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On the set of discrete subgroups of bounded covolume in a semisimple group

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Abstract. In this note G is a locally compact group which is the product of finitely many groups $\mathscr{G}_s(k_s)(s\in S)$, where k_s is a local field of characteristic zero and $\mathscr{G}_s$ an absolutely almost simple k_s-group, of k_s-rank $\geqslant 1$. We assume that the sum of the r_s is $\geqslant 2$ and fix a Haar measure on G. Then, given a constant $c > 0$, it is shown that, up to conjugacy, G contains only finitely many irreducible discrete subgroups L of covolume $\leqslant c$ (4.2). This generalizes a theorem of H C Wang for real groups. His argument extends to the present case, once it is shown that L is finitely presented (2.4) and locally rigid (3.2).

Keywords. Discrete subgroups; bounded covolume; semisimple group.

1. Introduction

Let first G be a connected semisimple Lie group with finite centre and no compact factors, of $\mathbb{R}$-rank $\geqslant 2$. Fix a Haar measure on G, hence on any quotient of G by a discrete subgroup. The total measure $v(G/L)$ of G/L is called the covolume of L, and will be denoted $c(L)$. By a result of H C Wang [15], given a constant $c > 0$, the number of conjugacy classes of irreducible discrete subgroups for which $c(L) \leqslant c$ is finite. Recently, Serre and Tits asked whether this is true in the p-adic case. In this paper, we show this is indeed the case. In fact, we prove more generally a similar assertion for discrete subgroups of products of semisimple groups over local fields of characteristic zero without compact factors.

This paper answers only the first of several questions of increasing generality pertaining to the finiteness of the number of pairs G, L with L of covolume bounded by a given constant, for a suitable a priori universal normalization of the Haar measure, when G or even the groundfield are allowed to vary. A number of results in that direction have since been obtained jointly with G Prasad and announced in *C R Acad. Sci.* **305** (1987), 357–362. These and others will be proved in two papers now in preparation.

In § 2 we fix our notation and assumptions and review, or extend to the present case, some known properties of discrete subgroups of finite covolume. In § 3 we show that Margulis' superrigidity implies local rigidity. Once this is done, H C Wang's argument may be used without change (§ 4).

2. Preliminaries

2.1. *Notation and assumptions*

S is a finite set. For each $s\in S$, there is given a local field k_s of characteristic zero and almost absolutely simple isotropic k_s-group $\mathscr{G}_s$. We view $G_s = \mathscr{G}_s(k_s)$ as a locally compact topological group, using the topology of k_s, and let G be the product of the G_s. For $T \subset S$, let pr_T be the projection of G onto $G_T = \prod_{s \in T} G_s$.

Let S_∞ (resp. S_f) be the set of s for which k_s is archimedean (resp. non-archimedean), $G_\infty = G_{S_\infty}$, $G_f = G_{S_f}$ and pr_∞ (resp. pr_f) be pr_T for $T = S_\infty$ (resp. S_f).

Let V be the set of places of $\mathbb{Q}$. It is the union of the set of primes and of the infinite place ∞. For $v \in V$, let $\mathbb{Q}_v$ be the completion of $\mathbb{Q}$ at v. We denote by S_v the set of $s \in S$ for which k_s contains $\mathbb{Q}_v$. We also write pr_v for pr_T when $T = S_v$. For $T \subset S_v$, let

$$R\mathscr{G}_T = \prod_{s \in T} R_{k_s/\mathbb{Q}_v} \mathscr{G}_S RG_T = \prod_{s \in T} R_{k_s/\mathbb{Q}_v} \mathscr{G}_s(\mathbb{Q}_v), \quad (T \subset S_v), \tag{1}$$

where $R_{k_s/\mathbb{Q}_v}$ refers to Weil's restriction of scalars [16:I]. The group $R\mathscr{G}_T$ is an algebraic $\mathbb{Q}_v$-group, and $RG_T = R\mathscr{G}_T(\mathbb{Q}_v)$ is in a natural way an analytic group over $\mathbb{Q}_v$. By [16:1·3·2], there is a canonical isomorphism of topological groups

$$G_T = RG_T. \tag{2}$$

We let r_s denote the k_s-rank of $\mathscr{G}_s$ and $r(G)$ the sum of the r_s ($s \in S$). By assumption $r_s \geqslant 1$ for all s.

2.2. A discrete subgroup L of G is "irreducible" if there is no partition $S = A \cup B$ into non-empty subsets such that $(L \cap G_A)(L \cap G_B)$ is of finite index in G.

2.3. *Lemma*

Let L be an irreducible subgroup of finite covolume of G.

(a) *For every $s \in S$, $\mathrm{pr}_s L$ is Zariski dense in $\mathscr{G}_s$.*
(b) *Let $T \subset S_v$. Then $\mathrm{pr}_T L$ is Zariski dense in $R\mathscr{G}_T$. Assume $T \neq S_v$ and v finite. Then the closure H of $\mathrm{pr}_T L$ contains the product of the groups G_s^+ ($s \in T$). In particular, it is open of finite index.*

[G_s^+ denotes the subgroup of G_s generated by the k_s-rational points of the connected unipotent k_s-subgroups of $\mathscr{G}_s$, cf. [3:§6].]

(a) follows from [9:1.12].
(b) The group $\mathrm{pr}_T L$ has the property (S) (cf. [1], [12:5.1]). Therefore its image on the right-hand side of (2) in § 2·1 is Zariski-dense by [1] if $v = \infty$, by [14] if v is finite. To prove the second assertion we consider first the case where $T \neq S_v$ is reduced to one place t. By [5:III, § 2, no. 2], H is a Lie subgroup of RG_v. If it were not open, then its Lie algebra would be a proper subspace of the Lie algebra of RG_t, which would be invariant under $\mathrm{pr}_T L$, in contradiction with (a). Therefore H is open. But $H \times \prod_{s \neq t} G_s$ is open in G, of finite covolume. Hence H has finite index in G_t. In particular, it is not compact. Then it contains G_t^+ by a theorem of Tits stated in [3:9.10], for which a proof is given in [11].

To go from this case to the general one, the argument is the same as that of Proposition 4.2 in [9]. We repeat it for the sake of completeness.

Let U be a compact open subgroup of $G_{T-\{t\}}$ and $H_U = H \cap (G_t \times U)$. Since the projection of $G_t \times U$ onto G_t is proper, H_U projects onto a closed subgroup H' of $G_{t'}$ in which $\mathrm{pr}_t L$ is obviously dense. Therefore H' contains G_t^+ and $G_t^+ \times U \subset H_U$. This being true for every U, we see that $G_t^+ = \cap_U (G_t^+ \times U) \subset H$.

2.4. PROPOSITION

Let L be a discrete subgroup of finite covolume of G. Then L is finitely presented.

If a subgroup of finite index of L is finitely presented, then L is itself finitely presented, as is well-known. Therefore we may assume L to be irreducible.

If $S = S_\infty$, the result is known: If one group G_s has $\mathbb{R}$-rank one, this is 13·20 in [12]. If not, then L is arithmetic [8], in which case this is proved e.g. in [2]. Let now $S \neq S_\infty$ and X_f the product of the Bruhat-Tits buildings of the groups G_s for $s \in S_f$. The group L operates on S_f. We claim that X_f is a finite polyhedron mod L. In fact either $S = S_v$, in which case L is discrete and cocompact, or $S \neq S_f$ and then $\mathrm{pr}_f L$ is dense in an open subgroup of finite index (2.3). In both cases our assertion is clear. The isotropy group K_σ in G_f of a face σ of X_f is a compact open subgroup. The isotropy group of σ in L is $L \cap (G_\infty \times K_\sigma)$. Its projection on G_∞ is discrete of finite covolume. By the above, it is finitely presented. Then L is finitely presented by Theorem 4 in [6].

Our next remark is a trivial extension to our situation of a result of Kazhdan–Margulis.

2.5. PROPOSITION

(a) *Let $S = S_f$. Then G has a compact open neighbourhood of 1 which meets any discrete subgroup at the identity only.*
(b) *The group G has a neighbourhood of the identity U such that if L is discrete in G, then a conjugate of L meets U only at 1. In particular $c(L)$ has a strictly positive minimum.*

In case (a) we just have to take for U a torsion-free compact open subgroup, which always exists.

If $G = G_\infty$, then our assertion is a well-known result of Kazhdan–Margulis [7; 12:XI]. Let U_∞ be such a neighbourhood in G_∞ and U_f a torsion-free compact open subgroup of G_f. Then $U = U_\infty \times U_f$ satisfies our condition.

2.6. *Lemma*

Let k be a local field, $\mathscr{H}$ an almost absolutely simple k-group, (σ, E) an absolutely irreducible k-representation of $\mathscr{H}$ and L a finitely generated Zariski dense subgroup of $\mathscr{H}(k)$. Assume that the set of traces $\mathrm{Tr}\,\sigma(x)$ $(x \in L)$ is bounded. Then $\mathscr{H}(k)$ is relatively compact in $GL(E)(k)$.

Let $\bar{k}$ be an algebraic closure of k. By Burnside's theorem, every $A \in \mathrm{End}\,E(\bar{k})$, is a linear combination of elements of $\sigma(H(\bar{k}))$. In turn, those are linear combinations of elements of L, since the latter is assumed to be Zariski-dense. Let then $\{t_i\}$ $(i = 1, \ldots, m = \dim^2 E)$ be a basis of End E consisting of elements of L, and let $\{u_j\}$ be the dual basis with respect to the trace form. We can write

$$u_j = \sum_i d_{ij} t_i, \quad (d_{ij} \in k).$$

We let $\mathscr{H}(k)$ operate on End E via σ by left multiplication. We may identify $\sigma(\mathscr{H}(k))$ with the orbit of the identity. It suffices therefore to see that the representation of L on End (E), defined by left multiplication, has bounded coefficients with respect to the basis

(u_j). This is easy: Let

$$h \cdot u_j = \sum h_{ji} u_i, \quad (h \in L).$$

Multiplying on the right by t_i and taking traces, we get

$$h_{ji} = \mathrm{tr}\,(h \cdot u_j \cdot t_i) = \sum_a d_{aj}\,\mathrm{tr}\,(h \cdot t_a \cdot t_j).$$

Hence $|h_{ji}|$ has a universal bound in view of our assumptions.
Note. This argument just copies a known one of E B Vinberg.

3. Local rigidity

3.1. We fix an irreducible discrete subgroup L of finite covolume of G. Let $\mathbf{a} = (a_1, \ldots, a_m)$ be a generating set for L and w_k $(k \in K)$ be a finite set of defining relations for L (2.4). As usual, the set $R(L, G)$ of homomorphisms of L into G is identified with the set of m-tuples $\mathbf{g} = (g_1, \ldots, g_m) \in G^m$ which satisfy the relations $w_k(g_1, \ldots, g_m) = 0$ $(k \in K)$. It is a closed subset of G^m, invariant under the group Aut G of automorphisms of G, acting componentwise in particular under the group Int G of inner automorphisms, where $x \in G$ acts by $\mathbf{g} \mapsto (xg_1x^{-1}, \ldots, xg_mx^{-1})$. The group L is said to be *locally rigid* if Int $G(\mathbf{a})$ contains a neighbourhood of $\mathbf{a}$ in $R(L, G)$.

3.2 Theorem. *Let L be as in 3.1 and assume that $r(G) \geqslant 2$. Then L is locally rigid.*

For $\mathbf{y} = (y_i) \in R(L, G)$, the map $a_i \mapsto y_i$ $(i = 1, \ldots, m)$ extends to a homomorphism $\alpha_{\mathbf{y}}$ of L onto the subgroup $L_{\mathbf{y}}$ generated by the components y_i of $\mathbf{y}$.

(a) Let $s \in S$. We claim first that for $\mathbf{y} \in R(L, G)$ sufficiently close to $\mathbf{a}$, the group $\mathrm{pr}_s L_{\mathbf{y}}$ is Zariski-dense in $\mathscr{G}_s$ and not relatively compact in G_s.

Assume the first assertion to be false. There exists then a sequence $\mathbf{y}_i \to \mathbf{a}$ in $R(L, G)$ and, for each i, a proper subalgebra $\mathfrak{q}_i$ of the Lie algebra $\mathfrak{g}_s$ of $\mathscr{G}_s$ such that

$$\mathrm{Ad}\, y_{ij}(\mathfrak{q}_i) = \mathfrak{q}_i \quad (j = 1, \ldots, m).$$

Passing to a subsequence, we may assume the $\mathfrak{q}_i$ to have a constant dimension, say d, and then, again going over to a subsequence, we may assume that the $\mathfrak{q}_i(k_s)$ converge to a subspace $\mathfrak{q}$ in the Grassmannian of d-planes in $\mathfrak{g}_s(k_s)$. By continuity, Ad a_j leaves $\mathfrak{q}$ invariant for all j's, hence also for all $x \in L$, and, by (2.3) for all $x \in G_s$. But $\mathfrak{g}_s(k_s)$ is simple, whence a contradiction.

Write L_i for $L_{\mathbf{y}}$ when $\mathbf{y} = \mathbf{y}_i$. If now the $\mathrm{pr}_s L_i$ were all relatively compact, the traces of the elements of these groups in some linear realization of $\mathscr{G}_s$ would be uniformly bounded and so would be those of the elements of $\mathrm{pr}_s L$. By 2.6, $\mathrm{pr}_s L$ would be relatively compact. This however, would contradict the fact that G_s is not compact and $\mathrm{pr}_s L$ is dense in an open subgroup of G_s if $S \neq \{s\}$ (2·3), or has finite covolume if $S = \{s\}$.

(b) Let $v \in V$ and $T \subset S_v$. We assert now that for $\mathbf{y} \in R(L, G)$ sufficiently close to $\mathbf{a}$, the group $\mathrm{pr}_T L_{\mathbf{y}}$ is Zariski dense in $\mathscr{G}_T$.

If T consists of one place, this is (a). Assume then that T has at least two elements.

Let $\mathscr{M}_y$ be the Zariski closure of $\mathrm{pr}_T L_y$ and m_y the Lie algebra of RM_y. In view of (a), it maps on to the Lie algebra of RG_t for any $t \in T$. Therefore, for any $x \in L_y$ the characteristic polynomial $C(\lambda, \mathrm{Ad}\,\mathrm{pr}_t(x) | L(RG_t))$ divides the characteristic polynomial $C(\lambda, \mathrm{Ad}\,\mathrm{pr}_T(x) | \mathscr{M}_y)$.

Assume (b) to be false. The previous argument, carried out in RG_T, shows the existence of a sequence $y_i \in R(L, G)$ tending to $\mathbf{a}$, such that the m_{y_i} have a constant dimension and tend to a proper subspace $\mathfrak{q}$ of the Lie algebra $L(RG_T)$ of RG_T. For every $x \in L$, the elements $\alpha_{y_i}(x)$ tend to x. Since $\mathrm{pr}_T L$ is Zariski-dense in $\mathscr{G}_T$ (2.3), it follows that $\mathfrak{q}$ is an ideal of $L(RG_T)$, proper by our assumption. It is therefore equal to the Lie algebra of $RG_{T'}$ for some $T' \subset T$, with $T' \neq \phi, T$. Fix $t \in T$. We have seen that $C(\lambda, \mathrm{pr}_t(\alpha_{y_i}(x)) | L(RG_t))$ divides $C(\lambda, \mathrm{pr}_T(\alpha_{y_i}(x) | \mathscr{M}_y))$. By continuity, it follows that for all $x \in L$, $C(\lambda, \mathrm{Ad}\,\mathrm{pr}_t(x) | L(RG_t))$ divides $C(\lambda, \mathrm{Ad}\,\mathrm{pr}_T(x) | L(G_{T'}))$. By Zariski-density, this should then be true for any $x \in G_T$. But, for $t \in T$, $t \notin T'$, this is absurd. This contradiction proves (b).

(c) Let $v \in V$ be finite and $T \subset S_v$. We now claim that for $y \in R(L, G)$ sufficiently close to $\mathbf{a}$, the group $\mathrm{pr}_T L_y$ contains an open subgroup of finite index of RG_T.

We know that $\mathrm{pr}_T L$ is dense in an open subgroup of finite index of RG_T (2.3). We claim that $\mathrm{pr}_T L_y$ is not discrete if y is close enough to $\mathbf{a}$. If it were discrete, then there would be a sequence $y_i \to \mathbf{a}$, such that the closure of the union of the L_{y_i} would contain $\mathrm{pr}_T L$, hence, by (2.3), an open subgroup of finite index of RG_T, but this contradicts 2.5(a). Combined with (b), this shows that $\mathrm{pr}_T L_y$ is dense in an open subgroup H of RG_T for y close to $\mathbf{a}$. Let $t \in T$ and $T' = T - \{t\}$. The kernel N of the restriction of $\mathrm{pr}_{T'}$ to H is then open in G_t. Since G_t^+ is simple modulo center [12], it follows that $G_t^+ \subset N$. Therefore, H contains the product of the G_t^+ $(t \in T)$. Since G_t^+ has finite index in G_t, our assertion follows.

(d) We assume here that all $\mathscr{G}_s$ are of adjoint type, hence absolutely simple. We fix $y \in R(L, G)$ close enough to $\mathbf{a}$ so that (b) and (c) hold for L_y. We write L' and α for L_y and α_y. We want to prove that α extends to an automorphism of G.

Fix s. By [8], there exist $t = t(s) \in S$, a continuous homomorphism $\mu_s : k_t \to k_s$ and a k_s-morphism $v_s : {}^{\mu_s}\mathscr{G}_t \to \mathscr{G}_s$ such that the composition

$$\sigma_s : G \xrightarrow{\mathrm{pr}_t} G_t \xrightarrow{\mu_s^0} {}^{\mu_s}\mathscr{G}_t(\mu_s(k_t)) \xrightarrow{j} {}^{\mu_s}\mathscr{G}_t(k_s) \xrightarrow{v_s(k_s)} G_s \tag{3}$$

extends $\mathrm{pr}_s o\alpha : L \to \mathrm{pr}_s L'$.

We claim that in fact μ_s is an isomorphism. The morphism v_s is non-trivial, hence is a k_s-isomorphism, since both groups are absolutely k_s-simple. Therefore $v_s(k_s)$ is an isomorphism of topological groups. The third arrow in (3) is just induced by the inclusion $\mu_s(k_t) \to k_s$. If it were not surjective, then $\mathrm{Im}\, j$ would be a closed subgroup H if infinite index, not discrete, not relatively compact and $\mathrm{pr}_s L'$ would be contained in such a subgroup. The Lie algebra of H in $L(RG_s)$ would be proper, contradicting the Zariski density of $\mathrm{pr}_s(L')$. Therefore, σ_s is a continuous and open surjective homomorphism of G onto G_s, which extends $\mathrm{pr}_s o\alpha$, and whose kernel is the product of the factors G_u with $u \neq t$. Let now σ be the product of the σ_s. It is a continuous homomorphism of G into G which extends α. We want to prove that σ is an automorphism.

Now that j is the identity in k_s we see that $v_s(k_s) o \mu_s^0$ is an isomorphism of $G_{t(s)}$ onto G_s. It suffices therefore to show that $s \mapsto t(s)$ is a permutation of S, or, equivalently, that no G_s is contained in the kernel of σ.

Of course, $s \mapsto t(s)$ leaves each S_v stable. It is therefore equivalent to show that σ induces an automorphism of G_{S_v} onto itself for every $v \in V$, and for this that no G_s belongs to the kernel of $\sigma | G_v$. Assume to the contrary that $\sigma(G_s) = 1$ for some $s \in S_v$. Then Card $S_v \geqslant 2$. Let $T' = S_v - \{s\}$ and $T = S_v$. The homomorphism σ induces a continuous homomorphism of $G_{T'}$ into G_T hence also of $RG_{T'}$ into RG_T. These groups are analytic over $\mathbb{Q}_{v'}$ hence σ is analytic [5:III, § 8]. The image is then a proper Lie subgroup H, whose projection on each factor RG_s is equal to $RG_s (s \in T')$. Its Lie algebra $\mathfrak{h}$ maps onto the Lie algebra of RG_s under $\mathrm{pr}_s (s \in T')$. On the other hand, $\mathfrak{h}$ should be invariant under $\mathrm{pr}_{T'} L'$, hence under Zariski dense subgroup (see (b)), and should therefore be a proper ideal. But this contradicts the previous surjectivity assertion.

(e) The automorphism σ belongs to the group $A(G)$ of automorphisms of G which are compositions of permutations of factors, field isomorphisms and restrictions to rational points of morphisms of algebraic groups of the various factors. [These are in fact all automorphisms of G, but we need not know that.] In this group the product B of the groups $\mathrm{Aut}\, \mathscr{G}_s(k_s)$ is open of finite index. Therefore $\mathrm{Int}\, G = \Pi_{s \in S} \mathrm{Int}\, G_s$ is also open of finite index in $A(G)$.

(f) We can now prove the theorem. Let $\mathscr{G}'_s = \mathrm{Ad}\, \mathscr{G}_s$ and let $\pi_s : \mathscr{G}_s \to \mathscr{G}'_s$ be the canonical isogeny $(s \in S)$. The morphism $\pi_s(k_s) : G_s \to G'_s$ has finite kernel, hence is proper, and its image is open of finite index [3:3.19], therefore the same is true for the product $\pi : G \to G'$ of the $\pi_s(k_s)$. In view of (e), $\pi(G)$ is also open of finite index in $A(G')$. In fact, π is a local homeomorphism.

Since π is proper and with image open of finite index, $L' = \pi(L)$ is discrete, of finite covolume in G'. With the notation and conventions of 3.1, let $\mathbf{a}' \in R(L', G')$ be the point with components $\mathbf{a}'_i = \pi(a_i)$ $(i = 1, \ldots, m)$ in G'^m. By (d), $A(G')(\mathbf{a}')$ contains a neighbourhood of $\mathbf{a}'$ in $R(L, G')$. By homogeneity, $A(G')(\mathbf{a}')$ is open in $R(L', G')$. Since all spaces under consideration are locally compact, countable at infinity, the orbit map $\beta : x \mapsto x \cdot \mathbf{a}'$ induces a homeomorphism of $A(G')/H$, where H is the isotropy group of $\mathbf{a}'$, onto $A(G')(\mathbf{a}')$ [4:VII, App. 1]. In particular, for every neighbourhood U of 1 in $A(G')$, $\beta(U)$ is a neighbourhood of $\mathbf{a}'$ in $R(L', G')$.

The group $\pi(G)$, identified to a subgroup of $A(G')$ via the inclusion of $G' = \mathrm{Int}\, G'$ into $A(G')$, is also open of finite index. If we let it act on $R(L', G')$ via this homomorphism, it follows that $g \mapsto g \cdot \mathbf{a}'$ is an open map.

Fix now an open neighbourhood V of $\mathbf{a}$ in G^m which is mapped homeomorphically onto a neighbourhood V' of $\mathbf{a}'$ by π. Let U be a neighbourhood of 1 in G such that $\mathrm{Int}\, g \cdot \mathbf{a} \subset V$ for $g \in U$. If now $\mathrm{y} \in R(L, G)$ is such that $\pi(\mathrm{y}) \in U \cdot \mathbf{a}'$, then there exists $g \in U$ such that $g \cdot \mathbf{a} = \pi(\mathrm{y})$. It follows that $g \cdot \mathbf{a}$ is an element of V which maps onto $\pi(\mathrm{y})$, hence $g \cdot \mathbf{a} = \mathrm{y}$. Therefore L is locally rigid.

4. Covolumes

4.1. The group G is locally compact, therefore the space of closed subgroups of G, endowed with the topology defined in [4:VIII, § 5] is compact. For any neighbourhood U of the identity, the subspace N_U of discrete subgroups which meet U only at 1 form a compact subset [4:VIII, § 5, no. 3, Prop. 7]. Moreover, the function $L \mapsto c(L)$ is lower semi-continuous [4:VIII, § 5, no. 2, Prop. 4]. In particular if a sequence of elements L_i of N_U tends to L and $c(L_i) \leqslant c$ for all i, then $c(L) \leqslant c$. We recall that $L_i \mapsto L$ if and only if the following condition is fulfilled (*loc. cit.*, no. 6): For any compact set $C \subset G$ and any neighbourhood U of 1 in G, we have

$$L_i \cap C \subset L \cdot U \text{ and } L \cap C \subset L_i \cdot U \text{ for } i \text{ big enough.} \qquad (4)$$

4.2 Theorem. *Fix a constant $c > 0$. Assume $r(G) \geqslant 2$. Then the discrete subgroups of G with covolume $c(L) \leqslant c$ form finitely many conjugacy classes.*

In view of 2.3, 2.4, 3.2 and 4.1, the argument of H C Wang in the real case [15] goes over without change. For the sake of completeness, we describe it briefly.

Assume 4.2 to be false. Then we can find an infinite sequence L_i of nonconjugate discrete subgroups with covolume $\leqslant c$. Passing to a subsequence, we may assume that $c(L_i)$ has a limit $b \leqslant c$. Replacing L_i by a conjugate, if necessary, we may assume that $L_i \cap U = 1$, where U is a suitable neighbourhood of 1 in G (2.5). Then a cofinal subsequence of the L_i's has a limit L and $c(L) \leqslant b$ (4.1). We now consider the setup of 3.1. Let $a_1, \ldots, a_m$ be a generating set for L and w_k $(k \in K)$ a finite defining set of relations (2.4). By 4.1, we can find $x_{ij} \in L_i$ such that

$$\lim_{i \to \infty} x_{ij} = a_j \quad (j = 1, \ldots, m).$$

For i big enough, we have then $w_k(x_{il}, \ldots, x_{im}) \in U \cap L_i$, hence $w_k(x_{il}, \ldots, x_{im}) = 1$ $(k \in K)$. Therefore the map $a_j \mapsto x_{ij}$ $(j = 1, \ldots, m)$ extends to a homomorphism of L onto the subgroup L_i' of L_i generated by the x_{ij}'s, and $x_i = (x_{il}, \ldots, x_{im})$ is a point of $R(L, G)$, which comes arbitrarily close to **a** if i is big enough. For such i's, L_i' is by 3.2 conjugate to L under an inner automorphism. In particular $c(L_i') = c(L)$, and L_i' has finite index in L_i. Then $c(L_i')/c(L_i)$ is an integer $\geqslant 1$. On the other hand, it is equal to $c(L)/c(L_i)$, hence tends to $c(L)/b$, which is $\leqslant 1$. Altogether we get $c(L_i') = c(L_i) = c(L)$, hence $L_i = L_i'$ and L_i is conjugate to L The L_i's are therefore pairwise conjugate for i big enough, whence a contradiction.

Remarks

(1) In the real case, the result of [15] is also valid if $r(G) = 1$, provided that G is not locally isomorphic to $SL_2(\mathbb{R})$ or $SL_2(\mathbb{C})$. [In fact, H C Wang ruled only $SL_2(\mathbb{R})$ out, but this was an oversight.] In the p-adic case, this is however false, since a torsion-free discrete subgroup of finite covolume of a simple p-adic group of relative rank one is cocompact and free.

(2) The previous argument also proves the following statement: Let L be a discrete finitely generated subgroup of G which is locally rigid, and $\{L_i\}$ a sequence of discrete subgroups of G which tend to L. If the L_i's are cocompact, then L is also cocompact.

In fact, the above proof shows that L is conjugate to a subgroup of L_i for i big enough.

Of the real groups G under consideration here, without restriction on the $\mathbb{R}$-rank, only groups locally isomorphic to $SL_2(\mathbb{R})$ or $SL_2(\mathbb{C})$ have discrete subgroups of finite covolume which are not locally rigid. On the other hand, they do contain sequences of discrete cocompact subgroups converging to a discrete non-cocompact subgroup of finite covolume: for $SL_2(\mathbb{C})$, such sequences are obtained by Dehn surgery. In $SL_2(\mathbb{R})$, such examples are easy to obtain geometrically: For instance we may consider a sequence of triangle groups (or rather of their subgroups of index two consisting of conformal transformations) with signatures $\pi/a, \pi/b_n, \pi/b_n$ where $a, b_n \in \mathbb{N}, a > 2$, and $b_n \to \infty$. The limit will be the subgroup of conformal transformations in the triangle group $(\pi/a, 0, 0)$. Therefore, for real groups, there is also a converse to the previous statement, so that the failure of local rigidity is necessary and sufficient for the existence of a sequence of discrete cocompact subgroups whose limit is not cocompact.

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