{-1}c^{-1}d^{-1}q^{1-2n})_n}, \tag{1}
$$

provided $bcde = q^{1-3n}$,

$$
{}_3\varphi_2\left(\begin{matrix} q^{-2n}, b, c \\ q^{1-2n}/b, q^{1-2n}/c \end{matrix}; \frac{q^{1-n}}{bc}\right) = \frac{(q^{-2n})_n (b^{-1}c^{-1}q^{1-2n})_n}{(b^{-1}q^{1-2n})_n (c^{-1}q^{1-2n})_n}. \tag{2}
$$

Bailey [3] defined a series to be *nearly poised* if $b_i = a_1 q/a_i$ for all but one value of i in $2 \leqslant i \leqslant r$. We shall instead give identities for what we term *almost poised* series where

$$
b_i = a_1 q^{\delta_i}/a_i, \quad 2 \leqslant i \leqslant r,
$$

where δ_i is 0, 1 or 2.

$$
{}_5\varphi_4\left(\begin{matrix} q^{-2n}, b, c, d, e \\ b^{-1}q^{-2n}, c^{-1}q^{-2n}, d^{-1}q^{1-2n}, e^{-1}q^{1-2n} \end{matrix}; \frac{q^{1-3n}}{bcde}\right)
$$

$$
= \frac{(q^{-2n})_n (b^{-1}c^{-1}q^{-2n})_n}{(b^{-1}q^{-2n})_n (c^{-1}q^{-2n})_n}\, {}_3\varphi_2\left(\begin{matrix} b, c, q^{-n} \\ d^{-1}q^{1-2n}, e^{-1}q^{1-2n} \end{matrix}; q\right), \tag{3}
$$

provided $bcde = q^{-3n}$,

$$
{}_3\varphi_2\left(\begin{matrix} q^{-2n}, b, c \\ b^{-1}q^{-2n}, c^{-1}q^{-2n} \end{matrix}; \frac{q^{-n}}{bc}\right) = \frac{(q^{-2n})_n (b^{-1}c^{-1}q^{-2n})_n}{(b^{-1}q^{-2n})_n (c^{-1}q^{-2n})_n}, \tag{4}
$$

$$
{}_3\varphi_2\left(\begin{matrix} q^{-2n}, b, c \\ b^{-1}q^{-2n}, c^{-1}q^{1-2n} \end{matrix}; \frac{q^{1-n}}{bc}\right) = \frac{(q^{-2n})_n (b^{-1}c^{-1}q^{1-2n})_n}{(b^{-1}q^{-2n})_n (c^{-1}q^{1-2n})_n}, \tag{5}
$$

$$
{}_3\varphi_2\left(\begin{matrix} q^{-2n}, b, c \\ b^{-1}q^{-2n}, c^{-1}q^{1-2n} \end{matrix}; \frac{q^{-n}}{bc}\right)
$$

$$
= \frac{(q^{-2n})_n (b^{-1}c^{-1}q^{1-2n})_n}{(b^{-1}q^{-2n})_n (c^{-1}q^{1-2n})_n}\left\{1 - c^{-1}q^{-n}\frac{(1-q^{-n})}{(1-b^{-1}c^{-1}q^{-2n})}\right\}, \tag{6}
$$

$$
{}_3\varphi_2\left(\begin{matrix} q^{-2n}, b, c \\ b^{-1}q^{-2n}, c^{-1}q^{2-2n} \end{matrix}; \frac{q^{1-n}}{bc}\right)
$$

$$
= \frac{(q^{-2n})_n (b^{-1}c^{-1}q^{2-2n})_n}{(b^{-1}q^{-2n})_n (c^{-1}q^{2-2n})_n}\left\{1 + c^{-1}q^{-n}\frac{(1-b^{-1})}{(1-c^{-1}q)}\right\}. \tag{7}
$$

It should be kept in mind that each of these represents several identities which can be obtained from the given identity by either replacing b, c and q by b^{-1}, c^{-1} and

q^{-1}, respectively, and then using the identity

$$\prod_{i=0}^{n-1}\frac{(1-a^{-1}q^{-i})}{(1-b^{-1}q^{-i})}=\frac{(a)_n}{(b)_n}\left(\frac{b}{a}\right)^n, \tag{8}$$

or by reversing the order of summation and using the identity

$$\begin{aligned}\frac{(a)_{2n-i}}{(b)_{2n-i}}&=\frac{(a)_{2n}(bq^{2n-i})_i}{(b)_{2n}(aq^{2n-i})_i}\\&=\frac{(a)_{2n}}{(b)_{2n}}\frac{(b^{-1}q^{1-2n})_i}{(a^{-1}q^{1-2n})_i}\left(\frac{b}{a}\right)^i,\end{aligned} \tag{9}$$

or by doing both. As an example, from (4) we derive three equivalent identities

$${}_3\varphi_2\left(\begin{matrix}q^{-2n},b,c\\ b^{-1}q^{-2n},c^{-1}q^{-2n}\end{matrix};\frac{q^{1-n}}{bc}\right)=\frac{(q^{-2n})_n(b^{-1}c^{-1}q^{-2n})_n}{(b^{-1}q^{-2n})_n(c^{-1}q^{-2n})_n}, \tag{10}$$

$$\begin{aligned}&{}_3\varphi_2\left(\begin{matrix}q^{-2n},b,c\\ b^{-1}q^{2-2n},c^{-1}q^{2-2n}\end{matrix};\frac{q^{3-n}}{bc}\right)\\&=\frac{(q^{-2n})_n(b^{-1}c^{-1}q^{2-2n})_n}{(b^{-1}q^{1-2n})_n(c^{-1}q^{1-2n})_n}q^{3n}\frac{(1-b^{-1}q^{1-2n})(1-c^{-1}q^{1-2n})}{(1-b^{-1}q)(1-c^{-1}q)},\end{aligned} \tag{11}$$

$$\begin{aligned}&{}_3\varphi_2\left(\begin{matrix}q^{-2n},b,c\\ b^{-1}q^{2-2n},c^{-1}q^{2-2n}\end{matrix};\frac{q^{2-n}}{bc}\right)\\&=\frac{(q^{-2n})_n(b^{-1}c^{-1}q^{2-2n})_n}{(b^{-1}q^{1-2n})_n(c^{-1}q^{1-2n})_n}q^{n}\frac{(1-b^{-1}q^{1-2n})(1-c^{-1}q^{1-2n})}{(1-b^{-1}q)(1-c^{-1}q)}.\end{aligned} \tag{12}$$

Proof. Jackson's identities (1) and (2) follow from the q-analog of Whipple's theorem ([6]):

$$\begin{aligned}&{}_8\varphi_7\left(\begin{matrix}a,q\sqrt{a},-q\sqrt{a},b,c,d,e,q^{-n}\\ \sqrt{a},-\sqrt{a},aq/b,aq/c,aq/d,aq/e,aq^{n+1}\end{matrix};\frac{a^2q^{2+n}}{bcde}\right)\\&=\frac{(aq)_n(aq/bc)_n}{(aq/b)_n(aq/c)_n}{}_4\varphi_3\left(\begin{matrix}aq/de,b,c,q^{-n}\\ bcq^{-n}/a,aq/d,aq/e\end{matrix};q\right).\end{aligned} \tag{13}$$

This equation also implies the Rogers–Ramanujan identities and plays a key role in many of Ramanujan's identities.

If we set $a=q^{-2n}$ in (12) we obtain

$$\begin{aligned}&{}_5\varphi_4\left(\begin{matrix}q^{-2n},b,c,d,e\\ b^{-1}q^{1-2n},c^{-1}q^{1-2n},d^{-1}q^{1-2n},e^{-1}q^{1-2n}\end{matrix};\frac{q^{2-3n}}{bcde}\right)\\&=\frac{(q^{-2n})_n(b^{-1}c^{-1}q^{1-2n})_n}{(b^{-1}q^{1-2n})_n(c^{-1}q^{1-2n})_n}{}_4\varphi_3\left(\begin{matrix}d^{-1}e^{-1}q^{1-2n},b,c,q^{-n}\\ bcq^{n},d^{-1}q^{1-2n},e^{-1}q^{1-2n}\end{matrix};q\right).\end{aligned} \tag{14}$$

Equation (1) is the case $bcde=q^{1-3n}$. Equation (2) is the case $de=q^{1-2n}$.

To obtain identities for almost poised but not well-poised series, we start with an almost poised version of (13) (see [1], Corollary 4.1):

$$ {}_6\varphi_5\left(\begin{matrix} a,b,c,d,e,q^{-n} \\ a/b,a/c,aq/d,aq/e,aq^n \end{matrix}; \frac{a^2q^{1+n}}{bcde}\right) - \frac{(1-b)(1-c)(1-q^n)}{(1-a/b)(1-a/c)(1-aq^n)}\left(\frac{a^2}{bc}\right) $$

$$ {}_6\varphi_5\left(\begin{matrix} a,bq,cq,d,e,q^{1-n} \\ aq/b,aq/c,aq/d,aq/e,aq^{n+1} \end{matrix}; \frac{a^2q^{2+n}}{bcde}\right) $$

$$ = \frac{(a)_n(a/bc)_n}{(a/b)_n(a/c)_n}\,{}_4\varphi_3\left(\begin{matrix} aq/de,b,c,q^{-n} \\ bcq^{1-n}/a,aq/d,aq/e \end{matrix};q\right). \tag{15} $$

Again setting $a=q^{-2n}$ gives us

$$ {}_5\varphi_4\left(\begin{matrix} q^{-2n},b,c,d,e \\ b^{-1}q^{-2n},c^{-1}q^{-2n},d^{-1}q^{1-2n},e^{-1}q^{1-2n} \end{matrix}; \frac{q^{1-3n}}{bcde}\right) $$

$$ = \frac{(q^{-2n})_n(b^{-1}c^{-1}q^{-2n})_n}{(b^{-1}q^{-2n})_n(c^{-1}q^{-2n})_n}\,{}_4\varphi_3\left(\begin{matrix} d^{-1}e^{-1}q^{1-2n},b,c,q^{-n} \\ bcq^{1+n},d^{-1}q^{1-2n},e^{-1}q^{1-2n} \end{matrix};q\right). \tag{16} $$

Equation (3) is the case $bcde=q^{-3n}$. Equation (4) is the case $de=q^{1-2n}$.

The remaining identities follow from a formula given by I Goulden and the present author [4] for the constant term with respect to x_1,x_2 and x_3 in

$$ f(x_1,x_2,x_3)=(x_1/x_2)_a(qx_2/x_1)_\alpha(x_1/x_3)_a(qx_3/x_1)_\beta(x_2/x_3)_b(qx_3/x_2)_\gamma, \tag{17} $$

where

$$ 0\leqslant b-\alpha\leqslant 1, $$

and

$$ |\beta-\gamma|\leqslant 1. $$

By the q-binomial theorem ([2]; Theorem 2.1) we have

$$ (z)_a(q/z)_b=\sum(-1)^iq^{i(i-1)/2}z^i\frac{(q)_{a+b}}{(q)_{a-i}(q)_{b+i}}, \quad -b\leqslant i\leqslant a. \tag{18} $$

The function $f(x_1,x_2,x_3)$ can thus be expressed as a triple summation

$$ f(x_1,x_2,x_3)=\sum x_1^{i+j}x_2^{-i+k}x_3^{-j-k}(-1)^{i+j+k}q^{(i(i-1)+j(j-1)+k(k-1))/2}* $$

$$ *\frac{(q)_{a+\alpha}(q)_{a+\beta}(q)_{b+\gamma}}{(q)_{a-i}(q)_{\alpha+i}(q)_{a-j}(q)_{\beta+j}(q)_{b-k}(q)_{\gamma+k}} \tag{19} $$

Setting $i=-j=k$ yields the constant term which, after simplification, can be expressed as

$$ \text{C.T. } f(x_1,x_2,x_3)=\frac{(q)_{a+\alpha}(q)_{a+\beta}(q)_{b+\gamma}(-1)^aq^{a(3a+1)/2}}{(q)_{2a}(q)_{a+\beta}(q)_{a+b}(q)_{\alpha-a}(q)_{\gamma-a}}* $$

$$ *\,{}_3\varphi_2\left(\begin{matrix} q^{-2a},q^{-a-\beta},q^{-a-b} \\ q^{\alpha-a+1},q^{\gamma-a+1} \end{matrix};q^{a+\beta+b+1}\right), \tag{20} $$

where C.T. denotes constant term with respect to x_1, x_2 and x_3.

We shall equate this with the expression given for the constant term given in Proposition 2.4 of [4].

Let $S(\delta, \varepsilon, \varphi)$ where $\delta, \varepsilon, \varphi \in \{0,1\}$ be the set of bijections σ from $\{1,2,3\}$ to $\{a,b,c\}$ such that

(i) if $\delta = 1$, then $\sigma^{-1}(a) < \sigma^{-1}(b)$,
(ii) if $\varepsilon = 1$, then $\sigma^{-1}(a) < \sigma^{-1}(c)$, and
(iii) if $\varphi = 1$, then $\sigma^{-1}(b) < \sigma^{-1}(c)$.

Let $I(\sigma)$ be $(3 - \sigma^{-1}(c))c$, if $\sigma^{-1}(a) < \sigma^{-1}(b)$ and

$$b + (3 - \sigma^{-1}(c))c, \quad \text{if } \sigma^{-1}(a) > \sigma^{-1}(b).$$

We define $F(\delta, \varepsilon, \varphi)$ to be

$$F(\delta, \varepsilon, \varphi) = \sum \frac{q^{I(\sigma)}}{(1 - q^{\sigma(1)})(1 - q^{\sigma(1)+\sigma(2)})}, \quad \sigma \in S(\delta, \varepsilon, \varphi). \tag{21}$$

As examples, we have

$$\begin{aligned} F(1,0,0) &= \frac{1}{(1-q^a)(1-q^{a+b})} + \frac{q^c}{(1-q^a)(1-q^{a+c})} + \frac{q^{2c}}{(1-q^c)(1-q^{a+c})} \\ &= \frac{1-q^{a+b+c}}{(1-q^a)(1-q^c)(1-q^{a+b})}, \end{aligned} \tag{22}$$

$$F(0,1,0) = \frac{1}{(1-q^a)(1-q^b)} + \frac{q^c}{(1-q^a)(1-q^{a+c})}, \tag{23}$$

$$F(1,0,1) = \frac{1}{(1-q^a)(1-q^{a+b})}, \tag{24}$$

$$F(1,1,0) = \frac{1}{(1-q^a)(1-q^{a+b})} + \frac{q^c}{(1-q^a)(1-q^{a+c})}. \tag{25}$$

Letting $c = \max(\beta, \gamma)$, Proposition 2.4 of [4] implies that

$$\text{C.T. } f(x_1, x_2, x_3) = \frac{(q)_{a+b+c-1}}{(q)_{a-1}(q)_{b-1}(q)_{c-1}} F(b-\alpha, c-\beta, c-\gamma). \tag{26}$$

Combining this with (19) and setting $B = q^{-a-b}$, $C = q^{-a-\beta}$ and $n = a$ yields

$$\begin{aligned} &{}_3\varphi_2\left(\begin{matrix} q^{-2n}, B, C \\ B^{-1}q^{1-2n+\alpha-b}, C^{-1}q^{1-2n+\gamma-\beta} \end{matrix}; \frac{q^{1-n}}{BC}\right) \\ &= \frac{(q^{-2n})_n (B^{-1}C^{-1}q^{1-2n+\gamma-\beta})_n}{(B^{-1}q^{1-2n+\alpha-b})_n (C^{-1}q^{1-2n+\gamma-\beta})_n} \frac{(q)_{a+b}}{(q)_{n+\alpha}} F(b-\alpha, c-\beta, c-\gamma). \end{aligned} \tag{27}$$

This is valid for $B = q^{-a-b}$, b any integer larger than a, and $C = q^{-1-\beta}$, β any integer larger than a. Since it is equivalent to a polynomial identity in B^{-1} and C^{-1}, it is valid for all values of B and C.

Equation (5) is the case $\alpha = b - 1$, $\beta = \gamma = c$. Equation (6) is the case $\alpha = b$, $\beta + 1 = \gamma = c$ with variables replaced by their inverses and the order of summation reversed. Equation (7) is the case $\alpha = b - 1$, $\beta + 1 = \gamma = c$. Equations (2) and (4) are also special cases of equation (27).

Conclusions and questions

Comparing (2), (4) and (5), one is struck by their similarity. Are there any other similar identities with the same simplicity?

What are the almost poised versions of the q-analog of Whipple's theorem which correspond to (5), (6) and (7)?

Can (1) and (3) be restated as constant term identities in a manner as simple as that used for (5)–(7)?

The proof of (15) is easily generalized to obtain identities in which two of the denominator parameters are of the form aq^i/b, aq^i/c, where i is an arbitrary integer. These identities, however, look immensely complicated. Can they be restated in a simple form?

What other almost poised identities exist?

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