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Unified theorems involving *H*-function transform and Meijer Bessel function transform

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Abstract. A theorem which reveals an interesting relationship between originals of related functions in H-function transform and Meijer Bessel function transforms is established. Another theorem interconnecting the two Meijer Bessel function transforms is also obtained. These theorems seem to be capable of unifying and extending a large number of results obtained by earlier workers. Finally, we evaluate an integral by way of illustration.

Keywords. H-function transform; Meijer Bessel function transform.

1. Introduction

Theorem 1 established in this paper exhibits an interesting relationship between the originals of the related functions in H-function transform and the Meijer Bessel function transform. Theorem 2 exhibits the relationship between originals of related functions in the Meijer Bessel function transforms. The earlier theorems [3, 2, 8] follow as particular cases of our findings. Several useful integrals involving various special functions can be evaluated as an application of our findings. We evaluate here only one integral by way of illustration giving us the Laplace transform of the function involving $S_2(\frac{1}{4}\pm(\omega/2), \frac{1}{4}\pm(\mu/2); at^{1/2})$ which can find application in solving boundary value problems.

2. Results

The following results obtained from the known results [5] will be required in the sequel

(I)
$$K\left\{2^{(1/2)-c}x^{c}H^{m,n}_{p,q+2}\left[t\left(\frac{x}{2}\right)^{\sigma x}\middle|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q},\left(\frac{1}{4}-\frac{c}{2}\pm\frac{\omega}{2},\frac{\sigma}{2}\right)\end{matrix}\right];\omega;s\right\}$$
$$= s^{-c-1}H^{m,n}_{p,q}\left[ts^{-\sigma}\middle|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right] \tag{1}$$

provided

$$\sigma>0,\quad \mathrm{Re}(s)>0,\quad \min\left[\mathrm{Re}\left\{\frac{b_j}{\beta_j}(j=1,\ldots,m)+(c\pm\omega+\tfrac{3}{2})\right\}\right]>0$$

and

$$\sum_{j=1}^{m}\beta_j-\sum_{j=m+1}^{q}\beta_j+\sum_{j=1}^{n}\alpha_j-\sum_{j=n+1}^{p}\alpha_j-\sigma_x>0.$$

(II)
$$K\left\{2^{-c}x^cH_{0,4}^{2,0}\left[\frac{t^2}{4}\left(\frac{x}{2}\right)^{2\sigma}\middle|\begin{matrix}\text{-----}\\ \left(\frac{1}{4}\pm\frac{\mu}{2},1\right),\left(\frac{1}{4}-\frac{c}{2}\pm\frac{\omega}{2},\sigma\right)\end{matrix}\right];\omega;s\right\}$$

$$=s^{-c-1}(ts^{-\sigma})^{1/2}K_\mu(ts^{-\sigma}) \tag{2}$$

provided

$$0<\sigma\leqslant 1,\quad \mathrm{Re}(s)>0\quad\text{and}\quad \min[\mathrm{Re}\{\sigma(\tfrac{1}{2}\pm\mu)+(c\pm\omega+\tfrac{3}{2})\}]>0,$$

where the H-function involved in each of the above equations stands for the well-known Fox's H-function which is quite general in nature and includes most of the commonly used functions as its special cases. The definition of H-function in terms of Mellin–Barnes type of contour integral, the various conditions satisfied by its parameters for the convergence of the defining integral, some of its special cases, the important properties and the asymptotic expansion can be referred to in a standard work on this function [7]. It has been assumed that the various H-functions occurring in this paper satisfy the conditions corresponding appropriately to those given by (2.2.11) on page 13, in the book referred above.

3. Main theorems

THEOREM 1.
If

$$h(s)=H\{f(x);s\}=\int_0^\infty H_{p,q}^{m,n}\left[sx\middle|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]f(x)\,\mathrm{d}x \tag{3}$$

and

$$s^{-c-1}h(s^{-\sigma})=K\{g(x);\omega;s\}=\int_0^\infty(sx)^{1/2}K_\omega(sx)g(x)\,\mathrm{d}x, \tag{4}$$

then

$$g(x)=2^{(1/2)-c}x^c\int_0^\infty H_{p,q+2}^{m,n}\left[t\left(\frac{x}{2}\right)^{\sigma}\middle|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q},\left(\frac{1}{4}-\frac{c}{2}\pm\frac{\omega}{2},\frac{\sigma}{2}\right)\end{matrix}\right]f(t)\,\mathrm{d}t \tag{5}$$

provided that the integrals involved in (3) to (5) are absolutely convergent,

$$\sigma>0,\quad \mathrm{Re}(s)>0,\quad \min\left[\mathrm{Re}\left\{\frac{b_j}{\beta_j}(j=1,\ldots,m)+(c\pm\omega+\tfrac{3}{2})\right\}\right]>0$$

and

$$\sum_{j=1}^{m}\beta_j-\sum_{j=m+1}^{q}\beta_j+\sum_{j=1}^{n}\alpha_j-\sum_{j=n+1}^{p}\alpha_j-\sigma>0,$$

where the transform occurring in (3) was introduced by Gupta and Mittal [7] and is known as the H-function transform while the transform occurring in (4) was introduced by Meijer [6] and is known as the Meijer Bessel function transform.

Proof. From (3), we easily obtain

$$s^{-c-1}h(s^{-\sigma})=\int_0^\infty s^{-c-1}H_{p,q}^{m,n}\left[ts^{-\sigma}\middle|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]f(t)\,\mathrm{d}t. \tag{6}$$

Now with the help of (1) we get

$$\begin{aligned}s^{-c-1}h(s^{-\sigma})&=\int_0^\infty K\left\{2^{(1/2)-c}x^cH_{p,q+2}^{m,n}\right.\\&\quad\times\left.\left[t\left(\frac{x}{2}\right)^\sigma\middle|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q},\left(\frac{1}{4}-\frac{c}{2}\pm\frac{\omega}{2},\frac{\sigma}{2}\right)\end{matrix}\right];\omega;s\right\}f(t)\,\mathrm{d}t\\&=\int_0^\infty\left\{\int_0^\infty(sx)^{1/2}K_\omega(sx)\,2^{(1/2)-c}x^cH_{p,q+2}^{m,n}\right.\\&\quad\times\left.\left[t\left(\frac{x}{2}\right)^\sigma\middle|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q},\left(\frac{1}{4}-\frac{c}{2}\pm\frac{\omega}{2},\frac{\sigma}{2}\right)\end{matrix}\right]\mathrm{d}x\right\}f(t)\,\mathrm{d}t.\end{aligned} \tag{7}$$

On inverting the order of integration in (7) which is easily seen to be permissible under the conditions stated with the theorem, we get

$$\begin{aligned}s^{-c-1}h(s^{-\sigma})&=\int_0^\infty(sx)^{1/2}K_\omega(sx)\left\{\int_0^\infty 2^{(1/2)-c}x^cH_{p,q+2}^{m,n}\right.\\&\quad\times\left.\left[t\left(\frac{x}{2}\right)^\sigma\middle|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q},\left(\frac{1}{4}-\frac{c}{2}\pm\frac{\omega}{2},\frac{\sigma}{2}\right)\end{matrix}\right]f(t)\,\mathrm{d}t\right\}\mathrm{d}x.\end{aligned} \tag{8}$$

Now comparing (8) with (4), we get the required result (5).

Now, if we take both the transforms to be Meijer Bessel function transforms defined earlier by (4) and proceeding along the lines followed in Theorem 1 and using the result (2) we arrive at the following theorem.

THEOREM 2.

If

$$h(s)=K\{f(x);\mu;s\}=\int_0^\infty(sx)^{1/2}K_\mu(sx)f(x)\,\mathrm{d}x, \tag{9}$$

and

$$s^{-c-1}h(s^{-\sigma}) = K\{g(x);\omega;s\} = \int_0^\infty (sx)^{1/2}K_\omega(sx)g(x)\,\mathrm{d}x, \tag{10}$$

then

$$g(x) = 2^{-c}x^c \int_0^\infty H^{2,0}_{0,4}\left[\frac{t^2}{4}\left(\frac{x}{2}\right)^{2\sigma}\middle| \begin{matrix} \text{-----} \\ \left(\frac{1}{4}\pm\frac{\mu}{2},1\right),\left(\frac{1}{4}-\frac{c}{2}\pm\frac{\omega}{2},\sigma\right)\end{matrix}\right] f(t)\,\mathrm{d}t \tag{11}$$

provided that the integrals involved in (9) to (11) are absolutely convergent, $0<\sigma\leqslant 1$, $\mathrm{Re}(s)>0$ and $\min[\mathrm{Re}\{\sigma(\frac{1}{2}\pm\mu)+(c\pm\omega+\frac{3}{2})\}]>0$.

Special cases. If in Theorem 2, $\sigma=(n/s)$, then by simplifying and using a known result of [7], we get a known theorem [3]. Also, substituting $\mu=\omega=\frac{1}{2}$ in Theorem 2 and simplifying we get a theorem of [2]. Further, if $\sigma=1$ and $\mu=\omega=\frac{1}{2}$ in Theorem 2, we get a well-known Tricomi's theorem [8].

4. Application

In Theorem 2, take $\sigma=1$, $c=0$ and let

$$h(s) = \tfrac{1}{2}\alpha^k s^{-1/2}\Gamma\left(\frac{1+\mu}{2}-k\right)\Gamma\left(\frac{1-\mu}{2}-k\right)\exp\left(\frac{s^2}{8\alpha}\right)W_{k,\frac{\mu}{2}}\left(\frac{s^2}{4\alpha}\right). \tag{12}$$

From the known result [1], we have

$$\tfrac{1}{2}\alpha^k s^{-1/2}\Gamma\left(\frac{1+\mu}{2}-k\right)\Gamma\left(\frac{1-\mu}{2}-k\right)\exp\left(\frac{s^2}{8\alpha}\right)W_{k,\frac{\mu}{2}}\left(\frac{s^2}{4\alpha}\right)$$
$$= K\{x^{-(1/2)-2k}\exp(-\alpha x^2);\mu;s\}, \tag{13}$$

where $\mathrm{Re}(\alpha)>0$, $2\,\mathrm{Re}(k)<1-|\mathrm{Re}(\mu)|$.

Now interpreting (12) and (13) in conjunction with (9), we get

$$f(x) = x^{-(1/2)-2k}\exp(-\alpha x^2). \tag{14}$$

Again, from (12) we have

$$s^{-1}h(s^{-1}) = \tfrac{1}{2}\alpha^k s^{-1/2}\Gamma\left(\frac{1+\mu}{2}-k\right)\Gamma\left(\frac{1-\mu}{2}-k\right)$$
$$\times\exp\left(\frac{s^{-2}}{8\alpha}\right)W_{k,\frac{\mu}{2}}\left(\frac{s^{-2}}{4\alpha}\right). \tag{15}$$

Also from another known result [4], by taking $p=s$, $\sigma=\frac{1}{2}$, $n=s=1$, $a=1/4\alpha$, $m=\frac{1}{2}\mu$ therein, we get after some simplification

$$\tfrac{1}{2}\alpha^k s^{-1/2}\Gamma(\tfrac{1}{2}-k+\tfrac{1}{2}\mu)\Gamma(\tfrac{1}{2}-k-\tfrac{1}{2}\mu)\exp\left(\frac{s^{-2}}{8\alpha}\right)W_{k,\frac{\mu}{2}}\left(\frac{s^{-2}}{4\alpha}\right)$$

$$= K\left\{\alpha^k x^{-1/2} G_{1,4}^{2,1}\left[\frac{x^2}{16\alpha}\left|\begin{matrix}1+k\\ \frac{1\pm\mu}{2},\frac{1\pm\omega}{2}\end{matrix}\right.\right]\omega; s\right\}, \tag{16}$$

where $\mathrm{Re}(2\pm\mu\pm\omega)>0$, $\mathrm{Re}(s)>0$. Now interpreting (15) and (16) in conjunction with (10), we get

$$g(x) = \alpha^k x^{-1/2} G_{1,4}^{2,1}\left[\frac{x^2}{16\alpha}\left|\begin{matrix}1+k\\ \frac{1\pm\mu}{2},\frac{1\pm\omega}{2}\end{matrix}\right.\right]. \tag{17}$$

Finally, from (11), (14) and (17), we get after some simplification the following integral

$$\int_0^\infty t^{-(3/2)-2k}\exp(-\alpha t^2) S_2\left(\frac{1}{4}\pm\frac{\mu}{2},\frac{1}{4}\pm\frac{\omega}{2};\frac{tx}{4}\right)\mathrm{d}t$$

$$= \frac{\alpha^k}{4} x^{1/2} G_{1,4}^{2,1}\left[\frac{x^2}{16\alpha}\left|\begin{matrix}1+k\\ \frac{1\pm\mu}{2},\frac{1\pm\omega}{2}\end{matrix}\right.\right], \tag{18}$$

where $2\,\mathrm{Re}(k) < 1-|\mathrm{Re}(\mu)|$, $\mathrm{Re}(\alpha)>0$.

If in the above integral, we take $\alpha = p, (x/4) = a$ and change the variable of integration slightly, we obtain the following result

$$L\left\{t^{-k-(5/4)} S_2\left(\frac{1}{4}\pm\frac{\mu}{2},\frac{1}{4}\pm\frac{\omega}{2}; at^{1/2}\right); p\right\}$$

$$= \int_0^\infty t^{-k-(5/4)} S_2\left(\frac{1}{4}\pm\frac{\mu}{2},\frac{1}{4}\pm\frac{\omega}{2}; at^{1/2}\right)\exp(-pt)\,\mathrm{d}t$$

$$= a^{1/2} p^k G_{1,4}^{2,1}\left[\frac{a^2}{p}\left|\begin{matrix}1+k\\ \frac{1\pm\mu}{2},\frac{1\pm\omega}{2}\end{matrix}\right.\right], \tag{19}$$

where $2\,\mathrm{Re}(k) < 1-|\mathrm{Re}(\mu)|$, $\mathrm{Re}(p)>0$. The transform occurring in (19) is the well-known Laplace transform.

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