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# On some generalizations of Ramanujan's continued fraction identities

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**Abstract.** In this note we establish continued fraction developments for the ratios of the basic hypergeometric function ${}_2\phi_1(a,b;c;x)$ with several of its contiguous functions. We thus generalize and give a unified approach to establishing several continued fraction identities including those of Srinivasa Ramanujan.

**Keywords.** $q$-continued fractions; Rogers-Ramanujan continued fractions; basic hypergeometric series.

## 1. Introduction

In [6] Bhargava and Adiga established equivalents of the following two continued fractions I and II which contain as special cases several continued fraction identities stated in the 'lost' notebook of Srinivasa Ramanujan [14] and proved earlier by different methods by Andrews [2], [3], Bhargava and Adiga [5], Hirschhorn [10], [11] and Ramanathan [12].

$$
{}_2\phi_1(a,b;c;xq)/{}_2\phi_1(a,b;c;x) = h(a,b,c,xq)(1-x)/h(a,b,c,x)
$$

$$
= \frac{1}{1+}\,\frac{x(1-a)(1-b)}{1-x+}\,\frac{abxq-c}{1+\cdots}\,\frac{x(1-aq^n)(1-bq^n)}{1-x+}\,\frac{abxq^{2n+1}-cq^n}{1+\cdots} \tag{I}
$$

$$
= \frac{1}{1+}\,\frac{x(1-a)(1-b)}{D_1+}\,\frac{N_2}{D_2+\cdots}\,\frac{N_n}{D_n+\cdots} \tag{II}
$$

where

$$
D_n = q^{n-1}\{1+ax+bx-cq^{n-1}\} - x\left(1+\frac{1}{q}\right),\quad n=1,2,3,\ldots,
$$

$$
N_{n+1} = \frac{x}{q}(q^n-x)(1-aq^n)(1-bq^n),\quad n=1,2,3,\ldots
$$

and

$$
{}_2\phi_1(a,b;c;x) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(q)_n(c)_n}x^n,\quad |q|<1,\quad |x|<1 \tag{1}
$$

$$
h(a,b,c,x) = (c)_\infty(x)_\infty\,{}_2\phi_1(a,b;c;x) \tag{2}
$$

with

$$(c)_\infty = \prod_{n=0}^{\infty}(1-cq^n), \quad (c)_k = \frac{(c)_\infty}{(cq^k)_\infty} \quad (k\text{: integer}).$$

The purpose of this paper (§3) is to establish in a simple, unified way several more continued fraction expansions of the ratios ${}_2\phi_1(a,bq;c;x)/{}_2\phi_1(a,b;c;x)$, ${}_2\phi_1(a,b;c;xq)/{}_2\phi_1(a,b;c;x)$, ${}_2\phi_1(aq,b;c;x)/{}_2\phi_1(a,b;c;x)$, ${}_2\phi_1(a,b;cq;x)/{}_2\phi_1(a,b;c;x)$, ${}_2\phi_1(a,b;cq;xq)/{}_2\phi_1(a,b;c;x)$, ${}_2\phi_1(aq,b;c;x/q)/{}_2\phi_1(a,b;c;x)$, ${}_2\phi_1(a,bq;c;x/q)/{}_2\phi_1$ $(a,b;c;x)$, ${}_2\phi_1(a,bq;cq;x)/{}_2\phi_1(a,b;c;x)$. ${}_2\phi_1(aq,b;cq;x)/{}_2\phi_1(a,b;c;x)$. We also obtain some identities of Ramanujan [4], [13], [14] and Hirshhorn [10], [11] as special cases. Denis [7] has recently given an alternative proof of I on the lines of Andrews [2].

In §2 we obtain some easily verified functional relations satisfied by our auxiliary function $h$ given by (2). These functional relations at once yield our continued fractions. Thus, the use of $h$ considerably simplifies and brings about unity in our proofs. We note that (2) immediately gives the symmetry

$$h(a,b,c,x) = h(b,a,c,x), \quad |q|<1, \quad |x|<1. \tag{3}$$

The well-known E Heine's transformation satisfied by ${}_2\phi_1(a,b;c;x)$ takes the form

$$h(abx/c,b,bx,c/b) = h(a,b,c,x), \quad |q|<1, |c/b|<1, |x|<1. \tag{4}$$

Further, changing

$$a \text{ to } -\lambda/a, \quad b \text{ to } c, \quad c \text{ to } -bq \quad \text{and} \quad x \text{ to } -(aq/c), \tag{5}$$

the function $h$ is easily seen to be the same as the function $g(a,\lambda,b,c,q)$ treated in [6] in proving I and II. We indicate some special cases in §4.

## 2. Some functional relations satisfied by $h$

*Lemma 2.1. The function h given by (2) satisfies*

$$h(a,b,c,x) - h(aq,b,c,x) = ax(b-1)h(aq,bq,cq,x), \tag{6}$$

$$h(a,b,c,x) - h(a,b,cq,x) = c(x-1)h(a,b,cq,xq), \tag{7}$$

$$\begin{aligned} h(a,b,c,x) - h(a,b,c,xq) &= x(c-a-b)h(a,b,cq,xq) \\ &\quad + abx(1-xq)h(a,b,cq,xq^2). \end{aligned} \tag{8}$$

*Proof.* The identity (6) is equivalent to (11) given in [6]. However, it is easy to see that (6) follows easily from the definition (2) of $h$ and since

$$(a)_n - (aq)_n = \begin{cases} 0 & , \quad n=0, \\ -a(aq)_{n-1}(1-q^n), & n>0. \end{cases}$$

For proving (7) we observe that the left side of (7), on using (2) and separating $n=0$

terms from other terms, becomes

$$(x)_\infty(cq)_\infty\left[-c+\sum_{n=1}^{\infty}\frac{(a)_n(b)_n x^n}{(q)_n(cq)_n}(1-cq^n-1)\right]$$

$$=-c(x)_\infty(cq)_\infty\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(q)_n(cq)_n}(xq)^n$$

which equals the right side of (7). Now, the left side of (8) can be written as

$$(c)_\infty(xq)_\infty\left[(1-x)\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(q)_n}x^n-\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(q)_n}x^nq^n\right]$$

$$=(c)_\infty(xq)_\infty\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(q)_n}x^n-\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(q)_n}x^{n+1}-\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(q)_n}x^nq^n\right]$$

$$=(c)_\infty(xq)_\infty\left[\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(q)_{n-1}}x^n-\sum_{n=1}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(q)_{n-1}}x^n\right].$$

On combining the first and third summations,

$$=x(cq)_\infty(xq)_\infty\sum_{n=1}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(cq)_{n-1}(q)_{n-1}}x^{n-1}X$$

where $X=(-aq^{n-1}-bq^{n-1}+cq^{n-1}+abq^{2n-2})$, which, on splitting into four summations, is easily seen to equal the right side of (8).

We need the following additional functional relations.

*Lemma 2.2. The function h given by (2) satisfies*

$$h(a,b,c,x)-(1-x)h(a/q,b,c,xq)=x(1-b)h(a,bq,cq,x), \tag{9}$$

$$h(a,b,c,x)-h(a/q,bq,c,x)=x((a/q)-b)h(a,bq,cq,x), \tag{10}$$

$$h(a,b,c,x)-h(a,bq,cq,(x/q))=((ax/q)-c)h(a,bq,cq,x), \tag{11}$$

$$h(a,b,c,x)-h(aq,b,c,(x/q))=(x/q)(b-c)h(aq,b,cq,x). \tag{12}$$

*Proof*. Identities (9) to (11) are equivalent to identities (9), (10) and (12) (respectively) given in [6]. However, their proofs are easy. Equations (9) and (10) follow directly from (2) on using respectively,

$$(a)_n-q^n(a/q)_n=\begin{cases}0 & , \quad n=0\\ (a)_{n-1}(1-q^n), & n>0,\end{cases}$$

$$(a)_n(b)_n-(a/q)_n(bq)_n=\begin{cases}0 & , \quad n=0\\ (a)_{n-1}(bq)_{n-1}\left(\dfrac{a}{q}-b\right)(1-q^n), & n>0.\end{cases}$$

Identity (11) follows immediately from (10) on using the Heine's transformation (4);

then changing $abx/c$ to $A$, $b$ to $B$, $bx$ to $C$ and $c/b$ to $X$; and finally renaming $A$, $B$, $C$ and $X$ as $a$, $b$, $c$ and $x$ respectively.

Identity (12) follows from (2) on using the two easily verified identities

$$(a)_n - (aq)_n q^{-n} = \begin{cases} 0 & , \quad n=0, \\ -q^{-n}(aq)_{n-1}(1-q^n), & n>0, \end{cases}$$

$$\frac{(b)_n}{(c)_n} - \frac{(b)_{n-1}}{(c)_{n-1}} = \frac{q^{n-1}(b)_{n-1}(c-b)}{(c)_n}, \quad n>0.$$

## 3. Continued fraction developments

As shown in [6], I and II follow from the functional relations (13) to (15) given in Lemmas 3.1 and 3.2 below. We only indicate here how these Lemmas follow from (6), (9) and (11).

*Lemma 3.1.*

$$h(a,b,c,x) = (1-x)h(a,b,c,xq) + x(1-a)(1-b)h(aq,bq,cq,x), \tag{13}$$

$$h(a,b,c,x) = h(aq,bq,cq,(x/q)) + (abx-c)h(aq,bq,cq,x). \tag{14}$$

*Proof.* Change $a$ to $aq$ in (9) and add the resulting equation to (6) to get (13). Change $a$ to $aq$ in (11) and add to (6) to get (14).

**Theorem 3.1.** *Identity* I *of* § 1 *holds.*

*Proof.* See [6]

*Lemma 3.2.*

$$(1-x)h((a/q),b,c,xq) = \left[(1-c+bx) + \frac{x}{q}(a-1-q)\right]h(a,bq,cq,x) + \frac{x}{q}(1-a)(1-bq)h(aq,bq^2,cq^2,(x/q)). \tag{15}$$

*Proof.* Change $a$ to $aq$, $b$ to $bq$, $c$ to $cq$ and $x$ to $(x/q)$ in (9), change $b$ to $bq$, $c$ to $cq$ and $x$ to $(x/q)$ in (6), take the negative of (9) and add the resulting identities to (11) to get (15).

**Theorem 3.2.** *Identity* II *of* § 1 *holds.*

*Proof.* See [6].

*Lemma 3.3.*

$$h(a,bq,cq,(x/q)) = (1+c-bx-(ax/q))h(a,bq,cq,x) + (1-x)(abx-c)h(a,bq,cq,xq). \tag{16}$$

*Proof.* Change $a$ to $(a/q)$, $x$ to $xq$ in (6) and multiply the result by $(1-x)$; change $x$ to $xq$ in (11) and multiply the result by $(1-x)$; take the negative of (11); add these three equations to (9) to get (16).

**Theorem 3.3.**

$$\frac{{}_2\phi_1(a,b;c;xq)}{{}_2\phi_1(a,b;c;x)}=\frac{(1-x)h(a,b,c,xq)}{h(a,b,c,x)}=\frac{(1-x)}{K_0+}\,\frac{L_1}{K_1+\cdots}\,\frac{L_n}{K_n+\cdots} \tag{III}$$

*where*

$$K_n=1+\frac{c}{q}-(a+b)xq^n,\qquad n=0,1,2,\ldots$$

$$L_n=\left(abxq^{n-1}-\frac{c}{q}\right)(1-xq^n),\quad n=1,2,\ldots.$$

*Proof.* Changing $b$ to $(b/q)$, $c$ to $(c/q)$, $x$ to $xq^{n+1}$, (16) can be written as

$$W_n\equiv\frac{h(a,b,c,xq^n)}{h(a,b,c,xq^{n+1})}=K_n+\frac{L_{n+1}}{W_{n+1}}.$$

Iterating this with $n=0,1,2,\ldots$ and taking the reciprocal and multiplying by $(1-x)$ we have III.

*Lemma 3.4.*

$$\left(1-\frac{c}{bq}\right)h(a,b,c,x)=\left[\left(1-bx+\frac{c}{q}\right)+b^{-1}q^{-2}(abxq-cq-c)\right]h(a,bq,c,x)-b^{-1}q^{-2}(-c+abxq)(1-bq)h(a,bq^2,c,x). \tag{17}$$

*Proof.* Change $a$ to $aq$ in (15), apply (4) to each $h$ appearing in the resulting equation, put $A=(abx/c)$, $B=b$, $C=bx$, $X=(c/b)$ and finally replace $A$, $B$, $C$ and $X$ by $(a/q)$, $b$, $(c/q)$ and $x$ to get (17).

**Theorem 3.4.**

$$\frac{{}_2\phi_1(a,bq,c,x)}{{}_2\phi_1(a,b,c,x)}=\frac{h(a,bq,c,x)}{h(a,b,c,x)}=\frac{1-\dfrac{c}{bq}}{B_0+}\,\frac{A_1}{B_1+\cdots}\,\frac{A_n}{B_n+\cdots} \tag{III$'$}$$

*where*

$$A_{n+1}=b^{-2}q^{-3}(abxq^{n+1}-c)(1-bq^{n+1})(c-bq^{n+2}),\qquad n=0,1,2,\ldots,$$

$$B_n=q^{n-1}(q-bxq^{n+1}+c)+b^{-1}q^{-2}(abxq^{n+1}-cq-c),\quad n=0,1,2,\ldots$$

*Proof*. Change $b$ to $bq^n$ in (17) and multiply the resulting identity by $q^n$ to get

$$T_n \equiv \frac{\left(q^n - \frac{c}{bq}\right)h(a, bq^n, c, x)}{h(a, bq^{n+1}, c, x)} = B_n + \frac{A_{n+1}}{T_{n+1}}, \quad n = 0, 1, \ldots,$$

where, $A_{n+1}$ and $B_n$ are as above. Iterating the last equation with $n = 0, 1, 2, \ldots$ and taking the reciprocal and using (2), we have III′.

*Lemma 3.5.*

$$h(a, b, c, x) = [1 - x(a + b - c)]h(a, b, c, xq) + x(1 - xq)[ab - c(a + b - c)]h(a, b, cq, xq^2), \tag{18}$$

$$h(a, b, c, xq) = [1 - xq(a + b) + c(xq^2 + xq - 1)]h(a, b, cq, xq^2) + xq(1 - xq^2)[ab - cq(a + b - cq)]h(a, b, cq^2, xq^3). \tag{19}$$

*Proof*. Changing $x$ to $xq$ in (7) and using the result to eliminate $h(a, b, cq, xq)$ in (8) we have (18). Changing $c$ to $cq$ and $x$ to $xq$ in (18), changing $x$ to $xq$ in (7) and adding the results we get (19).

**Theorem 3.5.**

$$\frac{{}_2\phi_1(a, b; c; xq)}{{}_2\phi_1(a, b; c; x)} = \frac{h(a, b, c, xq)(1 - x)}{h(a, b, c, x)}$$

$$= \frac{1 - x}{1 - x(a + b - c) +} \; \frac{E_1}{F_1 + \cdots} \; \frac{E_n}{F_n + \cdots}, \tag{IV}$$

*where,*

$$E_{n+1} = xq^n(1 - xq^{n+1})(a - cq^n)(b - cq^n), \qquad n = 0, 1, \ldots$$

$$F_n = 1 - xq^n(a + b) + cq^{n-1}(xq^{n+1} + xq^n - 1), \quad n = 1, 2, \ldots.$$

*Proof*. Changing $c$ to $cq^{n-1}$ and $x$ to $xq^{n-1}$, (19) can be written as

$$L_n \equiv \frac{h(a, b, cq^{n-1}, xq^n)}{h(a, b, cq^n, xq^{n+1})} = F_n + \frac{E_{n+1}}{L_{n+1}}, \quad n = 1, 2, \ldots, \tag{20}$$

where $E_n$ and $F_n$ are as above. Further (18) can be written as

$$\frac{h(a, b, c, xq)}{h(a, b, c, x)} = \frac{1}{1 - x(a + b - c) +} \; \frac{E_1}{\dfrac{h(a, b, c, xq)}{h(a, b, cq, xq^2)}}. \tag{21}$$

Iterating (20) with $n = 1, 2, \ldots$, using (21) and (2) we have IV.

*Lemma 3.6.*

$$h(a,bq,cq,(x/q)) = (1-bx)h(a,bq,cq,x) + \frac{x}{q}(a-cq)(bq-1)h(a,bq^2,cq^2,x), \tag{22}$$

$$h(a,b,c,xq) = (1-c+ax-bxq)h(a,bq,cq,xq) + x(a-cq)(bq-1)h(a,bq^2,cq^2,xq). \tag{23}$$

*Proof.* Change $b$ to $bq$ and $c$ to $cq$ in (9) and multiply the resulting equation by $(c-(ax/q))$; change $a$ to $(a/q)$, $b$ to $bq$ and $c$ to $cq$ in (6) and multiply the resulting equation by $(1-x)$; change $a$ to $(a/q)$ and $x$ to $xq$ in (11) and multiply the result by $(1-x)$; take the negative of (11); and add all these equations to (9) to get (22). Change $x$ to $xq$ in (22); change $x$ to $xq$ in (11) and add to get (23).

**Theorem 3.6.**

$$\frac{{}_2\phi_1(a,b;c;xq)}{{}_2\phi_1(a,b;c;x)} = \frac{h(a,b,c,xq)(1-x)}{h(a,b,c,x)} = \frac{1-x}{1-bx+}\,\frac{P_1}{Q_1+}\cdots\frac{P_n}{Q_n+}\cdots, \tag{IV'}$$

*where*

$$P_n = x(a-cq^{n-1})(bq^{n-1}-1), \quad n=1,2,\ldots,$$

$$Q_n = 1+ax-q^{n-1}(bxq+c), \quad n=1,2,\ldots.$$

*Proof.* Changing $b$ to $(b/q)$, $c$ to $(c/q)$ and $x$ to $xq$ (22) can be rewritten as

$$\frac{h(a,b,c,xq)}{h(a,b,c,x)} = \frac{1}{(1-bx)+}\,\frac{x(a-c)(b-1)}{\dfrac{h(a,b,c,xq)}{h(a,bq,cq,xq)}}. \tag{24}$$

Changing $b$ to $bq^{n-1}$, $c$ to $cq^{n-1}$, (23) can be rewritten as

$$Y_n \equiv \frac{h(a,bq^{n-1},cq^{n-1},xq)}{h(a,bq^n,cq^n,xq)} = Q_n + \frac{P_{n+1}}{Y_{n+1}}$$

where $P_n$ and $Q_n$ are as in IV′. Iterating the last equation with $n=1,2,\ldots$ and using (24) we have IV′.

The iterate of $Y_n$ with $n=1,2,3,\ldots$ is also equivalent to Lemma 1 of § 12 of [1].

**Theorem 3.7.**

$$\frac{{}_2\phi_1(aq,b;c;x)}{{}_2\phi_1(a,b;c;x)} = \frac{h(aq,b,c,x)}{h(a,b,c,x)} = \frac{1-\dfrac{c}{aq}}{B'_0+}\,\frac{A'_1}{B'_1+}\cdots\frac{A'_n}{B'_n+}\cdots \tag{V}$$

*where $A'_n$ and $B'_n$ are obtained from $A_n$ and $B_n$ of* III′ *on interchanging $a$ and $b$.*

*Proof.* Interchanging $a$ and $b$ in III′ and then applying (3) to the numerator and to the denominator of the left side of the resulting identity we have V.

**Theorem 3.8.** *Let the continued fractions in the right hand sides of* I, II, III, IV, *and* IV′ *be denoted by* $C_1(a,b,c,x)$, $C_2(a,b,c,x)$, $C_3(a,b,c,x)$, $C_4(a,b,c,x)$ *and* $C_5(a,b,c,x)$ *respectively. Then,*

$$\frac{{}_2\phi_1(a,b;cq,x)}{{}_2\phi_1(a,b;c;x)}=\frac{h(a,b,cq,x)(1-c)}{h(a,b,c,x)}=\frac{1-c}{1-c\,C_i(a,b,cq,x)},\quad i=1,2,3,4,5. \tag{VI–X}$$

*Proof.* Identity (7) can be written as

$$\frac{h(a,b,cq,x)}{h(a,b,c,x)}=\frac{1}{1+}\;\frac{-c}{\dfrac{h(a,b,cq,x)}{(1-x)h(a,b,cq,xq)}}. \tag{25}$$

Changing $c$ to $cq$ in I, II, III, IV and IV′ and using the resulting identities in (25) gives the result.

**Theorem 3.9.**

$$\frac{{}_2\phi_1(a,b;cq;xq)}{{}_2\phi_1(a,b;c;x)}=\frac{h(a,b,cq,xq)(1-c)(1-x)}{h(a,b,c,x)}=\frac{(1-c)(1-x)}{F'_1+}\;\frac{E'_2}{F'_2+\cdots}\;\frac{E'_n}{F'_n+\cdots} \tag{XI}$$

*where* $E'_n$ *and* $F'_n$ *are obtained from* $E_n$ *and* $F_n$ *of* IV *by changing* $x$ *to* $x/q$.

*Proof.* Change $x$ to $x/q$ in (20) and iterate the resulting identity with $n=1,2,\ldots$ and take the reciprocal.

**Theorem 3.10.**

$$\frac{{}_2\phi_1\left(aq,b;c;\dfrac{x}{q}\right)}{{}_2\phi_1(a,b;c;x)}=\frac{h\left(aq,b,c,\dfrac{x}{q}\right)}{\left(1-\dfrac{x}{q}\right)h(a,b,c,x)}=\frac{1}{1-\dfrac{x}{q}+}\;\frac{\dfrac{x}{q}(b-c)\left(1-\dfrac{x}{q}\right)}{F''_1+}\;\frac{E''_2}{F''_2+\cdots}\;\frac{E''_n}{F''_n+\cdots} \tag{XII}$$

*where* $E''_n$ *and* $F''_n$ *are obtained from* $E_n$ *and* $F_n$ *of* IV *by changing* $a$ *to* $aq$ *and* $x$ *to* $x/q^2$.

*Proof.* Identity (12) can be written as

$$\frac{h\left(aq,b,c,\frac{x}{q}\right)}{h(a,b,c,x)}=\frac{1}{1+}\,\frac{\frac{x}{q}(b-c)}{\frac{h\left(aq,b,c,\frac{x}{q}\right)}{h(aq,b,cq,x)}}. \tag{26}$$

Change $a$ to $aq$ and $x$ to $x/q^2$ in (20), iterate the resulting equation with $n=1,2,\ldots$ and use (26) to get XII.

**Theorem 3.11.**

$$\frac{{}_2\phi_1\left(a,bq;c;\frac{x}{q}\right)}{{}_2\phi_1(a,b;c;x)}=\frac{h\left(a,bq,c,\frac{x}{q}\right)}{\left(1-\frac{x}{q}\right)h(a,b,c,x)}$$

$$=\frac{1}{1-\frac{x}{q}+}\,\frac{\frac{x}{q}(a-c)\left(1-\frac{x}{q}\right)}{\tilde{F}_1+}\,\frac{\tilde{E}_2}{\tilde{F}_2+\cdots}\,\frac{\tilde{E}_n}{\tilde{F}_n+\cdots} \tag{XIII}$$

*where the* $\tilde{E}_n$ *and* $\tilde{F}_n$ *are obtained from* $E''_n$ *and* $F''_n$ *of* XII *by interchanging* $a$ *and* $b$.

*Proof.* Interchange $a$ and $b$ in XII and use (3).

Theorem 3.12 is the $q$-analogue of Entry 20 of Chapter 12 of Ramanujan's second notebook [13], [4]. It is also the same as the following well known Heine's continued fraction [9]. We only indicate here in Lemma 3.7 how the required functional relations follow from (9) and (12).

*Lemma 3.7.*

$$h(a,b,c,x)-(1-c)h(a,bq,cq,x)$$
$$=x(c-b)(1-a)h(aq,bq,cq^2,x), \tag{27}$$
$$h(a,b,c,x)-(1-c)h(aq,b,cq,x)$$
$$=x(c-a)(1-b)h(aq,bq,cq^2,x). \tag{28}$$

*Proof.* It is enough to prove (27) since (28) follows from (27) on using the symmetry relation (3). Changing $b$ to $bq$ and $c$ to $cq$ in (12) the right side of (27) can be written as $(a-1)\{h(a,bq,cq,x)-h(aq,bq,cq,x/q)\}$. So, to prove (27) it is enough to prove

$$h(a,b,c,x)-(a-c)h(a,bq,cq,x)=(1-a)h(aq,bq,cq,x/q).$$

This on using (12) and the symmetry relation (3) is equal to

$$h\left(a, bq, c, \frac{x}{q}\right) - \left(1 - \frac{x}{q}\right) h(a, b, c, x) = \frac{x}{q}(1-a) h\left(aq, bq, cq, \frac{x}{q}\right).$$

Changing $b$ to $b/q$ and $x$ to $xq$ and using the symmetry relation (3) the last equation is equivalent to (9).

**Theorem 3.12.**

$$\frac{{}_2\phi_1(a, bq; cq; x)}{{}_2\phi_1(a, b; c; x)} = \frac{h(a, bq, cq, x)(1-c)}{h(a, b, c, x)}$$

$$= \frac{(1-c)}{(1-c)+} \frac{J_0}{(1-cq)+} \frac{R_0}{(1-cq^2)+\cdots}$$

$$\cdots \frac{J_n}{(1-cq^{2n+1})+} \frac{R_n}{(1+cq^{2n+2})+\cdots}, \qquad \text{(XIV)}$$

*where*

$$J_n = x(cq^n - b)(1 - aq^n)q^n, \quad n = 0, 1, 2, \ldots$$

*and*

$$R_n = x(cq^{n+1} - a)(1 - bq^{n+1})q^n, \quad n = -1, 0, 1, 2 \ldots$$

**Theorem 3.13.**

$$\frac{{}_2\phi_1(aq, b; cq; x)}{{}_2\phi_1(a, b; c; x)} = \frac{h(aq, b, cq, x)(1-c)}{h(a, b, c, x)} \qquad \text{(XV)}$$

$$= \frac{(1-c)}{(1-c)+} \frac{qR_{-1}}{(1-cq)+} \frac{q^{-1}J_1}{(1-cq^2)+\cdots}$$

$$\cdots \frac{qR_{n-1}}{(1+cq^{2n+1})+} \frac{q^{-1}J_{n+1}}{(1-cq^{2n+2})+\cdots}$$

*Proof.* Interchange $a$ and $b$ in XIV and use (3).

## 4. Some special cases and further deductions from (6) to (12)

Using the substitutions (5) in our identities and then letting $c$ to $\infty$ we get a number of identities of the Rogers–Ramanujan type [14]. In particular I, II, III and III′ and IV and IV′ respectively give the following identities treated in [2], [5]:

$$\frac{\sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}[n(n+1)]}(-\lambda/a)_n (aq)^n}{(q)_n(-bq)_n}}{\sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}[n(n+1)]}(-\lambda/a)_n a^n}{(q)_n(-bq)_n}}$$

$$=\frac{1}{1+}\,\frac{aq+\lambda q}{1+}\,\frac{bq+\lambda q^2}{1+\cdots} \tag{I$'_R$}$$

$$=\frac{1}{1+}\,\frac{aq+\lambda q}{1-aq+bq+}\,\frac{aq+\lambda q^2}{1-aq+bq^2+\cdots} \tag{II$'$}$$

$$=\frac{1}{1-b+aq+}\,\frac{b+\lambda q}{1-b+aq^2+}\,\frac{b+\lambda q^2}{1-b+aq^3+\cdots} \tag{III$'_H$}$$

$$=\frac{1}{1+aq+}\,\frac{\lambda q-abq^2}{1+aq^2+bq+}\,\frac{\lambda q^2-abq^4}{1+aq^3+bq^2+\cdots}. \tag{IV$'_R$}$$

Suffix $R$ in $I'_R$ and $\mathrm{IV}'_R$ signifies that the identities are due to Ramanujan [14]. Suffix $H$ signifies that the identity is due to Hirschhorn [10].

On making the substitutions (5) in V to XII and XIV and letting $c$ to $\infty$ gives respectively

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{\frac{1}{2}[n(n+1)]}(-\lambda q/a)_n a^n}{(q)_n(-bq)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{\frac{1}{2}[n(n+1)]}(-\lambda/a)_n a^n}{(q)_n(-bq)_n}}=\frac{1-\dfrac{ab}{\lambda}}{(1-b)-\dfrac{a}{\lambda q}(\lambda q+bq+b)+}$$

$$\frac{\dfrac{1}{\lambda^2 q}(\lambda q+a)(\lambda q+b)(\lambda q-ab)}{q(1-b)-\dfrac{a}{\lambda q}(\lambda q^2+bq+b)+\cdots}\,\frac{\dfrac{1}{\lambda^2 q}(\lambda q^n+a)(\lambda q^n+b)(\lambda q^n-ab)}{q^n(1-b)-\dfrac{a}{\lambda q}(\lambda q^{n+1}+bq+b)+\cdots} \tag{V$'$}$$

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{\frac{1}{2}[n(n+1)]}(-\lambda/a)_n a^n}{(q)_n(-bq^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{\frac{1}{2}[n(n+1)]}(-\lambda/a)_n a^n}{(q)_n(-bq)_n}}$$

$$=\frac{1+bq}{1+}\,\frac{bq}{1+}\,\frac{aq+\lambda q}{1+}\,\frac{bq^2+\lambda q^2}{1+\cdots}\,\frac{aq^{n+1}+\lambda q^{2n+1}}{1+}\,\frac{bq^{n+2}+\lambda q^{2n+2}}{1+\cdots} \tag{VI$'$}$$

$$=\frac{1+bq}{1+}\,\frac{bq}{1+}\,\frac{aq+\lambda q}{(1-aq+bq^2)+\cdots}\,\frac{aq^{2n}+\lambda q^{3n}}{q^n(1-aq+bq^{n+2})+\cdots} \tag{VII$'$}$$

$$=\frac{1+bq}{1+}\,\frac{bq}{1-bq+aq+}\,\frac{bq+\lambda q}{1-bq+aq^2+\cdots}\,\frac{bq+\lambda q^n}{1-bq+aq^{n+1}+\cdots} \tag{VIII$'$}$$

$$=\frac{1+bq}{1+}\,\frac{bq}{1+aq+}\,\frac{\lambda q-abq^3}{1+aq^2+bq^2+\cdots}\,\frac{\lambda q^n-abq^{2n+1}}{1+aq^{n+1}+bq^{n+1}+\cdots} \tag{IX$'$ and X$'$}$$

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{\frac{1}{2}[n(n+1)]}(-\lambda/a)_n(aq)^n}{(q)_n(-bq^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{\frac{1}{2}[n(n+1)]}(-\lambda/a)_n a^n}{(q)_n(-bq)_n}}=\frac{1+bq}{1+aq+bq+}\frac{\lambda q-abq^3}{1+aq^2+bq^2+\cdots}$$

$$\cdots\frac{\lambda q^{n-1}-abq^{2n+1}}{1+aq^n+bq^n+\cdots} \tag{XI'}$$

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{\frac{1}{2}[n(n+1)]}(-\lambda q/a)_n(a/q)^n}{(q)_n(-bq)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{\frac{1}{2}[n(n+1)]}(-\lambda/a)_n a^n}{(q)_n(-bq)_n}}=\frac{1}{1+}\frac{-a}{1+a+bq+}\frac{\lambda q-abq^2}{1+aq+bq^2+\cdots}$$

$$\cdots\frac{\lambda q^{n-1}-abq^{2n-2}}{1+aq^{n-1}+bq^n+\cdots} \tag{XII'}$$

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{\frac{1}{2}[n(n+1)]}(-\lambda/a)_n(aq)^n}{(q)_n(-bq^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{\frac{1}{2}[n(n+1)]}(-\lambda/a)_n a^n}{(q)_n(-bq)_n}}=\frac{1+bq}{1+bq+}\frac{aq+\lambda q}{1+bq^2+}\frac{\lambda q^2-abq^4}{1+bq^3+\cdots}$$

$$\cdots\frac{aq^{n+1}+\lambda q^{2n+1}}{1+bq^{2n+2}+}\frac{\lambda q^{2n+2}-abq^{3n+4}}{1+bq^{2n+3}+\cdots} \tag{XIV'}$$

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{\frac{1}{2}[n(n+1)]}(-\lambda q/a)_n a^n}{(q)_n(-bq^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{\frac{1}{2}[n(+1)]}(-\lambda/a)_n a^n}{(q)_n(-bq)_n}}=\frac{1+bq}{1+bq+}\frac{\lambda q-abq^2}{1+bq^2+}\frac{aq+\lambda q^2}{1+bq^3+\cdots}$$

$$\cdots\frac{\lambda q^{2n+1}-abq^{3n+2}}{1+bq^{2n+2}+}\frac{aq^{n+1}+\lambda q^{2n+2}}{1+bq^{2n+3}+\cdots}. \tag{XV'}$$

Theorem 1.2 yields the $q$-analogue of Entry 21 of Chapter 12 of Ramanujan's second notebook [4, eq. 21.1]. We only state the analogue and omit the proof.

$$\frac{(1-b)x}{(c)_\infty(-x)_\infty}h(q,bq,cq,-x)$$

$$=\frac{(1-b)x}{1-c+}\frac{(1-c)(1-bq)x}{1-cq+}\frac{(bq-cq)\left(\dfrac{c}{q}-cq\right)x}{(1-cq^2)\dfrac{c}{q}+\cdots}$$

$$\cdots\frac{(bq^{n+1}-cq^{2n+1})(cq^{n-1}-cq^{2n+1})x}{(1-cq^{2n+2})cq^{n-1}+}\frac{(1-cq^n)(1-bq^{n+1})xcq^{2n+1}}{(1-cq^{2n+3})+\cdots}.$$

In [8] Frank has obtained expansions of the type

$$1+\frac{xd_1}{xf_1+1+}\cdots\frac{xd_n}{xf_n+1+}\cdots$$

(or its reciprocal) for the ratios, mainly, of each of ${}_2\phi_1(a,bq;cq;x)$, ${}_2\phi_1(aq,b;c;x)$, ${}_2\phi_1(a,b;cq;x)$, ${}_2\phi_1(aq,bq;cq;x)$, ${}_2\phi_1(aq,bq;cq;x/q)$, ${}_2\phi_1(a,b;c;xq)$ with ${}_2\phi_1(a,b;c;x)$. The functional relations needed for obtaining these expansions are apart from (3),

$$h(a,b,c,x)-h(aq,b,c,x)=ax(b-1)h(aq,bq,cq,x)$$

$$h(a,b,c,x)-(1-c)h(a,bq,cq,x)=x(c-b)(1-a)h(aq,bq,cq^2,x)$$

$$h(a,bq,c,x)-h(aq,b,c,x)=x(b-a)h(aq,bq,cq,x)$$

and

$$h(a,b,c,x)-(1-c)h(a,b,cq,x)=cx(1-a)(1-b)h(aq,bq,cq^2,x).$$

The first of these is the same as (6). The other three can be shown to follow from (6)–(12). In fact the second relation is the same as (27).

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