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On the proof of the reciprocity law for arithmetic Siegel modular functions*

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Abstract. Earlier we obtained a new proof of Shimura's reciprocity law for the special values of arithmetic Hilbert modular functions. In this note we show how from this result one may derive Shimura's reciprocity law for special values of arithmetic Siegel modular functions. To achieve this we use Shimura's classification of the special points of the Siegel space, Satake's classification of the equivariant holomorphic imbeddings of Hilbert–Siegel modular spaces into a larger Siegel space, and, finally, a corrected version of some of Karel's results giving an action of the Galois group $\mathrm{Gal}(\mathbf{Q}_{ab}/\mathbf{Q})$ on arithmetic Siegel modular forms.

Keywords. Reciprocity law; arithmetic Siegel modular functions; Siegel space.

1. Introduction

In [5] we obtained a new proof of Shimura's reciprocity law [14] for the special values of certain arithmetic Hilbert modular functions. In this note we show how from this result one may derive the reciprocity law for special values of arithmetic Siegel modular functions as formulated in [16]. To achieve this we use Shimura's results classifying the special points of the (adelized) Siegel space, Satake's results classifying the equivariant holomorphic imbeddings of a product of Hilbert–Siegel modular spaces into a larger Siegel space, and, finally, a corrected version [7] of Karel's results [6] giving an action of the Galois group $\mathrm{Gal}(\mathbf{Q}_{ab}/\mathbf{Q})$ on arithmetic Siegel modular forms which commutes with translation in the finite adele group of the group of symplectic similitudes.

We begin by formulating the reciprocity law for special values of Hilbert modular functions as stated in [5]. Let k be a finite totally real extension of $\mathbf{Q}$ of degree n and K be a purely imaginary quadratic (CM-) extension of k. The group $GL_{2+}(k)$ of two-by-two non-singular matrices over k having totally positive determinants acts on the product $\mathfrak{H}^n$ of n copies of the upper half complex plane $\mathfrak{H}$. Let $\Sigma = (\sigma_1, \ldots, \sigma_n)$ be the set of the n real imbeddings of k with some fixed ordering and $\tilde{\Sigma} = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n) = \tilde{\Sigma}(\tau)$ be an ordered extension of these to n complex imbeddings of K such that for some non-real element τ of K we have $\mathrm{Im}(\tilde{\sigma}(\tau)) > 0$ for every $\tilde{\sigma} \in \tilde{\Sigma}$. Let $\Xi = \Xi_{\mathbf{A}}$ be the set of special points of $G_+(\mathbf{A})$ or of the pro-algebraic variety $V_\infty = G_+(\mathbf{Q})\backslash G_+(\mathbf{A})/\mathbf{Z}^- \mathbf{K}_\infty$, where G is the algebraic group $R_{k/\mathbf{Q}} GL_2$, $\mathbf{Z}^-$ is the closure of the group $Z(\mathbf{Q})_f$ in $G(\mathbf{A}_f)$, and $\mathbf{K}_\infty$ is the stabilizer in $G_+(\mathbf{R})$ of some point of $\mathfrak{H}^n$. There is a natural action

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$(x,\xi)\to x*\xi$, $\xi\in\Xi$, $x\in\mathbf{I}(K)$, of the ideles $\mathbf{I}(K)$ of K on Ξ which is described in § 2 of [4]. Suppose that $(\tau)=(\tilde{\sigma}_1(\tau),\ldots,\tilde{\sigma}_n(\tau))\in\mathfrak{H}^n$ is the "infinite" component of ξ. Let $(K^*(\tilde{\Sigma}),\tilde{\Sigma}^*)$ be the CM-dual of $(K,\tilde{\Sigma})$ and $N_{\tilde{\Sigma}^*}$ be the type norm; then $N_{\tilde{\Sigma}^*}$ maps $\mathbf{I}(K^*(\tilde{\Sigma}))$ into $\mathbf{I}(K)$. One defines the notion of arithmetic Hilbert modular function as in [3]. The main result of [5] is then

Theorem. *Let Φ be an arithmetic Hilbert modular function in the* **Q**-*structure on* V_∞. *Then in the above notation, one has the relation*

$$\Phi(N_{\tilde{\Sigma}^*}(a)*\xi)=\mathfrak{a}(a)\cdot\Phi(\xi),\ a\in\mathbf{I}(K^*(\tilde{\Sigma})), \tag{1}$$

where $\mathfrak{a}$ *is the Artin symbol,* $\mathfrak{a}(a)=[a,K^*(\Sigma)]$.

This result, due originally to Shimura [14], was proved in [5] by "elementary" methods. In the present article, we shall sketch how to use this result to prove rather directly the reciprocity law for arithmetic Siegel modular functions, which we shall state later on, and which is essentially that obtained in [16; §§ 2.4–5].

2. Classification of special points on the Siegel space

Denote by $\mathfrak{S}_n$ the Siegel upper half space of degree n consisting of all n by n complex symmetric matrices $Z=X+iY$ such that Y is positive definite, $Y\gg 0$. Let J_n denote the $2n$ by $2n$ skew-symmetric matrix

$$\begin{bmatrix}0 & 1_n\\ -1_n & 0\end{bmatrix}.$$

The group $Gp(n,\mathbf{R})$ of symplectic similitudes, defined by

$$Gp(n,\mathbf{R})=\{X\in M_{2n}(\mathbf{R})\mid {}^tX\cdot J_n\cdot X=\nu(X)\cdot J_n,\ \nu(X)\in\mathbf{R}-\{0\}\}$$

contains the normal subgroup $Gp^+(n,\mathbf{R})$ of index two of all $X\in Gp(n,\mathbf{R})$ for which $\nu(X)>0$, and the latter group acts transitively on $\mathfrak{S}_n$ in a well known manner by linear fractional transformations, while if $\nu(X)<0$, then X interchanges $\mathfrak{S}_n$ and $\mathfrak{S}_n^-=\{-Z\mid Z\in\mathfrak{S}_n\}$.

A point $\xi\in\mathfrak{S}_n$ is called a special point if it is the unique isolated fixed point in $\mathfrak{S}_n$ of some element $X\in Gp^+(n,\mathbf{Q})$. Given ξ, let $S=S(\xi)$ be the group of all $g\in Gp(n,\mathbf{Q})$ such that $g\cdot\xi=\xi$, and let $Y=Y(\xi)$ be the **Q**-subalgebra of $M_{2n}(\mathbf{Q})$ generated by S. The classification of special points on $\mathfrak{S}_n$ has been carried out by Shimura in [15] by studying the structure of the algebra $Y=Y(\xi)$ for special points ξ. Let $B=M_2(\mathbf{Q})$; then we may identify $M_{2n}(\mathbf{Q})$ with $M_n(B)$ in a certain way and define an involution ι on B by

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{\iota}=\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$$

which extends to an involution ρ on $M_n(B)$: $X^\rho={}^tX^\iota$. Then $Y=Y(\xi)$ is stable under ρ which is a positive involution on Y. Therefore, Y is a semisimple algebra, hence a direct sum of simple algebras

$$Y = Y_1 \oplus \cdots \oplus Y_t,$$

each Y_j is stable under ρ, and, as we shall see, is a total matric algebra over some CM-field K_j, with maximal totally real subfield denoted by k_j. $Z = K_1 \oplus \cdots \oplus K_t$ is the centre of Y, and the fixed algebra of ρ in Z is $Z_0 = k_1 \oplus \cdots \oplus k_t$. Moreover ξ is the unique isolated fixed point of the group

$$Z_1 = \{z \in Z \mid zz^\rho = \mathbf{1}_{2n}\} \subset Z^\times.$$

Let $m_j = [k_j : \mathbf{Q}]$ and $q_j^2 = \dim_{K_j} Y_j$. In the present case, the totally real field F of [15] is just $\mathbf{Q}$ and the quaternion algebra B over F is $M_2(\mathbf{Q})$. Hence, each algebra Y_j, which has to be in the same Brauer class over K_j as $B \otimes_{\mathbf{Q}} K_j$ [15:§4.7], is actually a total matric algebra over K_j and $Y_j \cong M_{q_j}(K_j)$. We have inclusions

$$K_j \longrightarrow M_2(k_j) \longrightarrow M_{2m_j}(\mathbf{Q})$$

coming from the regular representation of K_j as an algebra over k_j and of k_j as a $\mathbf{Q}$-algebra; thus we also have

$$Y_j = M_{q_j}(K_j) \longrightarrow M_{2n_j}(\mathbf{Q}), \quad n_j = m_j q_j,$$

and $n = n_1 + \cdots + n_t$. Moreover, the centralizer $\mathfrak{C}(\xi)$ of Z_0 in M_{2n} (as an algebraic matric algebra defined over $\mathbf{Q}$) is the direct sum over j of the centralizer $\mathscr{Z}_j$ of k_j in M_{2n_j}, $j = 1, \ldots, t$. The centralizer of k_j in M_{2n_j} (as an algebraic total matric algebra over $\mathbf{Q}$) is just $R_{k_j/\mathbf{Q}} M_{2q_j}$, so that

$$\mathscr{Z}_j(\mathbf{Q}) \cong M_{2q_j}(k_j).$$

Then the centralizer of Z_0 in Sp_n is the direct product

$$\prod_{j=1}^{t} R_{k_j/\mathbf{Q}} Sp_{q_j} \longrightarrow Sp_n; \tag{2}$$

and if we let $G_j = R_{k_j/\mathbf{Q}} Sp_{q_j}$, $K_{j1}{}^\times = \{\alpha \in K_j \mid \alpha\bar{\alpha} = 1\}$, then we have a diagram

$$Z_1 = K_{11}{}^\times \times \cdots \times K_{t1}{}^\times \longrightarrow G_1(\mathbf{R}) \times \cdots \times G_t(\mathbf{R}), \tag{3}$$

and the centralizer of $K_{j1}{}^\times$ in $G_j(\mathbf{R})$ is a maximal compact subgroup of $G_j(\mathbf{R})$ having a unique fixed point ξ_j on $(\mathfrak{S}_{q_j})^{m_j}$ on which $G_j(\mathbf{R})$ acts (and on which the Hilbert–Siegel modular group [1, 10] of degree q_j for k_j acts discontinuously). Thus we have another diagram

$$\begin{array}{ccccccc} Z_1 \longrightarrow G_1(\mathbf{R}) & \times \cdots \times & G_t(\mathbf{R}) & \xrightarrow{\varphi} & Sp(n, \mathbf{R}) \\ \downarrow & & \downarrow & & \downarrow \\ (\xi_1, \ldots, \xi_t) \in (\mathfrak{S}_{q_1})^{m_1} & \times \cdots \times & (\mathfrak{S}_{q_t})^{m_t} & \xrightarrow{f} & \mathfrak{S}_n \in \xi, \end{array} \tag{4}$$

$$f(\xi_1, \xi_2, \ldots, \xi_t) = \xi,$$

so that the special point $\xi \in \mathfrak{S}_n$ is the image, under an equivariant holomorphic morphism f, of the special point

$$(\xi_1, \ldots, \xi_t) \in (\mathfrak{S}_{q_1})^{m_1} \times \cdots \times (\mathfrak{S}_{q_t})^{m_t}.$$

That equivariant holomorphic morphism comes from a rational homomorphism

$$\varphi: G_1 \times \cdots \times G_t \longrightarrow Sp_n$$

of algebraic groups, all defined over **Q**. It is clear from Satake's papers [11, 12] that φ satisfies not only the condition (H_1), but also (H_2) and (H_2'). In particular, the extension of f carries the 0-dimensional (point) rational boundary components of the product

$$\prod_j (\mathfrak{S}_{q_j})^{m_j}$$

into 0-dimensional rational boundary components of $\mathfrak{S}_n$.

Next one needs to consider the special point ξ_j on the factor $(\mathfrak{S}_{q_j})^{m_j}$ and show that it in turn is the image of a special point on a product of spaces of "Hilbert modular type" as considered in [5]. For this, we temporarily fix j and drop it as a subscript, and let K be a CM-field with maximal totally real subfield k of degree m over **Q**, and for a positive integer q, form the group $G'' = R_{k/\mathbf{Q}} Sp_q$, so that $G''_+(\mathbf{R})$ acts on a product $\mathfrak{S}_q^m$, and any subgroup of it commensurable with the Hilbert–Siegel modular group of degree q over k[1, 10] acts discontinuously on this product. Let (temporarily in this special case) $Y = M_q(K)$ and $G = R_{k/\mathbf{Q}} Sp_q$.

Let $K_1^{\times} = \{\alpha \in K \mid \alpha\bar{\alpha} = 1\}$, $Y = Y(\xi) \cong M_q(K)$ provided with the positive involution δ defined by $y^{\delta} = {}^t y^{\iota}$ in the notation of [15, §4]; then $Y_1 = \{y \in Y \mid y y^{\delta} = 1_q\}$ is the group of **Q**-rational points of the maximal compact subgroup of $G(\mathbf{R})$ fixing $\xi = (\xi^{(1)}, \ldots, \xi^{(m)}) \in \mathfrak{S}_q^m$. We have

$$K_1^{\times} \longrightarrow Y_1 \longrightarrow (R_{k/\mathbf{Q}} Sp_q)(\mathbf{Q}), \tag{5}$$

and $\xi^{(i)}$ is the unique fixed point of $K_1^{\times}$ acting thus on the ith factor of the Cartesian power $\mathfrak{S}_q^m$, $i = 1, \ldots, m$. In other words, for each $i = 1, \ldots, m$ we have imbeddings

$$K_1^{\times} \xrightarrow{\theta_i} Sp(q, k^{\sigma_i}), \quad i = 1, \ldots, m,$$

where $\sigma_1, \ldots, \sigma_m$ are the (real) imbeddings of k in **C**. Thus $\xi^{(i)}$ is the fixed point of $\theta_i(K_1^{\times})$ on the ith factor. Now we let $k_i = k^{\sigma_i}$ and then we may view $Sp(q, k_i)$ as the symplectic group $Sp(V_i, A_i)$, where V_i is a vector space of dimension $2q$ defined over k_i and A_i is a non-degenerate, skew-symmetric bilinear form on V_i all defined over k_i; and all of this is just the ith conjugate of $Sp(V, A)$, where V is a vector space defined over k, $\dim_k V = 2q$, etc., and supplied with an imbedding $\theta: K_1^{\times} \to Sp(q, k)$.

The character group $X_*(K_1^{\times})$ has a generator α, and the action of $\theta(K_1^{\times})$ on $\bar{V}$ has just two eigencharacters, α and $\bar{\alpha}$. Let V_{α} and $V_{\bar{\alpha}}$ be the two corresponding eigenspaces in $V_{\mathbf{C}}$. It is easy to verify that each of these is totally isotropic with respect to A. Let $v_1, \ldots, v_q$ be a basis of V_{α} consisting of elements rational over K, and $\bar{v}_1, \ldots, \bar{v}_q$ be the (complex) conjugate basis of $V_{\bar{\alpha}}$. Then for each $\nu = 1, \ldots, q$, the two-dimensional subspace $W_{\nu} = \mathbf{C} v_{\nu} + \mathbf{C} \bar{v}_{\nu}$ is stable under complex conjugation and is a (non-degenerate) hyperbolic plane with a k-structure spanned by the two k-rational vectors

$w_\nu = v_\nu + \bar{v}_\nu$ and $w_\nu' = (-D)^{-1/2}(v_\nu - \bar{v}_\nu)$ (where $D \in k$ is totally positive such that $K = k(\sqrt{-D})$). Thus, $W_\nu = \mathbf{R}w_\nu + \mathbf{R}w_\nu'$ is a real hyperbolic plane with respect to the restriction of A and W_ν is defined over k. Choosing $v_1, \dots, v_q$ in sequence, we may assume that v_ν is orthogonal to the complex conjugates of $v_{\nu'}$ for $\nu' \neq \nu$ for all ν, and hence that the hyperbolic planes W_ν are mutually orthogonal with respect to the skew-symmetric form A. In this way we obtain an *orthogonal* decomposition

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_q$$

of V into mutually orthogonal hyperbolic planes defined over k; hence we have inclusions of k-groups

$$K_1^\times \longrightarrow SL(W_1) \times \cdots \times SL(W_q) \longrightarrow Sp(V, A),$$

where the first imbedding on the left comes from the facts that $W_{\nu\mathbf{C}}$ is spanned by eigenvectors of $K_1^\times$ and W_ν is rational over k and where the second homomorphism comes from an isometry of symplectic spaces of $W_1 \oplus W_2 \oplus \cdots \oplus W_q$ onto V. By application of the groundfield reduction functor $R_{k/\mathbf{Q}}$, we further obtain a sequence of imbeddings

$$K_1^\times \longrightarrow R_{k/\mathbf{Q}}SL_2 \times \cdots \times R_{k/\mathbf{Q}}SL_2 \longrightarrow R_{k/\mathbf{Q}}Sp_q.$$

Now let $G_1 = R_{k/\mathbf{Q}}SL_2$, $G_2 = R_{k/\mathbf{Q}}Sp_q$. We obtain thus a diagram

$$\begin{array}{ccccc} K_1^\times \longrightarrow & G_1(\mathbf{R}) \times \cdots \times & G_1(\mathbf{R}) & \longrightarrow & G_2(\mathbf{R}) \\ & \downarrow & \downarrow & & \downarrow \\ & \mathfrak{H}^m \times \cdots \times & \mathfrak{H}^m & \longrightarrow & \mathfrak{S}_q^m, \quad \mathfrak{H} = \mathfrak{S}_1, \end{array}$$

such that under the mapping of the lower line, the special point $(\xi(1), \dots, \xi(q))$ of the product on the left is mapped to the special point $\xi \in \mathfrak{S}_q^m$, where $\xi(\nu)$ is a special point of the ν-th factor with respect to an imbedding of $K_1^\times$ in $G_1(\mathbf{R})$, G_1 carries a $\mathbf{Q}$-structure such that all the Hilbert modular subgroups of $SL_2(k)$ are contained in $G_1(\mathbf{Q})$ and such that $K_1^\times \to G_1(\mathbf{Q})$ is an imbedding of the group of $\mathbf{Q}$-points of an $\mathbf{R}$-compact $\mathbf{Q}$-torus.

Combining this discussion with the preceding observations, one infers that for the given special point $\xi \in \mathfrak{S}_n$ (with respect to the given imbedding of $Z_1 = K_{11}^\times \times \cdots \times K_{s1}^\times$ in $Sp(n, \mathbf{Q})$), there exist totally real number fields k_i, with $[k_i:\mathbf{Q}] = d_i$, $i = 1, \dots, s$ (not necessarily all different), and for each i a special point $\xi(i)$ of $\mathfrak{H}^{d_i}$ with respect to the $\mathbf{Q}$-structure on $S_i = R_{k_i/\mathbf{Q}}SL_2$ acting on $\mathfrak{H}^{d_i}$ and an imbedding $\psi_i': K_{i1}^\times \longrightarrow S_i(\mathbf{Q})$, as well as an equivariant holomorphic imbedding

$$f': \mathfrak{H}^{d_1} \times \cdots \times \mathfrak{H}^{d_s} \longrightarrow \mathfrak{S}_n, \quad \mathfrak{H} = \mathfrak{S}_1, \tag{6}$$

coming from a rational homomorphism φ' of $\mathbf{Q}$-groups

$$\varphi': R_{k_1/\mathbf{Q}}SL_2 \times \cdots \times R_{k_s/\mathbf{Q}}SL_2 \longrightarrow Sp_n, \tag{7}$$

with

$$n_1 + \cdots + n_s = n, \quad n_j = [k_j:\mathbf{Q}], \quad i = 1, \dots, s,$$

which is defined over $\mathbf{Q}$, and is induced by an isometry of a product of symplectic spaces. Let H_j be the direct product of G_m and S_j. The homomorphism φ' has certain properties which we now describe.

First of all, any two such φ' (for the same $k_1, \ldots, k_s$) are conjugate by an element of $Sp(n, \mathbf{Q})$, since both are induced by isometries into the symplectic space V. Secondly, it is easy to see that φ' and f' satisfy Satake's conditions (H_2) and (H_2'), and in particular the extension of f' to the Satake compactification [12] (more correctly, to the projective limit of Satake compactifications with respect to products of arithmetic groups commensurable with the respective Hilbert modular groups) carries rational point boundary components of the left side to rational point boundary components of the right side. Thus, by a suitable choice of coordinates on these domains, we may suppose that f' is a *linear* mapping with coefficients that are rational in a suitable sense (i.e. with respect to the given $\mathbf{Q}$-structures).

3. The homomorphism of the graded algebra of (arithmetic) modular forms

In [4] we have already introduced a notion of arithmetic modular form on the adele group of $R_{k/\mathbf{Q}}GL_2$, where k is a totally real field, and used this to formulate the version stated above of Shimura's reciprocity law for Hilbert modular functions. This notion is based on the theory of the Fourier expansions of Eisenstein series on the adele group. In [6] Karel has introduced a similar but more general notion of arithmetic modular form on the adele group of an algebraic group whose real points act on a rational symmetric tube domain. Though there is an error in that paper, Karel has subsequently corrected that [7], and proposes to publish the corrections as part of a larger undertaking . We subsume here that portion of his corrected work which is necessary for our purposes. Namely, on the graded algebra $\mathscr{A}$ of modular forms on the adele group of the group Gp_n of symplectic similitudes defined over $\mathbf{Q}$, one may introduce a $\mathbf{Q}_{ab}$-structure and an operation of the Galois group $\mathscr{G} = \mathrm{Gal}(\mathbf{Q}_{ab}/\mathbf{Q})$ on $\mathscr{A}$ which commutes with translation by elements of $Gp_n(\mathbf{A}_f)$, the set of whose invariants $\mathbf{A} = \mathscr{A}^{\mathscr{G}}$ determines the $\mathbf{Q}$-structure on $\mathscr{A}$. The field of homogeneous quotients of degree zero of $\mathbf{A}$ is denoted by $\mathbf{M}$ and is called the field of arithmetic modular functions.

To describe the action of $\mathscr{G}$ on $\mathscr{A}$ and define the $\mathbf{Q}$-arithmetic modular forms and functions, we recall [7] the existence of a certain $\mathbf{Q}$-rational homomorphism $\mu: G_m \to G$, defined in relation to a maximal $\mathbf{R}$-split torus T of G, defined over $\mathbf{Q}$, and containing a maximal $\mathbf{Q}$-split torus of G, where G is a reductive linear algebraic group defined over $\mathbf{Q}$ and such that $G_+(\mathbf{R})$ acts on a symmetric rational tube domain. One requires that all the positive non-compact roots of T take the same value α on $\mu(\alpha)$ for $\alpha \in G_m$. Then the centralizer of $\mu(G_m)$ in G is a reductive complement in a maximal $\mathbf{Q}$-parabolic subgroup P of G to the unipotent radical $U(P)$ which is generated by the positive non-compact root spaces for T; and for $\alpha \in G_m$, $\mathrm{Ad}(\mu(\alpha))$ acts on the Lie algebra of $U(P)$ through multiplication by α, and $P_+(\mathbf{R})$ is the subgroup of $G_+(\mathbf{R})$ which fixes a certain rational zero-dimensional boundary component ∞. If $G = Gp_n$, then the torus $\mu(G_m)$ is conjugate over $G(\mathbf{Q})$ to the group of all diagonal matrices $\mathrm{diag}(s, s, \ldots, s, 1, 1, \ldots, 1)$ (with n s's and n 1's), $s \in G_m$; and in particular if $n = 1$ and $G = GL_2$, this is the group of all diagonal matrices $\mathrm{diag}(s, 1)$.

One identifies the Galois group $\mathscr{G} = \mathrm{Gal}(\mathbf{Q}_{ab}/\mathbf{Q})$ with the group $\hat{\mathbf{Z}}^\times$ of rational integral idelic units. If f is a modular form arithmetic over $\mathbf{Q}_{ab}$ with respect to the open compact subgroup $\mathbf{K}$ of $G(\mathbf{A}_f)$, then f has a Fourier expansion about the distinguished point ∞ at infinity corresponding to P on each component of $G(\mathbf{A})/\mathbf{K}_\infty \mathbf{Z}^-$, where $\mathbf{Z}^-$ is the closure of $Z(\mathbf{Q})_f$ in $G(\mathbf{A}_f)$. Then if $\tau \in \mathscr{G}$ is made to act on the Fourier coefficients of each of these Fourier expansions (whose coefficients all lie in $\mathbf{Q}_{ab}$), one obtains the set of Fourier expansions of a $\mathbf{Q}_{ab}$-arithmetic modular form $\tau * f$ with respect to $\mu(\tau)_\mathbf{K} = \mu(\tau)\mathbf{K}\mu(\tau)^{-1}$, hence if R denotes right translation in $G(\mathbf{A})$, f^τ defined by

$$f^\tau(x) = (R(\mu(\tau)^{-1})\cdot(\tau * f))(x) = (\tau * f)(x\mu(\tau)^{-1}), \quad x \in G(\mathbf{A}),$$

is a modular form in $\mathscr{A}(\mathbf{K}, \mathbf{Q}_{ab})$. If we define thus the action of $\mathscr{G}$ on $\mathscr{A}(\mathbf{Q}_{ab})$, then the set $\mathbf{A} = \mathscr{A}^\mathscr{G}$ of invariants of $\mathscr{G}$ defines a $\mathbf{Q}$-structure on $\mathscr{A}$, and its elements are called $\mathbf{Q}$-arithmetic modular forms. The homogeneous quotients of degree 0 of $\mathbf{A}$ ($f/g, f, g \in \mathbf{A}_m, g \neq 0$) are called $\mathbf{Q}$-arithmetic modular functions.

When $G = R_{k/\mathbf{Q}} GL_2, k$ totally real, define $\mu' : G_m \to GL_2 : s \to \operatorname{diag}(s, 1)$ as above (up to conjugacy over $GL_2(k)$), and

$$\mu = R_{k/\mathbf{Q}}\mu' : G_m \longrightarrow G = R_{k/\mathbf{Q}} GL_2.$$

When $k = k_j$, denote μ (resp. G) by μ_j (resp. G_j). Let H be the algebraic subgroup of $G = \prod_j G_j$ generated by the subgroups $S = \prod_j S_j$ and $\mu(G_m)$, where $\mu: G_m \to G$ is defined as $\prod_j \mu_j$. Then H is the direct product of S and $\mu(G_m)$, and contains as a subgroup the direct product H_j of S_j and $\mu(G_m)$, which is isomorphic over $\mathbf{Q}$ to $\mu_j(G_m)\cdot S_j \subset G_j$. Then the homomorphism $Z_1 = \prod_j K_{j1}{}^\times \hookrightarrow Sp_n$ is factored by way of the maps

$$K_{j1}{}^\times \hookrightarrow S_j \hookrightarrow Sp_n, \quad j = 1, \ldots, s;$$

and $\mu: G_m \to Gp_n$ is factored by way of H_j:

$$G_m \xrightarrow{\mu} H \xrightarrow{\varphi'} Gp_n,$$

where φ' satisfies (H_2) and (H_2'). Now $\mathbf{Q}$-arithmetic modular forms may be defined on $H(\mathbf{A})$ alone, since $H(\mathbf{A})$ is invariant under translation by $\operatorname{diag}(s, 1)$, where s is any idelic unit of $\mathbf{Q}$, and $H(\mathbf{A}) \subset (R_{k/\mathbf{Q}} GL_2)(\mathbf{A})$.

For the rest of this section, consider only objects connected with a single index j, and drop j from the notation. Now the point is to start from a $\mathbf{Q}$-arithmetic modular function f on $Gp_n(\mathbf{A})$ (finite at ξ, of course) with respect to $\mathbf{K}$, and pull it back by composition to a modular function f^* on $H(\mathbf{A})$ (actually, $H_j(\mathbf{A})$), which will be $\mathbf{Q}$-arithmetic on $H(\mathbf{A})$ because of the above considerations. Then we can apply Shimura's reciprocity law for Hilbert modular functions to f^*, because the union of the components of $V_\mathbf{K} = G_+(\mathbf{Q})\backslash G_+(\mathbf{A})/\mathbf{K}\mathbf{K}_\infty$ meeting $H(\mathbf{A})$ is stable under $\mathscr{G}$, and the special points of these components are stable as a set under the action of the ideles of the reflex field as defined earlier. (In fact, if α is an idele of the reflex field $K^*(\tilde{\Sigma})$ and $(K^*(\tilde{\Sigma}), \tilde{\Sigma}^*)$ is the CM-type dual to $(K, \tilde{\Sigma})$, then the type norm $N_{\tilde{\Sigma}^*}\alpha = \beta$ is an idele of K, and $\beta\bar{\beta}$, *an idele of* $\mathbf{Q}$. But it is through the type norm β that the idele α acts on the special points of the double coset space of $R_{k/\mathbf{Q}} GL_2(\mathbf{A})$, where k is the maximal totally real subfield of K.

4. Action of the ideles of the composite $\mathbf{K}^*$ of the reflex fields on the special points of $Gp_n(\mathbf{A})$

Now let $G' = Gp_n(\mathbf{A}), G = \prod_j G_j$, and let

$$\xi \in V'_\infty = G'(\mathbf{A})/\mathbf{K}_\infty \mathbf{Z}^- = \varprojlim_{\mathbf{K}} V'_{\mathbf{K}}, \quad V'_{\mathbf{K}} = G'(\mathbf{Q})\backslash G'(\mathbf{A})/\mathbf{K}\mathbf{K}_\infty,$$

be a special point; for simplicity of notation we may (*without loss of generality*) assume $\xi \in \mathfrak{S}_n$ is a special point with respect to the action of $Sp(n, \mathbf{Q})$ on $\mathfrak{S}_n$. Then, by what we have observed above, there exists an imbedding defined over $\mathbf{Q}$ of a product of groups defined over $\mathbf{Q}$ into Sp_n:

$$R_{k_1/\mathbf{Q}} SL_2 \times \cdots \times R_{k_s/\mathbf{Q}} SL_2 \hookrightarrow Sp_n,$$

where $[k_i:\mathbf{Q}] = n_i$, $i = 1, \ldots, s$, and $n = n_1 + \cdots + n_s$, such that under the associated imbedding of rational tube domains,

$$\mathfrak{H}^{n_1} \times \cdots \times \mathfrak{H}^{n_s} \longrightarrow \mathfrak{S}_n,$$

the special point ξ is the image of $(\xi_1, \ldots, \xi_s)$, where ξ_j is a special point on $\mathfrak{H}^{n_j}$ with respect to the action of S_j. Let K_j be the CM-field and $(K_j, \tilde{\Sigma}_j)$ be the CM-type associated to the special point $\xi_j \in \mathfrak{H}^{n_j}$ on which $(R_{k_j/\mathbf{Q}} GL_2)_+(\mathbf{R})$ acts. Let $(K_j^*, \tilde{\Sigma}_j^*)$ be its CM-dual. Now if Φ_j is an arithmetic modular function on $R_{k_j/\mathbf{Q}} GL_2(\mathbf{A})$ with respect to an open compact subgroup $\mathbf{K}_j$ of $R_{k_j/\mathbf{Q}} GL_2(\mathbf{A})$, regular at ξ_j, then

$$K_j^*(\Phi_j(\xi_j))/K_j^*$$

is an abelian extension with reciprocity law given by

$$[v, K_j^*] \cdot \Phi_j(\xi_j) = \Phi_j(v * \xi_j), \quad v \in \mathbf{A}(K_j^*)^\times.$$

Let K^* be the compositum of the fields $K_1^*, \ldots, K_s^*$. Now $K^*(\Phi_j(\xi_j))$ is an abelian extension of K^*, and if $v^* \in \mathbf{A}(K^*)^\times = \mathbf{I}(K^*)$, then applying the reciprocity law for special values of arithmetic Hilbert modular functions (as stated in [5]) and one of the well-known properties of the Artin symbol, one obtains

$$[v^*, K^*] \cdot \Phi_j(\xi_j) = [N_{K^*/K_j^*} v^*, K_j^*] \cdot \Phi_j(\xi_j) = \Phi_j(N_{\tilde{\Sigma}_j^*}(N_{K^*/K_j^*} v^*) * \xi_j),$$

also taking into account that $\Phi_j(\xi_j)$ generates an abelian extension of K_j^*. In this way we obtain an action of the ideles of K^* on the special points of $\mathfrak{H}^{n_1} \times \cdots \times \mathfrak{H}^{n_s}$, namely, if $v^* \in \mathbf{I}(K^*)$, and $(\xi_1, \ldots, \xi_s)$ is a special point as above, then $v^* * (\xi_1, \ldots, \xi_s) = (\xi_1', \ldots, \xi_s')$, where

$$\xi_j' = N_{\tilde{\Sigma}_j^*}(N_{K^*/K_j^*} v^*) * \xi_j, \quad j = 1, \ldots, s.$$

In the next section we shall see how to formulate Shimura's reciprocity law for arithmetic Siegel modular functions in terms of this action.

5. The reciprocity law for special values of Siegel's modular functions

Now let $G = \prod_{j=1,\ldots,s} G_j$, where $G_j = R_{k_j/\mathbf{Q}} GL_2$ and let

$$V_{\mathbf{K}} = G_+(\mathbf{Q})\backslash G_+(\mathbf{A})/\mathbf{K}\mathbf{K}_\infty, \quad \mathbf{K} = \prod_{j=1,\dots,s} \mathbf{K}_j,$$

where $\mathbf{K}_j$ is open, compact in $G_j(\mathbf{A}_f)$.

Let A be an affine $\mathbf{Q}$-open subset of $V_{\mathbf{K}}$ containing the image in $V_{\mathbf{K}}$ of the full orbit of the special point $(\xi) = (\xi_1, \dots, \xi_s)$ under the group $\mathbf{I}(K^*)$ as explained above. (This is possible since V is defined over $\mathbf{Q}$ and the image of that orbit is a finite set.) Let $V_j = G_{j+}(\mathbf{Q})\backslash G_{j+}(\mathbf{A})/\mathbf{K}_j\mathbf{K}_{\infty j}$, where $\mathbf{K}_{\infty j}$ has a meaning for $G_{j+}(\mathbf{R})$ analogous to that of $\mathbf{K}_\infty$ for $G_+(\mathbf{R})$.

As before, let H be the algebraic group $\mu(G_m)\cdot\prod_j S_j$ which is defined over $\mathbf{Q}$, where $\mu(G_m)$ is defined with respect to the maximal torus T defined by

$$T = Z'\cdot\prod_j R_{k_j/\mathbf{Q}}T_j'',$$

T_j'' being a k_j-split maximal torus of SL_2 defined over k_j, and Z' being the center of G. (Then $T_j' = R_{k_j/\mathbf{Q}}T_j''$ is a maximal torus of maximal $\mathbf{Q}$-rank in S_j, and T splits over $\mathbf{R}$.)

$H(\mathbf{A})$ is an open subgroup of $G(\mathbf{A})$, and its image $V_{\mathbf{K},H}$ in $V_{\mathbf{K}}$ is an open subvariety of the latter and defined over $\mathbf{Q}$ for reasons noted earlier. If f is a $\mathbf{Q}$-arithmetic modular form on $G'(\mathbf{A})$ ($G' = Gp_n$), then f may be pulled back to an arithmetic modular form f^* on $H(\mathbf{A})$. On the other hand, each of the groups $H_j = \mu(G_m)\cdot R_{k_j/\mathbf{Q}}SL_2$ may be viewed as imbedded in H, and f^* may be pulled back to $H_j(\mathbf{A})$ for each j. As is easily seen, the reciprocity law for special values of Hilbert modular functions also applies to the restrictions $f_j = f^*|H_j$, $j = 1, \dots, s$, and in the same form as expressed in the Theorem stated in the introduction.

We may then assume the open affine set A is contained in $V_{\mathbf{K},H}'$. Let $\{\Phi_{ji}|j = 1, \dots, s; i = 1, \dots, N_j\}$ be a system of generators of the affine coordinate algebra of $A_0 = A\cap(V_{\mathbf{K}_1}\times\cdots\times V_{\mathbf{K}_j})$, where each Φ_{ji} is an arithmetic modular function on $G_j(\mathbf{A})$. Then a $\mathbf{Q}$-arithmetic modular function Φ on $\mathfrak{S}_n$ whose pullback is regular on A_0 is a linear combination of products

$$\Phi_{1i_1}\Phi_{2i_2}\times\cdots\times\Phi_{si_s}$$

with rational coefficients. But the reciprocity law for such a product may be inferred from what we have already done. Thus we have, in conclusion,

Theorem S. *If* $v^*\in\mathbf{A}(K^*)^\times$, *if* Φ *is a* $\mathbf{Q}$-*arithmetic Siegel modular function with respect to the open compact subgroup* $\mathbf{K}'$ *of* $Gp_n(\mathbf{A}_f)$, *if* ξ *is a special point of*

$$V_\infty' = Gp_n(\mathbf{Q})\backslash Gp_n(\mathbf{A})/\mathbf{K}'_\infty\mathbf{Z}'^-,$$

where $\mathbf{Z}'^-$ *and* $\mathbf{K}'_\infty$ *have meanings analogous to the earlier ones, then there exists an almost direct product* H *of algebraic groups defined over* $\mathbf{Q}$:

$$H = \mu(G_m)\cdot\prod_{j=1,\dots,s} R_{k_j\mathbf{Q}}SL_2$$

operating on a product

$$\mathscr{P} = \prod_{j=1,\dots,s}\mathfrak{H}^{n_j}$$

of upper half-planes, together with equivariant morphisms $\varphi: H \to Gp_n$ *and* $f: \mathscr{P} \to V'_\infty$ *and a special point* $(\xi_1, \ldots, \xi_s) \in \mathscr{P}$ *such that* $f(\xi_1, \ldots, \xi_s) = \xi$.

Let $\xi_{\mathbf{K}}$ *denote the image of* ξ *in*

$$V_{\mathbf{K}}' = Gp_n(\mathbf{Q}) \backslash Gp_n(\mathbf{A}) / \mathbf{K}'\mathbf{K}'_\infty.$$

Then, with $[,]$ *denoting the Artin symbol, we have*

$$[v^*, K^*] \cdot \Phi(\xi_{\mathbf{K}}) = [v^*, K^*] \cdot \Phi \circ f(\xi_1, \ldots, \xi_s)$$

$$= (\Phi \circ f)(N_{\bar{\Sigma}_1^*}(N_{K^*/K_1^*} v^*) * \xi_1, \ldots, N_{\bar{\Sigma}_s^*}(N_{K^*/K_s^*} v^*) * \xi_s).$$

(Note that the argument appearing in the function $\Phi \circ f$ on the right-hand side is a special point of $V_\infty = G_+(\mathbf{Q}) \backslash G_+(\mathbf{A}) / \mathbf{Z}^- \mathbf{K}_\infty$ (with G as above) whose image in $V_{\mathbf{K}}$ lies in $V_{\mathbf{K},H}$, and whose image under f is the image of a special point of V_∞'.)

It would seem that from this result one should be able very simply to derive some of the main theorems for canonical models connected with $Gp_n(\mathbf{A})$ and thence, by imbeddings, for the canonical models connected with domains that can be equivariantly imbedded in Siegel space.

References

[1] Baily W L Jr, On the Hilbert–Siegel modular space, *Am. J. Math.* **81** (1959) 846–874

[2] Baily W L Jr, On the theory of Hilbert modular functions I, Arithmetic groups and Eisenstein series, *J. Algebra* **90** (1984) 567–605

[3] Baily W L Jr, Arithmetic Hilbert modular forms, *Proc. Int. Symp. on automorphic forms of several variables* (eds) Y Morita and I Satake (Boston: Birkhäuser) (1983) pp. 49–92

[4] Baily W L Jr, Arithmetic Hilbert modular functions II *Rev. Mat. Iberoamericana* **1** (1985) 85–119

[5] Baily W L Jr, Reciprocity laws for special values of Hilbert modular functions, *Proc. KIT Mathematics Workshop* (eds) J K Koo and D Y Suh (Korea: Taejon) (1986)

[6] Karel M, Eisenstein series and fields of definition, *Compos. Math.* **37** (1978) 121–169

[7] Karel M, Corrections to my paper 'Eisenstein series and fields of definition' (unpublished ms) (1984)

[8] Karel M, Special values of elliptic modular functions, *Proc. Int. Symp. on automorphic forms of several variables* (eds) Y Morita and I Satake (Boston: Birkhäuser) (1983) pp. 179–225

[9] Karel M, Special values of Hilbert modular functions, *Rev. Mat. Iberoamericana* (1987)

[10] Klingen H, Eisensteinreihen zur Hilbertschen Modulgruppen *n*-ten Grades, *Nachr. Akad. Wiss. Göttingen II Math.-Phys. Klasse* (1960) Nr 4

[11] Satake I, Symplectic representations of algebraic groups satisfying a certain analyticity condition, *Acta. Math.* **117** (1967) 215–279

[12] Satake I, A note on holomorphic imbeddings and compactification of symmetric domains, *Am. J. Math.* **90** (1968) 231–247

[13] Satake I, On some properties of holomorphic imbeddings of symmetric domains, *Am. J. Math.* **91** (1969) 289–305

[14] Shimura G, Construction of class fields and zeta functions of algebraic curves, *Ann. Math.* **85** (1967) 58–159.

[15] Shimura G, Algebraic number fields and symplectic discontinuous groups, *Ann. Math.* **86** (1967) 503–592

[16] Shimura G, On canonical models of arithmetic quotients of bounded symmetric domains, *Ann. Math.* **91** (1970) 144–222