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Asymptotic properties of solutions of difference equations

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Abstract. Sufficient conditions for some m-th order finite difference equations are presented which have a solution behaving in a precisely specified way like a given polynomial.

Keywords. Difference equation; asymptotic behaviour.

1. Introduction

In this paper we examine the asymptotic behaviour of solutions of certain classes of m-th order difference equations. Motivated with the work of Trench [9] on differential equations we give conditions which imply that the equation

$$\Delta^m y_n + f(n,y_n) = 0 \quad m \geq 1, n \in N \tag{E}$$

has a solution which behaves like a given polynomial of degree $< m$ as $n \to \infty$.

Some asymptotic properties of solutions of second order difference equations were considered in [3–5, 8]. Results similar to those contained in this paper were presented in [1], [2], [6], [7]. Here $y_n = y(n), R\colon = (-\infty, \infty)$, $R_0\colon = [0, \infty)$, $R_+\colon = (0, \infty)$, $N\colon = \{n_0, n_0 + 1, \ldots\}$, n_0 is a given non-negative integer. For a function $x\colon N \to R$, we define the difference operators Δ^i as follows

$$\Delta^0 x_n = x_n,\ \Delta^k x_n = \Delta(\Delta^{k-1} x_n) = \Delta^{k-1} x_{n+1} - \Delta^{k-1} x_n,\ k \geq 1.$$

We write $x_n = O(z_n)$ and $x_n = o(z_n)$ to indicate that

$$\overline{\lim_{n\to\infty}} \left|\frac{x_n}{z_n}\right| < \infty \text{ or } \lim_{n\to\infty} \frac{x_n}{z_n} = 0$$

respectively. By a solution of (E) we mean any function x defined on N, which fulfils (E) for all sufficiently large n. Note that the above definition of the solution is different from this where x fulfils (E) for all $n \in N$.

2. Main result

THEOREM

Let $q:N \to R_+$ be nonincreasing, p be a given polynomial of degree $<m$ and suppose that there is a constant M such that the functions $f(n,.)$, $n \in N$ are continuous on the sets

$$U_n := \{u: |u-p_n| \leq Mq_n\}$$

and

$$|f(n,u)-f(n,v)| \leq g_n|u-v|, \tag{1}$$

if $u,v \in U_n$, where $g:N \to R_0$ and

$$\sum_{j=n_0}^{\infty} j^{m-1} g_j q_j \tag{2}$$

is convergent,

$$\overline{\lim_{n\to\infty}} \frac{1}{q_n} \sum_{j=n}^{\infty} j^{m-1} g_j q_j = c_1 < (m-1)!. \tag{3}$$

Suppose also that

$$\sum_{j=n_0}^{\infty} j^{m-1} f(j,p_j) \tag{4}$$

converges—perhaps conditionally—and

$$\overline{\lim_{n\to\infty}} \frac{1}{q_n} \left|\sum_{j=n}^{\infty} f(j,p_j)\right| = c_2, \tag{5}$$

where

$$c_2 + c_1 M < M(m-1)!. \tag{6}$$

Then (E) has a solution x which satisfies the asymptotic relation

$$\Delta^i x_n = \Delta^i p_n + O(n^{-i} q_n), \; O \leq i \leq m-1. \tag{AR}$$

Proof. Let $m(q)$ be the Banach space of sequences $h:N \to R$ such that $h_n = O(q_n)$ with thc norm

$$||h|| = \sup_{n \geq n_0} \{ |h_n|/q_n \}. \tag{7}$$

Let

$$m_M(q) := \{h \in m(q): ||h|| \leq M\}.$$

Take $\varepsilon > 0$ such that

$$c_2 + c_1 M + \varepsilon < (m-1)!M, \quad c_1 + \frac{\varepsilon}{2M} < (m-1)!. \tag{8}$$

By (3) and (5) there exist $n_1, n_2 \in N$ such that

$$\frac{1}{q_n} \sum_{j=n}^{\infty} j^{m-1} g_j q_j \le c_1 + \frac{\varepsilon}{2M} = c_3 \tag{9}$$

for all $n \ge n_1$, and

$$\frac{1}{q_n} \left| \sum_{j=n}^{\infty} f(j,p_j) \right| \le c_2 + \frac{\varepsilon}{2} = c_4 \tag{10}$$

for all $n \ge n_2$. Take $n_3 = \max\{n_1, n_2, m-1\}$ and define operator T, $\hat{h} = Th$ by

$$\hat{h}_n = \begin{cases} 0 & \text{for } n < n_3 \\ (-1)^{m-1} \sum_{j=n}^{\infty} \dfrac{(j+m-1-n)^{(m-1)}}{(m-1)!} f(j,p_j+h_j) & \text{for } n \ge n_3 \end{cases} \tag{11}$$

for $h \in m_M(q)$, where by $(n)^{(k)}$ we indicate product $n(n-1)\ldots(n-k+1)$ for $k \ge 1$, $(n)^{(0)} = 1$.

We will show that T is a contraction mapping of $m_M(q)$ into itself. Let $h \in m_M(q)$. We must first prove that the series in (11) converges. Denote

$$I(n,h) = \sum_{j=n}^{\infty} j^{m-1} f(j,p_j+h_j). \tag{12}$$

We note that assumption (4) implies that

$$I(n,O) = \sum_{j=n}^{\infty} j^{m-1} f(j,p_j) \tag{13}$$

converges. By virtue of (1), (2) and (7) we get for any $u, v \in m_M(q)$

$$\sum_{j=n}^{\infty} |j^{m-1}[f(j,p_j+u_j) - f(j,p_j+v_j)]| \le \sum_{j=n}^{\infty} j^{m-1} g_j |u_j - v_j|$$

$$\le ||u-v|| \sum_{j=n}^{\infty} j^{m-1} g_j q_j \le c_5 - \text{constant}.$$

Therefore the series

$$\sum_{j=n}^{\infty} j^{m-1}[f(j,p_j+u_j) - f(j,p_j+v_j)] \tag{15}$$

converges for any $u, v \in m_M(q)$. Taking $u = h$, $v \equiv 0$, by convergence of the series

(13) and (15) we obtain convergence of (12) for arbitrary $h \in m_M(q)$. It should be observed that the sequence

$$\left\{\frac{(j+m-1-n)^{(m-1)}}{j^{m-1}}\right\}_{j=n}^{\infty} = \left\{\prod_{k=0}^{m-2}\left(1+\frac{(m-1-n-k)}{j}\right)\right\}_{j=n}^{\infty}$$

is bounded and increasing.

Furthermore

$$\lim_{j\to\infty}\prod_{k=0}^{m-2}\left(1+\frac{(m-1-n-k)}{j}\right) = 1. \tag{16}$$

Using this fact and the convergence of (12), by Abel's test we obtain the convergence of the series

$$\sum_{j=n}^{\infty}\frac{(j+m-1-n)^{(m-1)}}{j^{m-1}}\, j^{m-1} f(j,p_j+h_j)$$

$$= \sum_{j=n}^{\infty}(j+m-1-n)^{(m-1)} f(j,p_j+h_j). \tag{17}$$

We shall estimate $\hat{h}_n$ for $n \geq n_3$. Differentiating (12) we obtain

$$\Delta I(n,h) = -n^{m-1} f(n,p_n+h_n). \tag{18}$$

On account of (18), summing by parts we have

$$\sum_{j=n}^{t}(j+m-1-n)^{(m-1)} f(j,p_j+h_j)$$

$$= -\sum_{j=n}^{t}(j+m-1-n)^{(m-1)}\frac{\Delta I(j,h)}{j^{m-1}}$$

$$= -\sum_{j=n}^{t}\left[\prod_{k=0}^{m-2}\left(1+\frac{(m-1-n-k)}{j}\right)\right]\Delta I(j,h)$$

$$= -\left[\prod_{k=0}^{m-2}\left(1+\frac{(m-1-n-k)}{t}\right)\right] I(t+1,h)$$

$$+\left[\prod_{k=0}^{m-2}\left(1+\frac{(m-1-n-k)}{n}\right)\right] I(n,h)$$

$$+\sum_{j=n}^{t-1} I(j+1,h)\,\Delta\prod_{k=0}^{m-2}\left(1+\frac{(m-1-n-k)}{j}\right).$$

Passing with t to infinity, by (16), convergence of the series (17), and

$$\lim_{t\to\infty} I(t+1,h) = 0$$

we get convergence of the series

$$\sum_{j=n}^{\infty} I(j+1,h)\Delta \prod_{k=0}^{m-2}\left(1+\frac{(m-1-n-k)}{j}\right)$$

and equality

$$= \sum_{j=n}^{\infty} (j+m-1-n)^{(m-1)} f(j,p_j+h_j)$$

$$= \frac{(m-1)!I(n,h)}{n^{m-1}}$$

$$+ \sum_{j=n}^{\infty} I(j+1,h)\Delta \prod_{k=0}^{m-2}\left(1+\frac{m-1-n-k}{j}\right). \tag{19}$$

Hence we see that to obtain an estimation of (17) we need any bound of $I(n,h)$. To this we denote

$$s_n = \sup_{k\geq n}\frac{1}{q_k}\left[M\sum_{j=k}^{\infty} j^{m-1}g_jq_j + \left|\sum_{j=k}^{\infty} j^{m-1}f(j,p_j)\right|\right]. \tag{20}$$

Therefore by (14) and (20) we get

$$|I(n,h)| = \left|\sum_{j=n}^{\infty} j^{m-1}[f(j,p_j+h_j)-f(j,p_j)] + \sum_{j=n}^{\infty} j^{m-1}f(j,p_j)\right|$$

$$\leq \|h\|\sum_{j=n}^{\infty} j^{m-1}g_jq_j + \left|\sum_{j=n}^{\infty} j^{m-1}f(j,p_j)\right| \leq q_n\left\{\frac{1}{q_n}\left[M\sum_{j=n}^{\infty} j^{m-1}g_jq_j\right.\right.$$

$$\left.\left.+\left|\sum_{j=n}^{\infty} j^{m-1}f(j,p_j)\right|\right]\right\}.$$

$$\leq q_ns_n \text{ for } n\geq n_0. \tag{21}$$

Applying (8), (9), and (10) to (20) yields

$$s_n = M\sup_{k\geq n}\frac{1}{q_k}\sum_{j=k}^{\infty} j^{m-1}g_jq_j + \sup_{k\geq n}\frac{1}{q_k}\left|\sum_{j=k}^{\infty} j^{m-1}f(j,p_j)\right|$$

$$\leq M\left(c_1+\frac{\varepsilon}{2M}\right)+c_2+\frac{\varepsilon}{2} < (m-1)!M. \tag{22}$$

So by (19), (21), (22) we can estimate $|\hat{h}_n|$ as follows

$$|\hat{h}_n| = \left|\sum_{j=n}^{\infty} \frac{(j+m-1-n)^{(m-1)}}{(m-1)!} f(j, p_j+h_j)\right|$$

$$= \frac{1}{(m-1)!} \left| \frac{(m-1)!I(n,h)}{n^{m-1}} \right.$$

$$\left. + \sum_{j=n}^{\infty} I(j+1,h)\Delta \prod_{k=0}^{m-2} \left(1+\frac{(m-1-n-k)}{j}\right)\right|$$

$$\leq \frac{1}{(m-1)!} \left[\frac{(m-1)!q_n s_n}{n^{m-1}} + \sum_{j=n}^{\infty} q_{j+1}s_{j+1}\Delta \prod_{k=0}^{m-2}\left(1+\frac{m-1-n-k}{j}\right)\right]$$

$$\leq \frac{1}{(m-1)!} \left[\frac{(m-1)!q_n s_n}{n^{m-1}} + q_n s_n \sum_{j=n}^{\infty} \Delta \prod_{k=0}^{m-2}\left(1+\frac{m-1-n-k}{j}\right)\right].$$

Since

$$\sum_{j=n}^{\infty} \Delta \prod_{k=0}^{m-2}\left(1+\frac{m-1-n-k}{j}\right) = \lim_{t\to\infty}\left[\prod_{k=0}^{m-2}\left(1+\frac{m-1-n-k}{t+1}\right)\right.$$

$$\left. - \prod_{k=0}^{m-2}\left(1+\frac{m-1-n-k}{n}\right)\right]$$

$$= 1-\frac{(m-1)!}{n^{m-1}}$$

then

$$|\hat{h}_n| \leq \frac{1}{(m-1)!}[s_n q_n] < q_n M, \quad n \geq n_0. \tag{23}$$

Hence we infer $\hat{h} \in m_M(q)$. This means that the operator T defined by (11) maps $m_M(q)$ into itself. We shall show that T is a contraction mapping. Assume $u, v \in m_M(q)$ and $\hat{u} = Tu$, $\hat{v} = Tv$. Then by (1), (7) taking into account that $j \geq n \geq n_3 \geq m-1$ and $(j)^{(k)} \leq j^k$ we deduce

$$|\hat{u}_n - \hat{v}_n| = \frac{1}{(m-1)!}\left|\sum_{j=n}^{\infty}(j+m-1-n)^{(m-1)}f(j,p_j+u_j)\right.$$

$$\left. - \sum_{j=n}^{\infty}(j+m-1-n)^{(m-1)}f(j,p_j+v_j)\right|$$

$$\leq \frac{1}{(m-1)!}\sum_{j=n}^{\infty}(j+m-1-n)^{(m-1)}|f(j,p_j+u_j)-f(j,p_j+v_j)|$$

$$\le \frac{1}{(m-1)!}\sum_{j=n}^{\infty} j^{(m-1)} g_j |u_j - v_j| \le \frac{1}{(m-1)!}\sum_{j=n}^{\infty} j^{m-1} g_j q_j \sup_{t \ge n_0} \frac{|u_t - v_t|}{q_t}$$

$$= ||u-v|| \frac{1}{(m-1)!}\sum_{j=n}^{\infty} j^{m-1} g_j q_j .$$

Hence by (8) and (9) we have

$$\sup_{n \ge n_0} \frac{|\hat{u}_n - \hat{v}_n|}{q_n} \le \frac{||u-v||}{(m-1)!} \sup_{n \ge n_3} \frac{1}{q_n} \sum_{j=n}^{\infty} j^{m-1} g_j q_j .$$

$$\le ||u-v|| \frac{c_1 + (\varepsilon/2M)}{(m-1)!} = \theta ||u-v||$$

where $\theta \in (0,1)$.

The above inequality yields

$$||\hat{u} - \hat{v}|| \le \theta ||u-v||,$$

where $\theta \in (0,1)$. Consequently, there is $w \in m_M(q)$ such that $w = Tw$. We find the equation for which w is a solution and next we find the solution x of (E) by means of w. Since the series $\hat{h}_n$ and $\hat{h}_{n+1}$ converge so do the series $\Delta \hat{h}_n = \hat{h}_{n+1} - \hat{h}_n$ and

$$\Delta \hat{h}_n = (-1)^{m-1} \sum_{j=n+1}^{\infty} \frac{(j+m-1-n-1)^{(m-1)}}{(m-1)!} f(j, p_j + h_j)$$

$$-(-1)^{m-1} \sum_{j=n}^{\infty} \frac{(j+m-1-n)^{(m-1)}}{(m-1)!} f(j, p_j + h_j)$$

$$= (-1)^{m-1}\left[\sum_{j=n}^{\infty} \frac{(j+m-1-n-1)^{(m-1)}}{(m-1)!} f(j, p_j + h_j) - \sum_{j=n}^{\infty} \frac{(j+m-1-n)^{(m-1)}}{(m-1)!} f(j, p_j + h_j)\right]$$

$$= (-1)^{m} \sum_{j=n}^{\infty} \frac{(j+m-1-n)^{(m-1)} - (j+m-2-n)^{(m-1)}}{(m-1)!} f(j, p_j + h_j)$$

$$= (-1)^{m} \sum_{j=n}^{\infty} \frac{(j+m-2-n)^{(m-2)}}{(m-2)!} f(j, p_j + h_j).$$

Repeating the above reasoning we obtain convergence of $\Delta^r \hat{h}_n$ for $0 \le r \le m-1$ and

$$\Delta^r \hat{h}_n = (-1)^{m-1+r} \sum_{j=n}^{\infty} \frac{(j+m-1-n-r)^{(m-r-1)}}{(m-r-1)!} \times$$

$$f(j, p_j+h_j),\ 0 \le r \le m-1. \tag{24}$$

For $r = m-1$, (24) is

$$\Delta^{m-1}\hat{h}_n = \sum_{j=n}^{\infty} f(j, p_j+h_j)$$

We thus get

$$\Delta^m \hat{h}_n = \sum_{j=n+1}^{\infty} f(j, p_j+h_j) - \sum_{j=n}^{\infty} f(j, p_j+h_j)$$

$$= -f(n, p_n+h_n). \tag{25}$$

For $h = w$ we get from (25)

$$\Delta^m w_n = -f(n, p_n+w_n). \tag{26}$$

Setting $x_n = P_n + w_n$, recalling $\Delta^m p_n = 0$, (26) yields

$$\Delta^m x_n = -f(n, x_n)$$

i.e. x is the solution of (E) for $n \ge n_3$. To conclude the proof, we shall concentrate on relation (AR). For this we need any bound of $\Delta^r w_n$. From (24) and (18)

$$\Delta^r w_n = (-1)^{m+r} \sum_{j=n}^{\infty} \frac{(j+m-1-n-r)^{(m-r-1)}}{(m-r-1)!} \frac{\Delta I(j,w)}{j^{m-1}}$$

$$= \frac{(-1)^{m+r}}{(m-r-1)!} \sum_{j=n}^{\infty} \frac{(j+m-1-n-r)^{(m-r-1)}}{j^{m-r-1}} j^{-r} \Delta I(j,w)$$

$$= \frac{(-1)^{m+r}}{(m-r-1)!} \sum_{j=n}^{\infty} j^{-r} \left[\prod_{k=1}^{m-r-1} \left(1+\frac{k-n}{j}\right) \right] \Delta I(j,w).$$

Since

$$\sum_{j=n}^{t} j^{-r} \left[\prod_{k=1}^{m-r-1} \left(1+\frac{k-n}{j}\right) \right] \Delta I(j,w)$$

$$= t^{-r} \left[\prod_{k=1}^{m-r-1} \left(1+\frac{k-n}{t}\right) \right] I(t+1,w)$$

$$-n^{-r} \prod_{k=1}^{m-r-1} \left(1+\frac{k-n}{n}\right) I(n,w)$$

$$-\sum_{j=n}^{t-1} I(j+1,w)\Delta\left[j^{-r}\prod_{k=1}^{m-r-1}\left(1+\frac{k-n}{j}\right)\right].$$

By using arguments similar to (17) we get

$$\Delta^r w_n = \frac{(-1)^{m+r}}{(m-r-1)!}\left\{-n^{-r}\frac{(m-r-1)!}{n^{m-r-1}} I(n,w)\right.$$

$$\left.-\sum_{j=n}^{\infty} I(j+1,w)\Delta\left[j^{-r}\prod_{k=1}^{m-r-1}\left(1+\frac{k-n}{j}\right)\right]\right\}.$$

$$= \frac{(-1)^{m+r+1}}{(m-r-1)!}\left\{\frac{(m-r-1)!}{n^{m-1}} I(n,w)\right.$$

$$+\sum_{j=n}^{\infty} I(j+1,w)\left[(j+1)^{-r}\Delta\prod_{k=1}^{m-r-1}\left(1+\frac{k-n}{j}\right)\right.$$

$$\left.\left.+\left(\prod_{k=1}^{m-r-1}\left(1+\frac{k-n}{j}\right)\right)\Delta j^{-r}\right]\right\}.$$

Hence, by virtue of (21)

$$|\Delta^r w_n| \le \frac{1}{(m-r-1)!}\left\{\frac{(m-r-1)!}{n^{m-1}}|I(n,w)|\right.$$

$$+\sum_{j=n}^{\infty}|I(j+1,w)|\left|\left[(j+1)^{-r}\Delta\prod_{k=1}^{m-r-1}\left(1+\frac{k-n}{j}\right)\right.\right.$$

$$\left.\left.\left.+\left(\prod_{k=1}^{m-r-1}\left(1+\frac{k-n}{j}\right)\right)\Delta j^{-r}\right]\right|\right\}$$

$$\le \frac{1}{(m-r-1)!}\left\{\frac{(m-r-1)!}{n^{m-1}} q_n s_n\right.$$

$$+q_n s_n\sum_{j=n}^{\infty}(j+1)^{-r}\Delta\prod_{k=1}^{m-r-1}\left(1+\frac{k-n}{j}\right)$$

$$\left.\left.+\left(\prod_{k=1}^{m-r-1}\left(1+\frac{k-n}{j}\right)\right)\Delta\left(-j^{-r}\right)\right]\right\}.$$

But

$$\lim_{t\to\infty}\left\{\sum_{j=n}^{t}(j+1)^{-r}\Delta\prod_{k=1}^{m-r-1}\left(1+\frac{k-n}{j}\right)\right.$$

$$+\sum_{j=n}^{t}\left[\prod_{k=1}^{m-r-1}\left(1+\frac{k-n}{j}\right)\right]\Delta\left(-j^{-r}\right)\Bigg\}$$

$$\leq \lim_{t\to\infty}\left\{\frac{1}{(n+1)^r}\left[\prod_{k=1}^{m-r-1}\left(1+\frac{k-n}{t+1}\right)-\prod_{k=1}^{m-r-1}\left(1+\frac{k-n}{n}\right)\right]\right.$$

$$\left.+\sum_{j=n}^{t}\Delta\left(-j^{-r}\right)\right\}=\frac{1}{(n+1)^r}\left[1-\frac{(m-r-1)!}{n^{m-r-1}}\right]+\frac{1}{(n+1)^r}$$

$$\leq\frac{1}{n^r}\left[1-\frac{(m-r-1)!}{n^{m-r-1}}\right]+\frac{1}{n^r}=\frac{2}{n^r}-\frac{(m-r-1)!}{n^{m-1}};$$

therefore

$$|\Delta^r w_n|\leq\frac{1}{(m-r-1)!}\left[\frac{(m-r-1)!}{n^{m-1}}q_ns_n+\frac{2}{n^r}q_ns_n\right.$$

$$\left.-\frac{(m-r-1)!}{n^{m-1}}q_ns_n\right]=\frac{2}{m-r-1!}\frac{q_ns_n}{n^r}. \quad (27)$$

For $r = m-1$, $\Delta^{m-1}w_n$ is estimated as follows

$$|\Delta^{m-1}w_n|=\left|\sum_{j=n}^{\infty}\frac{\Delta I(j,w)}{j^{m-1}}\right|\leq\frac{2}{n^{m-1}}q_ns_n. \quad (28)$$

Equation (27) which hold for $O<r<m-1$, (28) together with (23) gives us

$$|\Delta^r w_n|\leq\frac{2(m-1)!M}{(m-r-1)!}q_nn^{-r},\ O<r\leq m-1. \quad (29)$$

Recalling $x_n = p_n+w_n$, by (23), and (29) we obtain

$$\Delta^r x_n=\Delta^r p_n+O(n^{-r}q_n),\ O\leq r\leq m-1. \qquad \text{Q.E.D.}$$

3. Remarks and examples

Remark 1. We have noticed that a solution x of (E), for which asymptotic relation (AR) holds, exists for n sufficiently large. For many applications we need the solution for all $n\in N$. We rewrite equation (E) in the equivalent form

$$(-1)^m y_n+f(n,y_n)=-\sum_{i=1}^{m}\binom{m}{i}(-1)^{m-i}y_{n+i}. \quad (30)$$

From this it is evident that if the functions z_n defined by

$$z_n(x):=(-1)^m x+f(n,x),\ n_0\leq n<n_3$$

are surjections of R onto R, then there exist a solution x of (E) defined for all $n \in N$ which satisfies the asymptotic relation (AR). To this we put in the equality (30) $n = n_3 - 1, y_{n+i} = x_{n_3+i-1}$, $i = 1, \ldots, m$, whose existence is ensured by the theorem. The solution may not be uniquely denoted by x_{n_3-1}. Repeating this way we find in succession $x_{n_3-2}, \ldots, x_{n_0}$. This solution satisfies (AR).

Remark 2. If $c_1 = c_2 = 0$, then by (20) $\lim_{n\to\infty} s_n = 0$. Therefore from (23), (27), and (28) we have $\lim_{n\to\infty} \Delta^r w_n = 0$, $0 \le r \le m-1$. Hence the solution x of (E) satisfies the asymptotic relation

$$\Delta^r x_n = \Delta^r p_n + o(n^{-r} q_n). \tag{AR1}$$

Remark 3. If instead of (2) we have

$$\sum_{j=n_0}^{\infty} j^{m-1} g_j < \infty \tag{31}$$

then (3) holds for any nonincreasing q, with $c_1 = 0$. If (2) holds and $\lim_{n\to\infty} q_n \neq 0$ then (31) is fulfilled. Furthermore by (4) it follows from (5) that $c_2 = 0$. So in this case the solution x has the asymptotic behaviour (AR1).

Remark 4. In some distances we may take M as arbitrarily large. Then (6) is no restriction.

Example. Suppose

$$\sum_{j=n_0}^{\infty} j^{2m-2} a_j \text{ and } \sum_{j=n_0}^{\infty} j^{m-1} b_j \tag{32}$$

converge—perhaps conditionally—and

$$\sum_{j=n_0}^{\infty} j^{m-1} |a_j| \tag{33}$$

converges. Let p be any polynomial of degree $< m$. Then the theorem implies that the equation

$$\Delta^m y_n + a_n y_n = b_n \tag{E1}$$

has a solution x such that

$$\Delta^r x_n = \Delta^r p_n + o(n^{-r}), \quad 0 \le r \le m-1. \tag{34}$$

To see this take $q \equiv 1$ and $f(n,u) = a_n u - b_n$. Then (1) holds with $g_n = |a_n|$. Hence by (32), (33) the assumptions (2) to (6) are satisfied with $c_1 = c_2 = 0$. Therefore, by Remark 2 we get the existence of a solution x of (E1) for which (34) holds.

As the second example we consider the equation

$$\Delta^m y_n + a_n (y_n)^s = b_n, \ s \geq 1. \tag{E2}$$

If

$$\sum_{j=n_0}^{\infty} j^{m-1}|a_j| \text{ and } \sum_{j=n_0}^{\infty} j^{m-1}b_j \tag{35}$$

then by the theorem there exists a solution x of (E2) which has an asymptotic property

$$x_n = c + o(1)$$

for arbitrary positive constant c. In this case we can take $q \equiv 1$, $g_n = s(M+c)^{s-1}|a_n|$ and apply Remark 2 with $r = 0$, $p = c$.

Remark 5. Based on the above two examples we compare the result of this note with the other contained in earlier papers. Equation (E2) cannot be studied by Theorem 3.1 [2] because of some kind of submultiplicity assumptions for the function f contained there and by theorems contained in [8] because as it was noticed in this paper the case $s > 1$ was ruled out. Asymptotic properties of the solutions of equations (E1) and (E2) can be studied by Theorem 3.6 of [1]; however this theorem gives conditions under which the solution x is asymptotic to the polynomial of degree exactly $m-1$. Instead of convergence of the second series (35) it is supposed that the series

$$\sum_{j=n_0}^{\infty} |b_j| \tag{36}$$

converges. Examining equation (E1) by Theorem 2 of [6] we state that if the series (36) and (33) converge every solution of (E1) satisfies asymptotic relation

$$\Delta^r x_n = O(n^{m-r-1}), \ 0 \leq r \leq m-1.$$

By Theorem 3 of [6], if both the series of (32) converge absolutely, then every solution of (E1) is of the form

$$x_n = p_n + o(1),$$

where the polynomial p_n has calculated coefficients. The theorem presented in [7] gives conditions when the solutions are of the form such that

$$\lim_{n\to\infty} \frac{\Delta^r x_n}{\Delta^r z_n} = c, \ 0 \leq r \leq m-1,$$

where z is such that

$$\lim_{n\to\infty} \Delta^{m-1} z_n = \infty$$

and so z_n cannot be polynomial of degree $< m$. Note that the equations considered in [1], [2], [6], [7] are of general form than (E).

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