Proc. Indian Acad. Sci. (Math. Sci.), Vol. 95, No. 2, December 1986, pp. 133–140.

On a class of bilateral generating functions for certain special functions

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MS received 3 October 1986

Abstract. A general theorem unifying a novel class of bilateral generating functions of certain special functions is established. A number of applications of the theorem are also given.

Keywords. Bilateral generating functions; special functions.

1. Introduction

Several attempts have been made by many researchers [2, 4, 9, 11, 17] to formulate theories in connection with the unification of different classes of bilateral generating functions of certain special functions found in the literature. The present note is a further attempt to establish a general theorem on the unification of a novel class of bilateral generating functions of various special functions e.g. Laguerre, modified Laguerre, generalized Bessel, Gegenbauer and modified Jacobi polynomials. In fact we have obtained the following theorem:

THEOREM 1. For a set of functions $\{S_n^k(x) \mid n = 0, 1, 2, \ldots\}$ generated by

$$\sum_{n=0}^{\infty} A_n(m,k)\, S_{n+m}^{k-n}(x)t^n = \frac{\{f(x,t)\}^k g(x,t)}{\{h(x,t)\}^m}\, S_m^k\{q(x,t)\}, \tag{1}$$

where m is a non-negative integer, A_n the arbitrary constant and f, g, h, q are arbitrary functions of x and t, let

$$F(x,t) = \sum_{n=0}^{\infty} a_n\, S_{n+r}^{k-n}(x)t^n, \tag{2}$$

then the following bilateral generating relation for $S_r^k(x)$ holds:

$$\sum_{n=0}^{\infty} S_{n+r}^{k-n}(x)\sigma_n(y)t^n = \frac{\{f(x,t)\}^k g(x,t)}{\{h(x,t)\}^r}\, F\left(q(x,t), \frac{yt}{f(x,t)h(x,t)}\right), \tag{3}$$

where

$$\sigma_n(y) = \sum_{p=0}^{n} a_p\, A_{n-p}(r+p, k-p)y^p.$$

During the application of the above theorem, a large number of bilateral generating functions for various special functions are pointed out.

Proof of theorem 1

We have

$$\sum_{n=0}^{\infty} S_{n+r}^{k-n}(x)\sigma_n(y)t^n = \sum_{p=0}^{\infty} a_p(yt)^p \sum_{n=0}^{\infty} A_n S_{n+p+r}^{k-n-p}(x)t^n$$

$$= \frac{\{f(x,t)\}^k g(x,t)}{\{h(x,t)\}^r} \sum_{p=0}^{\infty} a_p \left(\frac{yt}{f(x,t)h(x,t)}\right)^p S_{r+p}^{k-p}(q(x,t))$$

$$= \frac{\{f(x,t)\}^k g(x,t)}{\{h(x,t)\}^r} F\left(q(x,t), \frac{yt}{f(x,t)h(x,t)}\right).$$

This completes the proof of the theorem.

COROLLARY 1. If we put $r = 0$, we get the following result. If

$$F(x,t) = \sum_{n=0}^{\infty} a_n S_n^{k-n}(x)t^n,$$

then

$$\sum_{n=0}^{\infty} S_n^{k-n}(x)\sigma_n(y)t^n = \{f(x,t)\}^k g(x,t) F\left(q(x,t), \frac{yt}{f(x,t)h(x,t)}\right).$$

where

$$\sigma_n(y) = \sum_{p=0}^{n} a_p A_{n-p} y^p.$$

In the next section we proceed to give the various applications of the theorem in the field of certain special functions.

2. Applications

(i) *On Laguerre polynomials*

We first consider the Laguerre polynomials satisfying the following generating relation [1, 7]

$$(1+t)^k \exp(-xt) L_r^{(k)}(x(1+t)) = \sum_{n=0}^{\infty} \frac{(r+1)_n}{n!} L_{n+r}^{(k-n)}(x)t^n. \tag{4}$$

The relation (4) is of type (1) with,

$$f(x,t) = 1+t,\ g(x,t) = \exp(-xt),\ h(x,t) = 1,\ q(x,t) = x(1+t)$$

and $A_n = \dfrac{(r+1)_n}{n!}$.

Now using the theorem, we get the following result on bilateral generating relation involving Laguerre polynomials:

THEOREM 2. If there exists a generating relation of the form

$$F(x,t) = \sum_{n=0}^{\infty} a_n L_{n+r}^{(k-n)}(x)t^n, \tag{5}$$

then

$$\sum_{n=0}^{\infty} L_{n+r}^{(k-n)}(x)\sigma_n(y)t^n = (1+t)^k \exp(-xt) F\left(x(1+t), \frac{yt}{1+t}\right), \tag{6}$$

where

$$\sigma_n(y) = \sum_{p=0}^{n} a_p \frac{(r+p+1)_{n-p}}{(n-p)!} y^p.$$

COROLLARY 2. If we put $r = 0$ in the above theorem we get the following result. If

$$F(x,t) = \sum_{n=0}^{\infty} a_n L_n^{(k-n)}(x)t^n,$$

then

$$\sum_{n=0}^{\infty} L_n^{(k-n)}(x)\sigma_n(y)t^n = (1+t)^k \exp(-xt) F\left(x(1+t), \frac{yt}{1+t}\right),$$

where

$$\sigma_n(y) = \sum_{p=0}^{n} a_p \frac{(p+1)_{n-p}}{(n-p)!} y^p.$$

(ii) *On modified Laguerre polynomials*

We now consider the following well-known generating function for the modified Laguerre polynomials [10]

$$\sum_{n=0}^{\infty} \frac{(r+1)_n}{n!} f_{n+r}^{k-n}(x)t^n = (1+t)^{k-1} \exp\left(\frac{xt}{1+t}\right) f_r^k\left(\frac{x}{1+t}\right). \tag{7}$$

The relation (7) is of type (1) with

$$f(x,t) = 1+t,\ g(x,t) = (1+t)^{-1} \exp\left(\frac{xt}{1+t}\right),$$

$$h(x,t) = 1,\ q(x,t) = \frac{x}{1+t} \text{ and } A_n = \frac{(r+1)_n}{n!}.$$

Therefore by the application of our theorem we get the following result on bilateral generating relation involving modified Laguerre polynomials.

THEOREM 3. If there exists a generating relation of the form

$$F(x,t) = \sum_{n=0}^{\infty} a_n f_{n+r}^{k-n}(x) t^n, \tag{8}$$

then

$$\sum_{n=0}^{\infty} f_{n+r}^{k-n}(x)\, \sigma_n(y) t^n = (1+t)^{k-1} \exp\left(\frac{xt}{1+t}\right) F\left(\frac{x}{1+t}, \frac{yt}{1+t}\right)$$

where

$$\sigma_n(y) = \sum_{p=0}^{n} a_p \frac{(r+p+1)_{n-p}}{(n-p)!} y^p. \tag{9}$$

COROLLARY 4. Putting $r = 0$ in the above theorem, we get the known result derived by Ghosh [15].

(iii) *On Bessel polynomials*

We now consider the generalized Bessel polynomials satisfying the following generating function [8]

$$\sum_{n=0}^{\infty} \frac{\beta^n}{n!} Y_{n+r}^{(k-n)}(x) t^n = (1-xt)^{1-k-r} \exp(\beta t)\, Y_r^{(k)}\left(\frac{x}{1-xt}\right). \tag{10}$$

The relation (10) is of the same type (1) with

$$f(x,t) = (1-xt)^{-1},\ g(x,t) = (1-xt)\exp(\beta t),\ h(x,t) = 1-xt,$$

$$q(x,t) = \frac{x}{1-xt}, \text{ and } A_n = \beta^n/n!.$$

Therefore by using our theorem, we get the following result involving Bessel polynomials.

THEOREM 4. If there exists a generating relation of the form

$$F(x,t) = \sum_{n=0}^{\infty} a_n Y_{n+r}^{(k-n)}(x) t^n \tag{11}$$

then

$$\sum_{n=0}^{\infty} Y_{n+r}^{(k-n)}(x)\sigma_n(y)t^n = (1-xt)^{1-k-r}\exp(\beta t)F\left(\frac{x}{1-xt}, yt\right)$$

where

$$\sigma_n(y) = \sum_{p=0}^{n} a_p \frac{\beta^{n-p}}{(n-p)!} y^p. \tag{12}$$

COROLLARY 4. Putting $r = 0$ in the above theorem we get the interesting result found derived with some misprints in [6].

(iv) *On Gegenbauer polynomials*

Next we consider the Gegenbauer polynomials satisfying the following generating relation [5]

$$\sum_{n=0}^{\infty} \binom{n+r}{r} \frac{(1-r-2k)_n}{(1-k)_n} C_{n+r}^{k-n}(x)t^n$$
$$= [1+4tx+4t^2(x^2-1)]^{k-1/2}\, C_r^k(x+2t(x^2-1)). \tag{13}$$

The relation (13) is of type (1) with

$$f(x,t) = 1+4tx+4t^2(x^2-1),\ g(x,t) = [1+4tx+4t^2(x^2-1)]^{-1/2}$$

$$h(x,t) = 1,\ q(x,t) = x+2t(x^2-1),\ A_n = \binom{n+r}{r}\frac{(1-r-2k)_n}{(1-k)_n}.$$

Now using theorem 1, we get the following result on the bilateral generating function of Gegenbauer polynomials

THEOREM 5. If there exists a generating relation of the form

$$F(x,t) = \sum_{n=0}^{\infty} a_n C_{n+r}^{k-n}(x)t^n \tag{14}$$

then

$$\sum_{n=0}^{\infty} C_{n+r}^{k-n}(x)\sigma_n(y)t^n = [1+4tx+4t^2(x^2-1)]^{k-1/2} \times$$
$$F\left(x+2t(x^2-1), \frac{yt}{1+4tx+4t^2(x^2-1)}\right) \tag{15}$$

where

$$\sigma_n(y) = \sum_{p=0}^{n} a_p \binom{n+r}{r+p} \frac{(1-r+p-2k)_{n-p}}{(1-k+p)_{n-p}} y^p.$$

COROLLARY 5. Putting $r = 0$ in theorem 5, we get the result derived in [14].

(v) *On Jacobi polynomials*

Next we consider the Jacobi polynomials satisfying the following generating relation [3]

$$\sum_{n=0}^{\infty} \frac{(r+1)_n}{n!} P_{n+r}^{(\alpha,k-n)}(x)t^n \tag{16}$$

$$= (1-t)^k \left\{1-(1+x)\frac{t}{2}\right\}^{-1-\alpha-k-r} P_r^{(\alpha,k)}\left(\frac{x-\frac{t}{2}(1+x)}{1-\frac{t}{2}(1+x)}\right).$$

The relation (16) is of type (1) with

$$f(x,t) = \frac{1-t}{1-\frac{t}{2}(1+x)}, \; g(x,t) = \left\{1-\frac{t}{2}(1+x)\right\}^{-1-\alpha},$$

$$h(x,t) = \left\{1-\frac{t}{2}(1+x)\right\},$$

$$q(x,t) = \left\{1-\frac{t}{2}(1+x)\right\} \Big/ \left\{1-\frac{t}{2}(1+x)\right\}, \; A_n = (r+1)_n/n!.$$

Now using theorem 1, we get the following result on the bilateral generating relation involving Jacobi polynomials.

THEOREM 6. If there exists a generating relation of the form

$$F(x,t) = \sum_{n=0}^{\infty} a_n P_{n+r}^{(\alpha,k-n)}(x)t^n, \tag{17}$$

then

$$\sum_{n=0}^{\infty} P_{n+r}^{(\alpha,k-n)}(x)\sigma_n(y)t^n \tag{18}$$

$$= (1-t)^k \left\{1-\frac{t}{2}(1+x)\right\}^{-1-\alpha-k-r} F\left(\frac{x-\frac{t}{2}(1+x)}{1-\frac{t}{2}(1+x)}, \frac{yt}{1-t}\right),$$

where

$$\sigma_n(y) = \sum_{p=0}^{n} a_p \frac{(r+p+1)_{n-p}}{(n-p)!} y^p.$$

COROLLARY 6. Putting $r = 0$ in the above theorem we get the known result derived in [12].

If in place of relation (16) we consider the following [16]

$$\sum_{n=0}^{\infty} \frac{(r+1)_n}{n!} P_{n+r}^{(k-n,\beta)}(x)t^n = (1+t)^k \left\{1 + \frac{t}{2}(1-x)\right\}^{-1-k-\beta-r} \times$$
$$P_r^{(k,\beta)}\left(\frac{x - \frac{t}{2}(1-x)}{1+\frac{t}{2}(1-x)}\right). \tag{19}$$

we get the following result.

THEOREM 7. If there exists a generating relation of the form

$$F(x,t) = \sum_{n=0}^{\infty} a_n P_{n+r}^{(k-n,\beta)}(x)t^n, \tag{20}$$

then

$$\sum_{n=0}^{\infty} P_{n+r}^{(k-n,\beta)}(x)\sigma_n(y)t^n = (1+t)^k \left\{1 + \frac{t}{2}(1-x)\right\}^{-1-k-\beta-r} \times$$
$$F\left(\frac{x - \frac{t}{2}(1-x)}{1+\frac{t}{2}(1-x)}, \frac{yt}{1+t}\right),$$

where

$$\sigma_n(y) = \sum_{p=0}^{n} a_p \frac{(r+p+1)_{n-p}}{(n-p)!} y^p.$$

COROLLARY 7. Putting $r = 0$ in the above theorem we get the result which has been derived in [13].

Conclusion

From the above discussion it is clear that one may apply theorem 1 in the case of other polynomials and functions existing in the field of special functions to obtain the bilateral generating relation involving the special function under consideration.

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