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Explicit Ramanujan-type approximations to pi of high order

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Abstract. We combine previously developed work with a variety of Ramanujan's higher order modular equations to make explicit, in very simple form, algebraic approximations to π which converge with orders including 7, 11, 15 and 23.

Keywords. Ramanujan-type approximations; recursive approximation to pi.

1. Introduction

Ramanujan's 1914 paper 'Modular equations and approximations to π' [6] contains a number of approaches to approximating pi. The three most significant approaches all involve elliptic integrals. The first approximates π as the logarithm of singular values of elliptic integrals; the second uses the derivative of Dedekind's eta function to produce algebraic approximations to pi; the third constructs rapidly convergent series for $1/\pi$ whose coefficients are invariants from elliptic or modular function theory. All three of these approaches are described in Borwein and Borwein ([2], Chapters 2, 5, 9) which we follow closely in the development of the material herein. As is described in [2] and [3] both of the latter two approaches implicitly involve recursive approximations to $1/\pi$.

Between 1983 and the present a number of record setting calculations of digits of pi have been undertaken. All rely on methods that relate directly to this material. Details of the calculations of Gosper, Bailey, Tamura and Kanada and Kanada may be found in [2] and [3].

In this paper we record the results of our systematic attempts to exploit Ramanujan's higher order modular equations as given by Bruce Berndt's edited versions of Chapters 19 and 20 of Ramanujan's second notebook [1].

2. Recursive approximations to pi

We take for granted a familiarity with the basic language of elliptic integrals and theta functions as delimited in [2, 5, 6, 7, 8]. Thus we have the *complete elliptic integral of the first kind*

$$K(k) := \int_0^{\pi/2} dt/(1 - k^2 \sin^2 t)^{\frac{1}{2}}$$

and the *complete elliptic integral of the second kind*

$$E(k) := \int_0^{\pi/2} (1 - k^2 \sin^2 t)^{\frac{1}{2}}\, dt.$$

We set $k' := \sqrt{(1-k^2)}$, $K'(k) := K(\sqrt{(1-k^2)})$, $E'(k) := E(\sqrt{(1-k^2)})$. We then have the key relations

$$K(k) = \frac{\pi}{2}\theta_3^2(q), \quad k := k(q) = \frac{\theta_2^2(q)}{\theta_3^2(q)} \quad \text{and} \quad q := \exp(-\pi K'/K), \tag{1}$$

where the *special theta functions* are given by

$$\theta_3(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \quad \text{and} \quad \theta_2(q) := \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}.$$

The *singular value function* is then defined by

$$k(N) := k[\exp(-\pi\sqrt{N})] \quad \text{or} \quad K'(k(N))/K(k(N)) = \sqrt{N} \tag{2}$$

for positive N. This uniquely defines k on $(0, \infty)$ as a decreasing function with $k(0) = \infty$, $k(1) = 1/\sqrt{2}$, and $k(\infty) = 0$. Also k is algebraic when N is rational [2]. In [2] a corresponding function α (*a singular value of the second kind*) was studied. It is defined by

$$\alpha(N) := (E'/K) - (\pi/4K^2) \qquad k := k(N) \tag{3}$$

and likewise is algebraic at rational values with $\alpha(1) = 1/2$, $\alpha(\infty) = 1/\pi$. Moreover, α satisfies recursions which allow for its computation both algebraically and numerically. For example,

$$\alpha(4N) = (1 + k(4N))^2\alpha(N) - 2\sqrt{N}k(4N) \tag{4}$$

and

$$k(4N) = [1 - k'(N)]/[1 + k'(N)]. \tag{5}$$

This type of relation can then be iterated and leads to high-order iterations for π (see [2], [3] and below). The function alpha is directly connected to *Ramanujan's multiplier* R_N which is defined in terms of the Eisenstein series

$$P(q) := 1 - 24\sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{2n}} \tag{6}$$

by

$$R_N(l, k) := \frac{NP(q^N) - P(q)}{\theta_3^2(q)\theta_3^2(q^N)} \quad k = k(q^N) \quad l = k(q). \tag{7}$$

Ramanujan gives closed forms for R_N for $N = 2, 3, 5, 7, 11, 15, 17, 19, 23, 31$ and 35, many of which we use below (see [2] and [6]). From now on we will freely write k and l for

$$k = k(N^2 r) = k[\exp(-\pi N\sqrt{r})], \quad l = k(r) = k[\exp(-\pi\sqrt{r})].$$

The pair of modulii k and l then solve the *Nth order modular equation* for K and so satisfy an algebraic equation

$$W_N(l, k) = 0.$$

The corresponding *multiplier* defined by

$$M_N(l,k) := K(k)/K(l) = \theta_3^2(q^N)/\theta_3^2(q) \tag{8}$$

is then also algebraic.

We will often suppress variables and subscripts. Thus we write $m := m_N = m_N(l,k)$ where

$$m_N(l,k) := [K(l)/K(k)] - M_N^{-1}(l,k). \tag{9}$$

In these terms the recursive approximations to pi given in Chapter 5 of [2] can be described by

$$\alpha(N^2 r) = m_N^2 \alpha(r) - \sqrt{r}\,\varepsilon_N(r), \tag{10a}$$

where

$$\varepsilon_N(r) := \frac{m_N^2 - N}{2} + \frac{m_N}{3}\left\{R_N + m_N(l^2 - \tfrac{1}{2}) - \frac{N}{m_N}(k^2 - \tfrac{1}{2})\right\} \tag{10b}$$

and

$$k = k[\exp(-\pi N\sqrt{r})], \quad l = k[\exp(-\pi\sqrt{r})].$$

The accuracy of these approximations is phenomenal. For $r > 1$ the error satisfies

$$0 < \alpha(r) - 1/\pi < 16\sqrt{r}\exp(-\pi\sqrt{r}) \tag{11}$$

and this is for all intents the asymptotic if '8' replaces '16'.

An Nth order iteration for π consists of the following. Commence with $k_0 := k(r)$ and $\alpha_0 := \alpha(r)$ for some given r and then repeatedly solve $W_N(k_n, k_{n+1}) = 0$ for k_{n+1} given k_n, while using (10) with $l = k_n$, $k = k_{n+1}$ to compute $\alpha_{n+1} := \alpha(N^{2n+2}r)$. Then (11) shows that the error between α_n and $1/\pi$ is of the order of $N^n \exp[(-\pi\sqrt{r})N^n]$ and that the number of decimal digits correct increases by more than a factor of N at each step.

3. Specific examples

To compute ε_N explicitly one needs tractable forms of R_N, and of $m_N - N/m_N$. The transformation of $q \to -q$ in the latter then provides a formula for $m_N(1 - l^2) - (1 - k^2)N/m_N$ and the component pieces may with some effort be folded together—use of modular symmetries somewhat simplifies the task. The resulting results are surprisingly simple, especially for $N \equiv 3 \pmod 4$. We list these cumulative results next. In each case the first entry gives a form of the modular equation W_N, the second $m_N - N/m_N$, while ε_N is given third.

For $N = 3, 5, 7$ alternative solvable modular equations for m_N are given in [2] and [3]. In each case the tabular entry can be converted into a iteration for π or used to approximate log or other quantities. We also observe that the formula for ε_N can be used to determine a closed form for $\mathrm{d}M_N(l,k)/\mathrm{d}k$.

Explicit examples with $N \equiv 3 \pmod 4$

$N=3$ (i) $(kl)^{1/2}+(k'l')^{1/2}=1$,

(ii) $m-3/m=2[(kl)^{1/2}-(k'l')^{1/2}]=2(kl-k'l')$,

(iii) $\varepsilon_3=(m^2+2m-3)/2$.

$N=7$ (i) $(kl)^{1/4}+(k'l')^{1/4}=1$,

(ii) $m-7/m=2[(kl)^{1/4}-(k'l')^{1/4}][2+(kl)^{1/2}+(k'l')^{1/2}]$,

(iii) $\varepsilon_7=(m^2+2m-7)/2+m[2(kl)^{1/2}+2(k'l')^{1/2}]$.

$N=11$ (i) $(kl)^{1/2}+(k'l')^{1/2}+2(4klk'l')^{1/6}=1$,

(ii) $m-11/m=2[(kl)^{1/2}-(k'l')^{1/2}][4+(kl)^{1/2}+(k'l')^{1/2}]$,

(iii) $\varepsilon_{11}=(m^2+2m-11)/2+m[1+(kl)^{1/2}+(k'l')^{1/2}]^2$.

$N=15$ (i) $P^3=4PQ-R$ where $P=1+(kl)^{1/4}+(k'l')^{1/4}$

$Q=(kl)^{1/4}+(k'l')^{1/4}+(klk'l')^{1/4}$ $R=4(klk'l')^{1/4}$,

(ii) $m-15/m=[(kl)^{1/4}-(k'l')^{1/4}](P^3+2P^2-2P+2)$,

(iii) $\varepsilon_{15}=(m^2-2m-15)/2+m[(kl)^{1/4}+(k'l')^{1/4}](4Q+4)$.

$N=23$ (i) $(kl)^{1/4}+(k'l')^{1/4}+\sqrt{2}(4klk'l')^{1/12}=1$ $[T=(kl)^{1/4}+(k'l')^{1/4}]$,

(ii) $m-23/m=[(kl)^{1/4}-(k'l')^{1/4}](T^4+3T^3+3T^2+7T+8)$,

(iii) $\varepsilon_{23}=(m^2+2m-23)/2+m(T^5+3T^3+T^2+4T+1)$.

In all these cases Ramanujan has provided a formula for R_N [6] while for $N=7,11,23$ he gives a form of m_N-N/m_N [1]. The calculation of m_N-N/m_N for $N=3$ is straightforward, while for $N=15$ we discovered the formula computationally. The modular equations used may be found in a variety of places. Many are derivable from or are included in the notebooks [1]. Ramanujan's formulae are usually but not always the simplest available. For example, Ramanujan gives

$$R_{23}(l,k)=11(1+kl+k'l')-16(4kk'll')^{1/6}[1+(kl)^{1/2}+(k'l')^{1/2}]-20(4kk'll')^{1/3},$$

which we recast as

$$R_{23}(l,k)=3(1+kl+k'l')+4[1+(kl)^{1/2}+(k'l')^{1/2}]^2-4(4kk'll')^{1/3}$$

by using W_{23}.

For comparison we also list the following:

$N=2$ (i) $k=(1-l')/(1+l')$,

(ii) $m=1+k=2/(1+l')$,

(iii) $\varepsilon_2=2k$.

$N=4$ (i) $\sqrt{k}=(1-(1-l^2)^{1/4})/(1+(1-l^2)^{1/4})$,

(ii) $m=(1+\sqrt{k})^2$,

(iii) $\varepsilon_4=4\sqrt{k}[1+\sqrt{k}+k]$.

$N = 5$ (i) $k^{3/2} - l^{3/2} + 4(kl)^{1/4}(1 - kl) + 5(kl)^{1/2}(k^{1/2} - l^{1/2}) = 0$,
(ii) $m + 5/m = 2[2 + kl + k'l']$,
(iii) $\varepsilon_5 = (m^2 - 5)/2 + [m(m^2 - 2m + 5)]^{1/2}$.

We also indicate the components for $N = 19$ where Ramanujan does not provide an expression for $m_N - N/m_N$ and we have not been able to determine a suitable formula, although R_{19} is simple.

$N = 19$ (i) $P^5 = 7P^2R - 16QR$ where $P = (kl)^{1/2} + (k'l')^{1/2} - 1$
$Q = (klk'l')^{1/2} - (kl)^{1/2} - (k'l')^{1/2}$ $R = -16(klk'l')^{1/2}$,
(ii) $R_{19} = 6[(1 + kl + k'l') + (kl)^{1/2} + (k'l')^{1/2} - (klk'l')^{1/2}] = 6P^2 - 18Q$
$m_{19} = \{1 + [(A - B - 1)^{1/2} - (AB)^{1/2}]^2\}[(1 + kl + k'l')/2]^{-1/2}$
$A = k'^{3/4}l'^{-1/4}$ $B = k^{3/4}l^{-1/4}$.

4. Additional comments

It is worth emphasising the staggering rate of convergence of the algorithms based on these recursions. The two most natural starting values for the Nth order algorithm are $r = 1$ and $r = 1/N$. As observed, for $r = 1$ we have $k_0 = 1/\sqrt{2}$ and $\alpha_0 = 1/2$. For $r = 1/N$ one knows that k_N is the Nth singular value and solves $W_N(k'_N, k_N) = 0$ while $M_N(k'_N, k_N) = 1/\sqrt{N}$. Thus we can use (10b) or the corresponding tabular entry to obtain

$$\alpha(N) - N\alpha(1/N) = \varepsilon_N(1/N)/\sqrt{N}.$$

In addition one always has

$$\alpha(N) + N\alpha(1/N) = \sqrt{N}.$$

Thus one may determine that

$$\alpha(N) = \tfrac{1}{2}(\sqrt{N} - \varepsilon_N/\sqrt{N}) \quad \text{and} \quad \alpha(1/N) = (\sqrt{N} + \varepsilon_N/\sqrt{N})/2N,$$

where $\varepsilon_N = \varepsilon_N(1/N)$.

Example. We show how this works for various N. Throughout we let $\delta(N) := \varepsilon_N/\sqrt{N}$. Note that $1/\pi = 0{\cdot}3183099\ldots$.

($N = 3$) Setting $l = k'$ in (i) yields $2k_3k'_3 = 1/2$.
Moreover, $m_3 = \sqrt{3}$ so that, using (iii), $\varepsilon_3(1/3) = \sqrt{3}$. Thus

$$\alpha(3) = (\sqrt{3} - 1)/2 = 0{\cdot}3660\ldots.$$

($N = 7$) Setting $l = k'$ in (i) yields $2k_7k'_7 = 1/8$.
Now $m_7 = \sqrt{7}$ so that, using (iii), $\delta_7(1/7) = (1 + 4(k_7k'_7)^{1/2}) = 2$ and

$$\alpha(7) = (\sqrt{7} - 2)/2 = 0{\cdot}3228\ldots.$$

($N = 11$) Setting $l = k'$ in (i) and letting $x := 2\sqrt{(k_{11}k'_{11})}$ produces the cubic equation $(1-x)^3 = (2x)^2$. This can be solved to produce

$$2kk' = (2\gamma - 5)/6$$

with

$$\gamma := (3\sqrt{33} + 17)^{1/3} - (3\sqrt{33} - 17)^{1/3}.$$

Also, (iii) becomes $\delta(11) = 1 + (1+x)^2 = (5 + 2\gamma + 2\gamma^2)/9$ and

$$\alpha(11) = (\sqrt{11} - \delta(11))/2 = 0{\cdot}3190\ldots.$$

($N = 15$) Setting $l = k'$ in (i) leads to

$$kk' = ((\sqrt{5} - 1)/4)^4 = (\sin \pi/10)^4.$$

If G denotes the golden mean, $(\sqrt{5} - 1)/2$, (iii) becomes

$$\delta(15) = G(4 + 4G + G^2) - 1 = \sqrt{5} + 1 \text{ and}$$

$$\alpha(15) = (\sqrt{15} - \sqrt{5} - 1)/2 = 0{\cdot}31845\ldots.$$

($N = 23$) Setting $l = k'$ in (i) and letting $x := 2(k_{23}k'_{23})^{1/4}$ produces the cubic equation $(1-x)^3 = x^2$. This can be solved to produce

$$2kk' = \tfrac{1}{8}\gamma^{12}$$

with

$$\gamma := \frac{1}{3}\left(\left(\frac{25 + 3\sqrt{69}}{2}\right)^{1/3} + \left(\frac{25 - 3\sqrt{69}}{2}\right)^{1/3} - 1\right).$$

Also, (iii) becomes $\delta(23) = 4 + 2x^3 = 4 + 2\gamma^9$. Finally, as $\gamma^3 + \gamma^2 = 1$

$$\alpha(23) = (\sqrt{23} - 2\gamma^2 - 4\gamma)/2 = 0{\cdot}31832\ldots.$$

We conclude by presenting the asymptotics of some of these iterations commencing with the previously discussed starting values, using (11). The effect of commencing with $r = N$ instead of $r = 1/N$ is to move one index along in the iteration count. The estimate is of $\pi - 1/\alpha_n$.

	Decimal digits correct after iteration n										
n	1	2	3	4	5	6	7	8	9	10	
$r = 1$	1	3	8	19	40	84	171	345	694	1392	$N = 2$
$r = 1/2$	0	2	5	13	28	58	120	243	490	983	
$r = 1$	2	10	34	107	327	990	2979	8946	26849	8[illegible]59	$N = 3$
$r = 1/3$	0	5	18	60	188	570	1718	5163	15499	46508	
$r = 1$	7	63	464	3271	22925						$N = 7$
$r = 1/7$	1	22	173	1233	8662						
$r = 1$	12	161	1811	19970	219727						$N = 11$
$r = 1/11$	2	47	543	6018	66246						
$r = 1$	28	717	16595	381801	8781586						$N = 23$
$r = 1/23$	4	147	3456	79606	1831081						

The first iteration with $r = 1/N$ produces $\alpha(N)$ as given explicitly above for 3, 7, 11, 23. The tenth iteration with $N = 11$ and $r = 1$ produces in excess of $1{\cdot}45{\cdot}10^8$ digits coincidence with $1/\pi$. The eighth iteration with $N = 23$ and $r = 1$ produces in excess of 10^{12} digits coincidence with $1/\pi$.

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