The characters of supercuspidal representations as weighted orbital integrals

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Abstract. Weighted orbital integrals are the terms which occur on the geometric side of the trace formula. We shall investigate these distributions on a $p$-adic group. We shall evaluate the weighted orbital integral of a supercuspidal matrix coefficient as a multiple of the corresponding character.

Keywords. Supercuspidal representation; weighted orbital integrals.

1. Introduction

Let $G$ be a reductive algebraic group over a non-Archimedean local field $F$ of characteristic 0. Suppose that $\pi$ is a (smooth) supercuspidal representation of $G(F)$ on a complex vector space $V$. Let $f(x)$ be a finite sum of matrix coefficients

$$\xi(x^{-1}v), \quad \forall x \in G(F), \quad v \in V, \quad \xi \in V^*.$$

Then $f$ is a locally constant function on $G(F)$ which is compactly supported modulo the split component $A_G$ of the centre of $G$. If $\Theta_\pi$ is the character of $\pi$, set

$$\Theta_\pi(f) = \int_{A_G(F)\backslash G(F)} f(x)\Theta_\pi(x)dx.$$

We are going to study the weighted orbital integrals of $f$.

Suppose that $M$ is a Levi component of some parabolic subgroup of $G$ which is defined over $F$. The set $\mathcal{P}(M)$ of all parabolic subgroups over $F$ with Levi component $M$ is parametrized by the chambers in a real vector space $a_M$. The weight factor for orbital integrals is a certain function $v_M(x)$ on $M(F)\backslash G(F)$ which arises in the theory of automorphic forms; it is defined as the volume of the convex hull in $a_M/a_G$ of a set of points indexed by $\mathcal{P}(M)$. Suppose that $\gamma$ is a $G$-regular element in $M(F)$ which is $M$-elliptic over $F$. This means that the centralizer of $\gamma$ in $M(F)$ is compact modulo $A_M(F)$. The object of this paper is to prove the following result.

**Theorem:** The weighted orbital integral

$$\int_{A_M(F)\backslash G(F)} f(x^{-1}\gamma x)v_M(x)dx$$
equals

$$(-1)^{\dim(A_M/A_G)} \Theta_n(f) \Theta_n(y).$$

Observe that the second expression depends on a choice of invariant measure on $A_G(F)/G(F)$; the first expression depends on choices of invariant measures on $a_M/a_G$ and $A_M(F)/G(F)$. There is a compatibility requirement between the implicit measure on $A_M(F)/G(F)$ and the measure on $a_M/a_G$.

The theorem is a $p$-adic version of a similar result for real groups ([1], Theorem 9.1). It tells us that the character values of $\pi$ on a non-compact torus can be recovered as the weighted orbital integrals of a matrix coefficient of $\pi$. There is reason to believe that the result is part of a larger theory. Kazhdan has suggested the possibility of proving a local trace formula for $G$. The idea would be to try to compute the trace of the left-right convolution operator of a pair of functions, acting on the discrete spectrum of $L^2(G(F))$. Our theorem could be regarded as a special case, in which one of the two functions is the matrix coefficient $f$. A different special case of this (as yet undiscovered) trace formula is provided by work of Waldspurger [6]. We hope to return to the question on another occasion.

For $G = SL(2)$, the theorem was first established by Kazhdan (unpublished). I am indebted to him for enlightening conversations.

2. Positive orthogonal sets

Let us recall the precise definition of $v_M(x)$. It depends on a special maximal compact subgroup $K$ of $G(F)$ which is in good position relative to $M$. (This means that the vertex of $K$ in the building of $G$ lies in the apartment of a maximal split torus of $M$.)

For any parabolic subgroup $P \in \mathcal{P}(M)$, with Levi decomposition $P = MN_p$, and any point $x \in G(F)$, we can write

$$x = n_p(x)m_p(x)k_p(x),$$

with $n_p(x) \in N_p(F)$, $m_p(x) \in M(F)$ and $k_p(x) \in K$. Set

$$H_p(x) = H_M(m_p(x)),$$

where $H_M$ is the usual map from $M(F)$ to the real vector space

$$a_M = \text{Hom}(X(M)_F, \mathbb{R}),$$

given by

$$\exp(\langle H_M(m), \chi \rangle) = |\langle \chi(m) \rangle|, \ m \in M(F), \ \chi \in X(M)_F.$$

There is a canonical map from $a_M$ onto $a_G$, whose kernel we denote by $a_G^M$. Since $X(M)_F$ embeds into the character group $X(A_M)$ of $A_M$, there is also a compatible embedding of $a_G$ into $a_M$, and therefore a canonical decomposition

$$a_M = a_M^G \oplus a_G.$$
The function $v_M(x)$ equals the volume of the convex hull of the projection of 
\[ \{-H_P(x) : P \in \mathcal{P}(M)\} \]
on $a_M/a_G \cong a_M^G$.

It is convenient to choose a suitable Euclidean metric $\| \cdot \|$ on $a_M$, and to use this to normalize the Haar measures on $a_M$, $a_G$ and $a_M/a_G$. These measures then determine Haar measures on $A_M(F)$, $A_G(F)$ and $A_M(F)/A_G(F)$. Indeed,

$$\kappa_M = A_M(F) \cap K$$

is the maximal (open) compact subgroup of $A_M(F)$, and $H_M$ maps $A_M(F)/\kappa_M$ injectively onto a lattice in $a_M$. We take the Haar measure on $A_M(F)$ such that

$$\text{vol}(\kappa_M) = \text{vol}(a_M/H_M(A_M(F))).$$

The cases of $A_G(F)$ and $A_M(F)/A_G(F)$ are similar, and we have

$$\text{vol}(\kappa_M/\kappa_G) = \text{vol}(a_M/H_M(A_M(F)) + a_G). \quad (2)$$

The points $\{-H_P(x)\}$ form a positive orthogonal set. In general, we say that a set $\mathcal{Y} = \{Y_p : P \in \mathcal{P}(M)\}$ of points in $a_M$ is a positive orthogonal set for $M$ if it has the following property. For any pair $P$ and $P'$ of adjacent groups in $\mathcal{P}(M)$, whose chambers in $a_M$ share the wall determined by a simple root $\alpha$ in $\Delta_P \cap (-\Delta_P)$ of $(P, A_M)$, we have

$$Y_P - Y_{P'} = r\alpha^\vee,$$

for a non-negative number $r$. As usual, $\Delta_P$ is the set of simple roots of $(P, A_M)$, and $\alpha^\vee \in a_M$ is the "co-root" associated to $\alpha$. Suppose that $\mathcal{Y}$ has this property. Then the volume in $a_M/a_G$ of the convex hull of $\{Y_p\}$ can be expressed analytically as

$$\lim_{\lambda \to 0} \sum_{P \in \mathcal{P}(M)} \exp[\lambda(Y_p)] \theta_P(\lambda)^{-1}, \quad (3)$$

where

$$\theta_P(\lambda) = \text{vol}(a_M^G/Z(\Delta_P))^{-1} \prod_{\alpha \in \Delta_P} \lambda(\alpha^\vee).$$

(See [3], p. 36.) As in [4], we shall write $d(\mathcal{Y})$ for the smallest of the numbers

$$\{\alpha(Y_p) : \alpha \in \Delta_P, P \in \mathcal{P}(M)\}.$$ 

Fix such a $\mathcal{Y}$, and let

$$Q = M_QN_Q, \quad M_Q \supseteq M,$$

be an element in the set $\mathcal{F}(M)$ of parabolic subgroups of $G$ over $F$ which contain $M$. 
is a positive orthogonal set for $M$, but relative to $M_Q$ instead of $G$. As above, $a_M$ is the direct sum of the spaces $a_{M_Q} = a_{Q}^t$ and $a_{M_Q} = a_Q$. We shall write $S_{M_Q}^L(\Psi)$ for the convex hull of (4) in $a_M$, taken modulo $a_Q$, and we shall let $\sigma_M^L(\cdot, \Psi)$ stand for the characteristic function of $S_{M_Q}^L(\Psi)$. The vectors (4) all project onto the same point $Y_Q$ in $a_Q$. Moreover, if we fix the Levi component $L = M_Q$ instead of $Q$, the set

$$\Psi_L = \{ Y_Q : Q \in \mathcal{P}(L) \}$$

is a positive orthogonal set for $L$. For simplicity, we shall usually denote $S_{M_Q}^L(\Psi_L)$ and $\sigma_M^L(\cdot, \Psi_L)$ by $S_L(\Psi)$ and $\sigma_L(\cdot, \Psi)$ respectively.

The following geometric property is a restatement of Lemmas 3.1 and 3.2 of [4].

**Lemma 1:** There is a positive constant $d_M$ with the following property. If $\Psi$ is any positive orthogonal set for $M$ and $L \supset M$ is as above, and if

$$H_M = H_M^L \oplus H_L, \quad H_M^L \in a_M^L, \quad H_L \in a_L,$$

is a point in $a_M$ such that

$$\| H_M^L \| \leq d_M \| \Psi \|,$$

then $H_M$ belongs to $S_M(\Psi)$ if and only if $H_L$ belongs to $S_L(\Psi_L)$. □

Another example of a positive orthogonal set is provided by the Weyl orbit of a point. Let $M_0 \subset M$ be a fixed Levi component of some minimal parabolic subgroup over $F$, and let $W_0$ be the Weyl group of $(G, A_{M_0})$. Our metric on $a_M$ is understood to be the restriction of a Euclidean metric $\| \cdot \|$ on $a_{M_0}$ which is invariant under $W_0$. Choose an element $T_{P_0} \in \mathcal{P}(M_0)$, and let $T_{P_0}$ be a point in $a_{M_0}$ which lies in the chamber associated to $P_0$. The Weyl group $W_0$ acts simply transitively on $\mathcal{P}(M_0)$, and

$$\hat{F} = \{ T_{P_0} = sT_{P_0} : P_0 = sP_0, s \in W_0 \}$$

is a positive orthogonal set for $M_0$. By the discussion above (with $(M, L)$ replaced by $(M_0, M)$), we obtain a positive orthogonal set

$$\hat{F}_M = \{ T_P : P \in \mathcal{P}(M) \}$$

for $M$, and it is not hard to show that

$$d(\hat{F}_M) \geq d(\hat{F}),$$

provided that $M$ is not equal to $G$. We are of course free to vary the original point $T_{P_0}$. In future we shall want to choose $T_{P_0}$ so that the number $d(\hat{F})$ is large, and of an order of magnitude comparable to the norm

$$\| \hat{F} \| = \| T_{P_0} \|.$$

We shall actually work with a combination of the two examples. For a given
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\( x \in G(F) \), set

\[ \mathcal{Y}(x, \mathcal{F}) = \{ Y_\rho(x, \mathcal{F}) = T_\rho - H_\rho(x); P \in \mathcal{P}(M) \}, \]  

(6)

where \( \bar{P} \) denotes the parabolic subgroup opposite to \( P \). Because it is a difference of positive orthogonal sets, rather than a sum, \( \mathcal{Y}(x, \mathcal{F}) \) need not be a positive orthogonal set. However, if \( d(\mathcal{F}) \) is large with respect to \( x \), the positivity of \( \mathcal{F} \) dominates, and \( \mathcal{Y}(x, \mathcal{F}) \) becomes a positive orthogonal set. We shall assume this in what follows.

3. The main geometric lemma

We shall now begin the proof of the theorem. Suppose that \( \mathcal{F} \) is defined by (5). Let \( u(x, \mathcal{F}) \) denote the characteristic function in \( A_\mathcal{G}(F) \setminus G(F) \) of the set of points

\[ x = k_1 h k_2, \quad k_1, k_2 \in K, \quad h \in A_\mathcal{G}(F) \setminus A_{\mathcal{M}_0}(F), \]

such that the projection onto \( a^G_{\mathcal{M}_0} \) of \( H_\mathcal{G}(h) \) lies in the convex hull \( S_{\mathcal{M}_0}(\mathcal{F}) \). Since \( K \) corresponds to a special vertex,

\[ G(F) = K A_{\mathcal{M}_0}(F) K. \]

We can consequently force \( u(x, \mathcal{F}) \) to be identically equal to 1 on any given compact subset of \( A_\mathcal{G}(F) \setminus G(F) \) simply by choosing \( \mathcal{F} \) so that \( d(\mathcal{F}) \) is sufficiently large.

Our starting point for the study of the matrix coefficient \( f \) is a simple consequence of results of Harish-Chandra.

Lemma 2: Suppose that \( f \) and \( \gamma \) are as in the theorem. Then

\[ \Theta_\mathcal{F}(f) \Theta_\gamma(g) = \int_{A_\mathcal{G}(F) \setminus G(F)} f(x^{-1} \gamma x) u(x, \mathcal{F}) \, dx, \]

whenever \( d(\mathcal{F}) \) is sufficiently large.

Proof: If \( g \) is any function in \( C^\infty_c(G(F)) \), Theorem 9 of [5] tells us that

\[ \Theta_\mathcal{F}(f) \Theta_\gamma(g) = \int_{A_\mathcal{G}(F) \setminus G(F)} \left( \int_{G(F)} f(x^{-1} k y x) g(y) \, dy \right) \, dx. \]  

(7)

Assume that

\[ g(y) = \text{vol}(K)^{-1} \int_{K} g_0(k y k^{-1}) \, dk, \]

where \( g_0 \) is supported on a small neighbourhood \( \Omega \) of \( \gamma \). The right hand side of (7) can then be written

\[ \text{vol}(K)^{-1} \int_{A_\mathcal{G}(F) \setminus G(F)} \left( \int_{G(F)} \int_{K} f(x^{-1} k^{-1} y k x) g_0(y) \, dk \, dy \right) \, dx. \]

It is a straightforward consequence of [5, Lemma 13] that the integrand in \( x \) is
supported on a compact subset of $A_n(F) \backslash G(F)$ which depends only on $\Omega$. Now let $g_0$ approach the Dirac measure at $\gamma$. The left hand side of (7) approaches

$$\Theta_n(f) \Theta_n(\gamma),$$

while the right hand side converges to

$$\text{vol}(K)^{-1} \int_{A_n(F) \backslash G(F)} \left( \int_K f(x^{-1}k^{-1}\gamma kx) dk \right) dx.$$

This last integrand in $x$ is compactly supported. We can therefore multiply it with $u(x, \mathcal{F})$ without changing its value, as long as $d(\mathcal{F})$ is sufficiently large. The expression becomes

$$\text{vol}(K)^{-1} \int_{A_n(F) \backslash G(F)} \left( \int_K f(x^{-1}k^{-1}\gamma kx) dk \right) u(x, \mathcal{F}) dx$$

$$= \text{vol}(K)^{-1} \int_{A_n(F) \backslash G(F)} \int_K f(x^{-1}k^{-1}\gamma kx) u(x, \mathcal{F}) dk dx$$

$$= \int_{A_n(F) \backslash G(F)} f(x^{-1}\gamma x) u(x, \mathcal{F}) dx,$$

since $u(x, \mathcal{F})$ is bi-invariant under $K$. This establishes the required formula.

In view of the lemma, we may write

$$\Theta_n(f) \Theta_n(\gamma) = \int_{A_n(F) \backslash G(F)} f(x^{-1}\gamma x) u(x, \mathcal{F}) dx$$

$$= \int_{A_n(F) \backslash G(F)} f(x^{-1}\gamma x) \left( \int_{A_n(F) \backslash A_n(F)} u(ax, \mathcal{F}) da \right) dx.$$  

By assumption, the centralizer of $\gamma$ in $G(F)$ is compact modulo $A_n(F)$. Therefore, the last integral over $x$ may be taken over a compact set of representatives of $A_n(F) \backslash G(F)$ in $G(F)$. Our task then is to evaluate the integral

$$\int_{A_n(F) \backslash A_n(F)} u(ax, \mathcal{F}) da.$$ 

The main step is to express the integral in terms of the set $\mathcal{H}(x, \mathcal{F})$ given by (6).

**Lemma 3:** For any compact subset $\Gamma$ of $G(F)$ and any $\delta > 0$, there is a positive constant $c(\Gamma, \delta)$ with the following property. If $x$ belongs to $\Gamma$, $a$ belongs to $A_n(F)$, and $\mathcal{F}$ is such that

$$d(\mathcal{F}) \geq \delta \| \mathcal{F} \| \geq c(\Gamma, \delta),$$

then $u(ax, \mathcal{F})$ equals 1 if and only if $H(a)$ belongs to $S_M(\mathcal{H}(x, \mathcal{F}))$. 

\[ \text{(8)} \]
Proof: If $Q$ is a group in $\mathcal{F}(M)$, we write $\tau_Q$ for the characteristic function of
\[ \{ H \in \mathfrak{a}_M : \alpha(H) > 0, \alpha \in \Delta_Q \}. \]

It is known that
\[ \sum_{Q \in \mathcal{F}(M)} \sigma_Q^0(H, \mathcal{F}) \tau_Q(H - T_Q) = 1, \tag{9} \]
for $\mathcal{F}$ as in (5), and any $H \in \mathfrak{a}_M$. This is a general property of positive orthogonal sets which is easily deduced, for example, from Langlands' combinatorial lemma ([1], Lemma 2.3), ([2], Lemma 6.3). We shall actually apply the result with $H = H_M(a)$, and $\mathcal{F}$ replaced by the set
\[ \epsilon \mathcal{F} = \{ \epsilon T_{p_0} : P_0 \in \mathcal{P}(M_0) \}, \]
for a certain $\epsilon > 0$. Having been given $\delta$, we choose $\epsilon$ so that $2\epsilon \delta^{-1}$ is smaller than the numbers $\delta_M$ and $\delta_{M_0}$ provided by Lemma 1.

Fix the elements $a \in A_M(F)$ and $x \in \Gamma$. The left hand side of (9) is a sum of characteristic functions, so there is a unique group $Q \in \mathcal{F}(M)$ such that
\[ a_{H_M(a), \epsilon \mathcal{F}} \tau_Q(H_M(a) - \epsilon T_Q) = 1. \]

Once $Q$ is determined, we can write
\[ ax = am_\delta(x) n_\delta(x) k_\delta(x) \]
\[ = ad(am_\delta(x)) n_\delta(x) \cdot am_\delta(x) k_\delta(x). \]

Consider a root $\alpha$ of $(Q, A_Q)$. Since $H_M(a)$ is the sum of a vector in $\mathfrak{a}_Q^+$, the positive chamber of $Q$, with a convex linear combination of points
\[ \{ \epsilon T_P : P \in \mathcal{P}(M), P \subset Q \}, \]
we have
\[ \alpha(H_M(a)) \geq \epsilon \inf_{P \in Q} \alpha(T_P) \geq \epsilon d(\mathcal{F}). \]

Having fixed $\epsilon$, we choose $c(\Gamma, \delta)$ so that $\epsilon c(\Gamma, \delta)$ is large. Then $\epsilon d(\mathcal{F})$ will be large whenever $\mathcal{F}$ satisfies (8), and $ad(a)$ will act by contraction on $n_\delta(x)$. In particular, we can force the point
\[ ad(am_\delta(x)) n_\delta(x) \]
to be close to 1, uniformly for $x$ in $\Gamma$. We may therefore assume that the point lies in the open compact subgroup $K$. Consequently, $ax$ belongs to the double coset
\[ Kam_\delta(x) K. \]

The next step is to write
\[ am_\delta(x) = k_1 h k_2, \quad h \in A_{M_0}(F), k_1, k_2 \in K \cap M_0(F). \tag{10} \]
Then $ax$ belongs to $KhK$. Observe also that

$$H_Q(h) = H_Q(a) + H_Q(m_Q(x)),$$

so that

$$H_Q(h) = H_Q(a) + H_Q(x). \quad (11)$$

We write

$$H_M(a) = H_M(a) + H_Q(a), \quad H_M(a) \in a_M^\ast,$$

for the decomposition of $H_M(a)$ relative to the direct sum $a_M = a_M^\ast \oplus a_Q$. Similarly

$$H_{M_0}(h) = H_{M_0}(h) + H_Q(h), \quad H_{M_0}(h) \in a_{M_0}^\ast.$$

Then there is a constant $c(\Gamma)$ such that

$$\| H_{M_0}(h) \| \leq \| H_M(a) \| + c(\Gamma),$$

for any $x \in \Gamma$ and $a \in A_M(F)$, and for $h$ defined by (10). This follows easily from the standard properties of height functions on $G(F)$. Now, we are assuming that

$$\sigma^Q_\Phi(H_M(a), \varepsilon, \mathcal{F}) = 1,$$

so that $H_M^Q(a)$ belongs to the convex set $S_M^Q(\varepsilon, \mathcal{F})$. It follows that $\| H_M^Q(a) \|$ is bounded by the norm of the projection of any of the vectors

$$\{ \varepsilon T_P : P \in \mathcal{P}(M), P \in Q \}$$

onto $a_M^\ast$. Therefore,

$$\| H_M^Q(a) \| \leq \varepsilon \| T_P \| \leq \varepsilon \| \mathcal{F} \|. \quad (12)$$

Choose $c(\Gamma, \delta)$ to be so large that $\varepsilon \delta^{-1} c(\Gamma, \delta)$ is greater than the constant $c(\Gamma)$ above. Then

$$\| H_{M_0}^Q(h) \| \leq 2 \delta^{-1} d(\mathcal{F}) \leq \delta_M d(\mathcal{F})$$

whenever $\mathcal{F}$ satisfies (8). Recall that the function

$$u(ax, \mathcal{F}) = u(h, \mathcal{F})$$

equals 1 if and only if $H_{M_0}(h)$ belongs to $S_{M_0}(\mathcal{F})$. It follows from Lemma 1 that $u(ax, \mathcal{F})$ equals 1 if and only if $H_Q(h)$ belongs to $S_{M_0}(\mathcal{F})$.

We are also assuming that

$$\tau_Q(H_Q(a) - \varepsilon T_Q) = \tau_Q(H_M(a) - \varepsilon T_Q) = 1.$$

In particular, $H_Q(a)$ lies in the positive chamber $a_Q^+$. More precisely,

$$\alpha(H_Q(a)) \geq e \alpha(T_Q) \geq e \delta(\mathcal{F}),$$

for any root $\alpha \in \Delta_Q$. We can make this number as large as we wish, for $\mathcal{F}$ satisfying (8),
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simply by taking \( c(\Gamma, \delta) \) large enough. Now \( H_Q(a) \) is related to \( H_Q(h) \) by equation (11). Since \( H_Q(x) \) remains bounded, we can assume that \( H_Q(h) \) also lies in \( a_Q^+ \). But according to ([1] Lemma 3.2), the intersection of \( a_Q^+ \) with \( S_{M_Q}(\mathcal{F}) \) is the set

\[
\{ H \in a_Q^+: \varpi(H - T_Q) < 0, \varpi \in \hat{\Delta}_Q \},
\]

where \( \hat{\Delta}_Q \) is the dual basis of \( \Delta_Q^\vee \). Thus, \( u(ax, \mathcal{F}) \) equals 1 if and only if each of the numbers

\[
\varpi(H_Q(h) - T_Q) = \varpi(H_Q(a) - Y_Q(x, \mathcal{F})), \quad \varpi \in \hat{\Delta}_Q,
\]

is negative. We have now only to retrace our steps. Since \( H_Q(a) \) lies in \( a_Q^+ \), the last condition is equivalent to the assertion that \( H_Q(a) \) lies in \( S_{M_Q}(\mathcal{Y}(x, \mathcal{F})) \). Moreover, \( d(\mathcal{F}) \) is large relative to \( x \), so we can assume that

\[
d(\mathcal{Y}(x, \mathcal{F})) \geq \frac{1}{2} d(\mathcal{F}).
\]

It follows from (12) that

\[
\| H_Q'(a) \| \leq \varepsilon \| \mathcal{F} \|
\]

\[
\leq \varepsilon \delta^{-1} d(\mathcal{F})
\]

\[
\leq 2 \varepsilon \delta^{-1} d(\mathcal{Y}(x, \mathcal{F}))
\]

\[
\leq \delta_M d(\mathcal{Y}(x, \mathcal{F})),
\]

whenever \( \mathcal{F} \) satisfies (8). Applying Lemma 1 again, we conclude that \( H_Q(a) \) belongs to \( S_{M_Q}(\mathcal{Y}(x, \mathcal{F})) \) if and only if \( H_M(a) \) belongs to \( S_M(\mathcal{Y}(x, \mathcal{F})) \). This is equivalent to the original condition that \( u(ax, \mathcal{F}) \) equals 1, so the proof of the lemma is complete. 

As an identity of characteristic functions, the lemma asserts that

\[
u(ax, \mathcal{F}) = \sigma_M(H_M(a), \mathcal{Y}(x, \mathcal{F})), \quad x \in A_M(F)/A_G(F),
\]

for \( x \) and \( \mathcal{F} \) as stated. It follows that \( \Theta_n(f)\Theta_n(y) \) equals

\[
\int_{A_M(F)/A_G(F)} f(x^{-1} \gamma x) \left( \int_{A_M(F)/A_G(F)} \sigma_M(H_M(a), \mathcal{Y}(x, \mathcal{F})) \, da \right) \, dx.
\]

However, the integral

\[
\int_{A_M(F)/A_G(F)} \sigma_M(H_M(a), \mathcal{Y}(x, \mathcal{F})) \, da
\]

is not equal to the volume of \( S_M(\mathcal{Y}(x, \mathcal{F})) \). For

\[
\{ H_M(a): a \in A_M(F)/A_G(F) \}
\]

is a lattice in \( a_M/a_G \), the integral is multiple of the number of lattice points in \( S_M(\mathcal{Y}(x, \mathcal{F})) \). We must find a way to relate this to the volume.

It will actually be convenient to replace \( A_M(F) \) by a subgroup. Suppose that \( A'_M \)
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is any subgroup of finite index in $A_M(F)$, which contains $A_0(F)$. Combining Lemmas 2 and 3 as above, we obtain

$$\Theta_\pi(f)\Theta_{\pi}((\gamma)) = \int_{A_M^0(G(F))} f(x^{-1}w(x)) \left( \int_{A_M^0(G(F))} \sigma_M(H_M(a), \psi(x, \mathcal{F})) \, da \right) \, dx,$$

a formula which holds whenever $\mathcal{F}$ satisfies the conditions (8).

4. Counting lattice points

For each reduced root $\beta$ of $(G, A_{M_0})$, we have the co-root $\beta^\vee$. Any such $\beta^\vee$ defines an element in the lattice

$$X_*(A_{M_0}) = \text{Hom}(X(A_{M_0}), \mathbb{Z})$$

in $a_{M_0}$. Suppose that $P \in \mathcal{P}(M)$ and that $\alpha$ is a root in $\Delta_P$. For any given $P_0 \in \mathcal{P}(M_0)$, with $P_0 \subset P$, there is a unique root $\beta \in \Delta_{P_0}$ whose restriction to $A_M$ equals $\alpha$; the “co-root” $\beta^\vee \in \Delta_P^\vee$ is, by definition, the projection of $\beta^\vee$ onto $a_M$. The lattice $\mathbb{Z}(\Delta_P^\vee)$ in $a_M^G$, generated by $\Delta_P^\vee$, is the projection of $\mathbb{Z}(\Delta_{P_0}^\vee)$ onto $a_M^G$. Since $\mathbb{Z}(\Delta_{P_0}^\vee)$ is independent of $P_0$, $\mathbb{Z}(\Delta_P^\vee)$ is independent of $P$. The lattice $\mathbb{Z}(\Delta_P^\vee)$ need not be contained in $X_*(A_M) = \text{Hom}(X(A_M), \mathbb{Z})$.

However, it is easily seen to be a subgroup of

$$\text{Hom}(X(M)_F, \mathbb{Z}),$$

which is in turn a finite extension of $X_*(A_M)$. Consequently, there is an integer $k$ such that $k\mathbb{Z}(\Delta_P^\vee)$ is a subgroup of $X_*(A_M)$.

Recall that

$$\exp(\langle H_M(m), \chi \rangle) = |\chi(m)|,$$

for any $\chi \in X(M)_F$ and $m \in M(F)$. It follows easily that $H_M(A_M(F))$ equals the lattice

$$\log(q_F)X_*(A_M)$$

in $a_M$, where $q_F$ is the degree of the residue field of $F$. Define

$$\Lambda_{M,k} = k \log(q_F)\mathbb{Z}(\Delta_P^\vee) = \log(q_F)\mathbb{Z}(\Delta_P^\vee)$$

for any $P \in \mathcal{P}(M)$ and any positive integer $k$. For any such $P$, the vectors

$$\mu_{\alpha,k} = k \log(q_F)\alpha^\vee, \quad \alpha \in \Delta_P,$$

form a $\mathbb{Z}$-basis of $\Lambda_{M,k}$. We fix $k$ so that $\Lambda_{M,k}$ is contained in $H_M(A_M(F))$. Set

$$A_{M,k} = \{a \in A_M(F); H_M(a) \in \Lambda_{M,k}\}.$$

Then

$$A'_{M,k} = A_{M,k}A_0(F)$$
is a subgroup of finite index in $A_M(F)$; it is this group which we will employ in the
formula (13). The first step will be to calculate the integral

$$\int_{A_{M,k}/A_G(F)} \sigma_M(H_M(a), \mathcal{Y}(x, \mathcal{F})) \, da.$$  

(14)

The kernel of $H_M$ in $A_{M,k}$ equals the group

$$\kappa_M = A_M(F) \cap K.$$

It follows easily that the quotient of $A_{M,k}/A_G(F)$ by $\kappa_M/\kappa_G$ is isomorphic under $H_M$ to $\Lambda_{M,k}$. We can therefore write (14) as the product of the volume of $\kappa_M/\kappa_G$ with the number of points in the intersection of $\Lambda_{M,k}$ with $S_M(\mathcal{Y}(x, \mathcal{F}))$. Consequently, (14) may be rewritten as

$$\text{vol}(\kappa_M/\kappa_G) \lim_{\lambda \to 0} \left\{ \sum \epsilon^{\mu(\xi)} \right\},$$

(15)

the sum being taken over $\xi$ in $\Lambda_{M,k} \cap S_M(\mathcal{Y}(x, \mathcal{F}))$. We shall calculate this by the method in ([1], §3).

Take $\lambda$ to be a point in $a^*_M$ whose real part $\lambda_\Re(\lambda)$ is regular. If $P \in \mathcal{P}(M)$, we shall write

$$\Delta_P = \{ \alpha \in \Delta_P : \lambda_\Re(\alpha^\vee) < 0 \}.$$

Let $\phi_P^\lambda$ denote the characteristic function of the set of $H \in a_M$ such that $\mu_\lambda(H) > 0$ for each $\alpha \in \Delta_P$, and $\mu_\lambda(H) \leq 0$ for any $\alpha$ in the complement of $\Delta_P$ in $\Delta_P$. (Recall that

$$\hat{\Delta}_P = \{ \mu_\lambda : \alpha \in \Delta_P \}$$

is the basis of $(a^*_M)^* \text{ which is dual to } \{ \alpha^\vee : \alpha \in \Delta_P \}.)$ It follows easily from Langlands' combinatorial lemma that

$$\sum_{P \in \mathcal{P}(M)} (-1)^{\lambda_\Re(\phi_P^\lambda(H - Y_P(x, \mathcal{F})))} = \text{vol}_{a_M}.$$ 

equals the characteristic function of $S_M(\mathcal{Y}(x, \mathcal{F})).$(See Lemma 3.2 of [1] for the special case that $H$ lies in the complement of a finite set of hyperplanes. The general case follows in the same way from [2], Lemma 6.3.) Therefore, the expression in the brackets in (15) equals

$$\sum_{\xi \in \Lambda_{M,k}} (-1)^{\lambda_\Re(\phi_P^\lambda(\xi - Y_P(x, \mathcal{F})))} \exp(\lambda(\xi)).$$

(16)

We shall write $Y_P^\lambda$ for the extreme point in

$$\{ \xi \in \Lambda_{M,k} : \phi_P^\lambda(\xi - Y_P(x, \mathcal{F})) = 1 \}.$$

(17)

That is,

$$Y_P^\lambda = Y_P(x, \mathcal{F}) + \sum_{\alpha \in \hat{\Delta}_P} t_\alpha \mu_{\alpha,k} - \sum_{\alpha \in \Delta_P - \Delta_P^\lambda} (1 - t_\alpha) \mu_{\alpha,k},$$

(18)
for positive numbers $t_a$, with $0 < t_a \leq 1$. The set (17) can then be written as

$$
\left\{ Y_p^+ + \sum_{x \in \Delta_p} n_a \mu_{t_x,k} \right\} = \sum_{x \in \Delta_p - \Delta_p} n_a \mu_{t_x,k},
$$

where each $n_a$ ranges over all positive integers. Expression (16) becomes a multiple geometric series, which equals

$$
(-1)^{\Delta_p} \exp [\lambda(Y_p^+)] \prod_{x \in \Delta_p} (1 - \exp [\lambda(\mu_{t_x,k})])^{-1} \prod_{x \in \Delta_p - \Delta_p} (1 - \exp [-\lambda(\mu_{t_x,k})])^{-1}.
$$

If $\lambda$ belongs to the negative chamber $- (a_+^*)$ of $P$ in $a_+^*$, we shall denote $Y_p^+$ simply by

$$
Y_p^+ = Y_p(x, \mathcal{F})^* = (T_p - H_p(x))^*.
$$

Then for general $\lambda$,

$$
Y_p^+ = Y_p^+ + \sum_{x \in \Delta_p - \Delta_p} \mu_{t_x,k}.
$$

Expression (16) may therefore be written as

$$
\exp [\lambda(Y_p^+)] \prod_{x \in \Delta_p} (\exp [\lambda(\mu_{t_x,k})] - 1)^{-1}.
$$

We have shown that (14) equals

$$
\text{vol}(\kappa_M/\kappa_G) \lim_{\lambda \to 0, P \in \mathcal{P}(M)} \sum_{x \in \Delta_p} (\exp [\lambda(Y_p^+)] \prod_{x \in \Delta_p} (\exp [\lambda(\mu_{t_x,k})] - 1)^{-1}).
$$

Let us rewrite this last formula for (14) as

$$
\text{vol}(\kappa_M/\kappa_G) \lim_{\lambda \to 0, P \in \mathcal{P}(M)} \sum_{x \in \Delta_p} c_p(\lambda, x, \mathcal{F}) d_p(\lambda) \theta_P(\lambda)^{-1},
$$

where

$$
c_p(\lambda, x, \mathcal{F}) = \exp [\lambda(Y_p^+)] = \exp [\lambda((T_p - H_p(x))^*)].
$$

and

$$
d_p(\lambda) = \theta_P(\lambda) \prod_{x \in \Delta_p} (\exp [\lambda(\mu_{t_x,k})] - 1)^{-1}.
$$

We leave the reader to check that

$$\{ Y_p^+: P \in \mathcal{P}(M) \}
$$

is a positive orthogonal set for $M$. This implies that $\{c_p(\lambda, x, \mathcal{F})\}$ is a $\mathcal{G}, M$ family, in the language of ([3], § 6). Moreover, $\{d_p(\lambda)\}$ is also a $\mathcal{G}, M$ family. Applying ([3], Lemma 6.3) to the product of $\mathcal{G}, M$ families in the expression above, we see that (14) equals

$$
\text{vol}(\kappa_M/\kappa_G) \sum_{Q \in \mathcal{P}(M)} c_Q^p(x, \mathcal{F}) d_Q.
$$
This follows the notation of ([3], §6). In particular,
\[ c^G_M(x, \mathcal{F}) = \lim_{\lambda \to 0} \sum_{P \in \mathcal{P}(M) : P \in Q} \exp \left[ \lambda \left( (T_P - H_P(x))^+ \right) \right] \theta_P(\lambda)^{-1}. \]

Next, we substitute the formula we have just established for (14) into the identity (13). We see that \( \Theta_n(f)\Theta_n(\gamma) \) equals
\[ \text{vol} (\kappa_M/\kappa_G) \sum_{Q \in \mathcal{F}(M)} d'_G \int_{A_{M,k}(G(F))} f(x^{-1} \gamma x)c^G_M(x, \mathcal{F})dx. \]
For any group \( Q \in \mathcal{F}(M) \) we have
\[ c^G_M(x, \mathcal{F}) = c^G_M(m_Q(x), \mathcal{F}). \]
It follows easily from this fact that
\[ \int_{A_{M,k}(G(F))} f(x^{-1} \gamma x)c^G_M(x, \mathcal{F})dx \]
is a multiple of
\[ \int_{N(F)} \int_{A_{M,k}(M(F))} f(k^{-1} m^{-1} \gamma mnk)c^G_M(m, \mathcal{F})dm dn dk. \]
Since \( f \) is a supercuspidal form, this expression vanishes for any \( Q \neq G \). Consequently,
\[ \Theta_n(f)\Theta_n(\gamma) = \text{vol} (\kappa_M/\kappa_G)d'_G \int_{A_{M,k}(G(F))} f(x^{-1} \gamma x)c_M(x, \mathcal{F})dx. \]
Now, by definition,
\[ d'_G = d'_G(0) = \lim_{\lambda \to 0} d_P(\lambda), \]
for any \( P \in \mathcal{P}(M) \). Therefore
\[ d'_G = \text{vol} \left( a^G_M/Z(\Lambda_P^c) \right)^{-1} \lim_{\lambda \to 0} \prod_{\alpha \in \Delta_P} (\lambda(\alpha^\vee)) \left( \exp \left[ \lambda(\mu_{\alpha,k}) \right] - 1 \right)^{-1} \]
\[ = \text{vol} \left( a^G_M/Z(\Lambda_P^c) \right)^{-1} \prod_{\alpha \in \Delta_P} (\lambda(\alpha^\vee)) \lambda(\mu_{\alpha,k})^{-1} \]
\[ = \text{vol} \left( a^G_M/\Lambda_{M,k} \right)^{-1}. \]
On the other hand, it follows from (2) that
\[ \text{vol}(\kappa_M/\kappa_G) = \text{vol}(a_M/H_M(A_M(F)) + a_G) \]
\[ = \text{vol}(a_M/\Lambda_{M,k} + a_G)|A_M(F)/A_{M,k}|^{-1} \]
\[ = \text{vol}(a^G_M/\Lambda_{M,k})|A_M(F)/A_{M,k}|^{-1}, \]
since the map
\[ A_M(F)/A_M' \rightarrow (H_M(A_M(F)) + a_G)/(A_M + a_G) \]
is an isomorphism. Our formula becomes
\[ \Theta_{\pi}(f) \Theta_{\gamma}(\gamma) = |A_M(F)/A'_M|^{-1} \int_{A_M'(G(F))} f(x^{-1} \gamma x) c_M(x, \mathcal{F}) \, dx. \] (19)
It is valid whenever \( \mathcal{F} \) satisfies the conditions (8)

5. Completion of the proof

The formula (19) is close to that of the theorem. The only problem is that it depends on \((T_p - H_{\beta}(x))^+\), rather than the vector \(T_p - H_{\beta}(x)\). To overcome this, we shall average \( \mathcal{F} \) over a certain compact domain.

Observe that \( \Lambda_{M,k} \) is the projection onto \( a_M \) of the lattice
\[ \Lambda_{M_0,k} = k \log(q) \mathbb{Z}(\Delta_{M_0}), \quad P_0 \in \mathcal{P}(M_0), \]
in \( a_{M_0} \). Choose an element \( P_0 \) in \( \mathcal{P}(M_0) \), and let \( \mathcal{D} \) denote the compact fundamental domain
\[ \left\{ u = \sum_{\beta \in \Lambda_{M_0}} u_{\beta} \mu_{\beta,k} : 0 \leq u_{\beta} \leq 1 \right\} \]
for \( \Lambda_{M_0,k} \) in \( a_{M_0} \). (Recall that \( \{ \mu_{\beta,k} \} \) is a basis of \( \Lambda_{M_0,k} \) consisting of positive multiples of the co-roots \( \Delta_{M_0} \)). Suppose that \( P \in \mathcal{P}(M) \). Then there is an element \( s \in W_0 \) such that \( P_0 = sP_0' \) contains \( P \). For each \( z \in \Delta_\beta \), let \( \beta(z) \) be the unique root in \( \Delta_{M_0} \) such that the restriction of \( s(z) \) onto \( a_M \) equals \( z \). Then \( \mu_{z,k} \) is the projection of \( s(\mu_{z(k)}, \Lambda_{M_0,k}) \) onto \( a_M \). Given a vector \( u \in \mathcal{D} \) as above, set
\[ u_{\beta} = \sum_{z \in \Delta_\beta} u_{\beta(z)} \mu_{z,k}. \]
This notation of course holds if \( M_0 \) is used instead of \( M \), and the set
\[ \mathcal{F}_u = \{ T_{P_0} - u_{P_0} : P_0 \in \mathcal{P}(M_0) \} \]
satisfies similar conditions to \( \mathcal{F} \). We may therefore replace \( c_M(x, \mathcal{F}) \) by \( c_M(x, \mathcal{F}_u) \) on the right hand side of (19).

Observe that
\[ c_p(\lambda, x, \mathcal{F}_u) = \exp[\lambda((T_p - u_{P_0} - H_{\beta}(x))^+)], \quad P \in \mathcal{P}(M). \]
Define
\[ c_p(\lambda, x, \mathcal{F}, u) = \exp[\lambda((T_p - u_{P_0} - H_{\beta}(x))^+ + u_{P_0})], \quad P \in \mathcal{P}(M). \]
so that
\[ c_p(\lambda, x, \mathcal{F}_u) = c_p(\lambda, x, \mathcal{F}, u) \exp[\lambda(u_{P_0})]. \]
This is a product of two \((G, M)\) families. We can therefore apply Lemma 6.3 of [3] to decompose \(c_M(x, \mathcal{F}_u)\) into a sum over \(Q \in \mathcal{F}(M)\). The second \((G, M)\) family is independent of \(x\). By arguing as in §4, we see that the contribution of any \(Q \neq G\) to the integral

\[
\int_{\lambda \in \mathcal{F}(M) \setminus \{G\}} f(x^{-1}yx) c_M(x, \mathcal{F}_u) \, dx
\]

vanishes. We may therefore replace \(c_M(x, \mathcal{F}_u)\) by \(c_M(x, \mathcal{F}, u)\), the term corresponding to \(Q = G\). Since this is valid for any \(u \in \mathcal{D}\), we may integrate over \(\mathcal{D}\) if we choose. It follows that (19) remains valid if the function \(c_M(x, \mathcal{F})\) is replaced by

\[
\int_{\mathcal{D}} c_M(x, \mathcal{F}, u) \, du.
\]

Now,

\[
\int_{\mathcal{D}} c_M(x, \mathcal{F}, u) \, du
\]

\[
= \int \lim_{\lambda \to 0} \sum_{p \in \mathcal{P}(M)} (c_p(\lambda, x, \mathcal{F}, u) \theta_p(\lambda)^{-1}) \, du
\]

\[
= \lim_{\lambda \to 0} \sum_{p \in \mathcal{P}(M)} \left( \int \sum_{\mathcal{D}} c_p(\lambda, x, \mathcal{F}, u) \, du \right) \theta_p(\lambda)^{-1}.
\]

Thus, we have only to compute

\[
\int_{\mathcal{D}} E((Y_p - u_p)^+ + u_p) \, du,
\]

where

\[
E((Y_p - u_p)^+ + u_p) = \exp \left[ \lambda((Y_p - u_p)^+ + u_p) \right],
\]

with

\[
Y_p = Y_p(x, \mathcal{F}) = T_p - H_p(x).
\]

This integral can be written as a multiple integral, over the cube

\[
\left\{ \prod_{a \in \mathcal{A}_p} r_a : 0 \leq r_a \leq 1 \right\},
\]

of the function

\[
E \left( \left( Y_p - \sum_{a} r_a \mu_{a,k} \right)^+ + \sum_{a} r_a \mu_{a,k} \right).
\]

Recall that \(Y_p^+\) is the unique point in \(\Lambda_{M,k} + a_G\) of the form

\[
Y_p + \sum_{a \in \mathcal{A}_p} t_a \mu_{a,k},
\]
where 0 < \( t_s \leq 1 \). Taking the integrals in \( r_s \) separately over the intervals \([0, 1 - t_s]\) and \([1 - t_s, 1]\), we can change variables; we obtain the integral over \( \{r_s\} \) of

\[
E \left( Y_p + \sum_s r_{sH_s} \right).
\]

It follows that (20) equals

\[
\int_{Q} E(Y_p + u_p)du.
\]

We have shown that

\[
\int_{Q} c_M(x, \mathcal{T}, u)du = \int_{Q} \tilde{v}_M(x, \mathcal{T}, u)du,
\]

where

\[
\tilde{v}_p(\lambda, x, \mathcal{T}, u) = \exp \left[ T_p + u_p - H_\lambda(x) \right]
\]

\[
= \exp \left[ -\lambda(H_\lambda(x)) \right] \exp \left[ \lambda(T_p + u_p) \right].
\]

This is again a product of \((G, M)\) families. We apply Lemma 6.3 of [3] once more, and decompose \( \tilde{v}_M(x, \mathcal{T}, u) \) into a sum over \( Q \in \mathcal{P}(M) \). Since the second \((G, M)\) family is independent of \( x \), the contribution of any \( Q \neq G \) to the integral

\[
\int_{\mathcal{A}_M \setminus \mathcal{G}(F)} f(x^{-1} \gamma x) \int_{Q} \tilde{v}_M(x, \mathcal{T}, u)du dx = \int_{\mathcal{A}_M \setminus \mathcal{G}(F)} \int_{\mathcal{A}_M \setminus \mathcal{G}(F)} f(x^{-1} \gamma x)\tilde{v}_M(x, \mathcal{T}, u)dx du
\]

vanishes. The term corresponding to \( Q = G \) is just \( \tilde{v}_M(x) \), where

\[
\tilde{v}_p(\lambda, x) = \exp \left[ -\lambda(H_\lambda(x)) \right], \quad P \in \mathcal{P}(M).
\]

This is of course independent of \( u \), so the integral over \( Q \) disappears. The formula (19) becomes

\[
\Theta_x(f)\Theta_x(\gamma) = |A_M(F)/A'_{M,k}|^{-1} \int_{\mathcal{A}_M \setminus \mathcal{G}(F)} f(x^{-1} \gamma x)\tilde{v}_M(x)dx.
\]

The \((G, M)\)-family \( \{\tilde{v}_p(\lambda, x)\} \) is slightly different from the original \((G, M)\) family

\[
v_p(\lambda, x) = \exp \left[ -\lambda(H_\lambda(x)) \right], \quad P \in \mathcal{P}(M).
\]

Observe, however, that

\[
\tilde{v}_M(x) = \lim_{\lambda \to \infty} \sum_{P \in \mathcal{P}(M)} \exp \left[ -\lambda(H_\lambda(x)) \right] \theta_P(\lambda)^{-1}
\]

\[
= (-1)^{\dim(A_M/A_\lambda)} \sum_{P \in \mathcal{P}(M)} \exp \left[ -\lambda(H_\lambda(x)) \right] \theta_P(\lambda)^{-1}
\]
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\[ (-1)^{\dim(A_M/A_G)} \sum_{\lambda \in \mathcal{P}(M)} \exp[-\lambda(H_p(x))] \theta_p(\lambda)^{-1} \]

\[ = (-1)^{\dim(A_M/A_G)} v_M(x), \]

since

\[ \theta_p(\lambda) = (-1)^{\dim(A_M/A_G)} \theta_p(\lambda). \]

In other words, \( \Theta_\alpha(f) \Theta_\gamma(\gamma) \) equals

\[ |A_M(F)/A'_M|^{-1} (-1)^{\dim(A_M/A_G)} \int_{A_M\backslash G(F)} f(x^{-1} \gamma x) v_M(x) \, dx. \]

Now, it is well known that the function \( v_M(x) \) is left invariant under \( M(F) \). In particular, the integrand is left invariant under \( A_M(F) \). We may therefore change the domain of integration to \( A_M(F) \backslash G(F) \), if we multiply by the index \( |A_M(F)/A'_M| \). We obtain the identity of \( \Theta_\alpha(f) \Theta_\gamma(\gamma) \) with

\[ (-1)^{\dim(A_M/A_G)} \int_{A_M(F) \backslash G(F)} f(x^{-1} \gamma x) v_M(x) \, dx. \]

This completes the proof of the theorem. \( \square \)

References