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On a subclass of Bazilevic functions

T N SHANMUGAM

Department of Mathematics, Anna University, Madras 600 025, India

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Abstract. A new subclass of Bazilevic functions is defined and some of its properties have been studied.

Keywords. Bazilevic functions; subclass.

1. Introduction

Let S denote the class of functions $f(z) = z + \sum_{2}^{\infty} a_n z^n$ analytic, univalent in the open unit disk $U = \{z: |z| < 1\}$. For functions g and G analytic in U, we say that g is subordinate to G, denoted as $g \prec G$, if there exists a Schwarz function $w(z)$ analytic in U such that $g(z) = G(w(z))$, $z \in U$. In this paper a new class of functions $B(\alpha, h_1, h_2)$ is defined as follows:

Definition 1.1. Let h_1, h_2 be two convex univalent, analytic functions defined on U such that $h_1(U), h_2(U)$ lies on the right-half plane and $h_1(0) = h_2(0) = 1$. We say a function $f \in S$ belongs to the class $B(\alpha, h_1, h_2)$ if $zf'(z)/f^{1-\alpha}(z)\, g^{\alpha}(z) \prec h_2(z)$, $z \in U$ where $zg'(z)/g(z) \prec h_1(z)$, $z \in U$ with $\alpha \geq 0$.

For $\alpha = 0$ and $h_1(z) = h_2(z) = h(z)$, this class $B(\alpha, h_1, h_2)$ is the class $S^*(h)$, which is the same as the class $S_a^*(h)$ with $a = 1$ defined in [7]. Further if $h(z) = (1 + Az)/(1 + Bz)$ $(-1 \leq B < A \leq 1)$, $S^*(h)$ is the class $S^*(A, B)$ studied by Janowski [4]. For $\alpha = 1$ and for different choices of $h_1(z)$ and $h_2(z)$ the class $B(\alpha, h_1, h_2)$ generalizes various classes of functions introduced in [2, 5, 6, 12, 13]. In particular, for $h_2(z) = (1 + Az)/(1 + Bz)$ $(-1 \leq B < A \leq 1)$ and $h_1(z) = (1 + Cz)/(1 + Dz)$ $(-1 \leq D \leq C \leq 1)$, $B(\alpha, h_1, h_2)$ is the class $C[A, B; C, D]$ introduced by Silvia [12]. Further, if $B = D = -1$ and $A = 1 - 2\lambda\, (0 \leq \lambda < 1)$ and $C = 1 - 2\rho\, (0 \leq \rho \leq 1)$, then $B(\alpha, h_1, h_2)$ is the class. $B(\alpha, \lambda, \rho)$ was introduced by Gupta and Jain [3]. In this paper it is proved that the class $B(\alpha, h_1, h_2)$ is invariant under some integral operators.

To establish the main results of this paper, we need the following results:

THEOREM A [1]. Let $\beta, \gamma \in C$ and h be a convex analytic univalent function in U with $h(0) = 1$ and $\operatorname{Re}(\beta h(z) + \nu) > 0$, $z \in U$. Let $p(z) = 1 + p_1 z + \ldots$ analytic in the unit disk. Then

$$p(z)+\frac{zp'(z)}{\beta p(z)+\nu} \prec h(z) \Rightarrow p(z) \prec h(z),\ z \in U.$$

A slight modification of the above result is as follows.

THEOREM B [8]. Let $\beta, \gamma \in \mathbf{C}$, $h(z)$ be a convex analytic univalent function in U with $h(0) = 1$ and $\mathrm{Re}(\beta q(z) + \gamma) > 0$, $z \in U$. If $p(z) = 1 + p_1 z + \dots$ is analytic in U then

$$p(z)+\frac{zp'(z)}{\beta q(z)+\gamma} \prec h(z) \Rightarrow p(z) \prec h(z),\ z \in U.$$

Let

$$f(z) = z + \sum_{2}^{\infty} a_n z^n,\ g(z) = z + \sum_{2}^{\infty} b_n z^n$$

then the Hadamards convolution product of f and g denoted by $(f^*g)\ (z)$ is

$$(f^*g)\ (z) = z + \sum_{2}^{\infty} a_n b_n z^n.$$

THEOREM C [10]. Let ϕ be a convex analytic function and $\phi(0) = 0$ and g is a star-like function in U. Then for F analytic in U with $F(0) = 1$, $[\phi^* Fg/\phi^* g]\ (U)$ is contained in the convex hall of $F(U)$.

2. Main theorems

THEOREM 2.1. Let $f \in S^*(h)$. Then for $\alpha > 0$ $\mathrm{Re}(c) > 0$, $F(z)$ defined as

$$F(z) = \left\{\frac{\alpha+c}{z^c}\int_0^z t^{c-1} f^{\alpha}(t)\ \mathrm{d}t\right\}^{1/\alpha}$$

is also an element of $S^*(h)$.

Proof. This follows easily by the application of Theorem A.

THEOREM 2.2. Let ϕ be a convex analytic function, then for $f \in S^*(h)$, $\phi * f \in S^*(h)$.

Proof. We have

$$\frac{z(\phi^*f)'(z)}{(\phi^*f)\ (z)} = \frac{(\phi^* zf')(z)}{(\phi^*f)\ (z)} = \frac{\left(\phi^* \frac{zf'}{f}\cdot f\right)(z)}{(\phi^*f)\ (z)}.$$

Since $f \in S^*(h)$ we have $zf'(z)/f(z) \prec h(z)$ in U; also ϕ is a convex function. Now an application of Theorem C yields that

$$\frac{z(\phi^*f)'(z)}{(\phi^*f)\ (z)} \prec h(z),$$

which establishes the theorem.

COROLLARY. If $f \in S^*(h)$, then

$$F_1(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)\, dt, \ \mathrm{Re}(c) > 0$$

$$F_2(z) = \int_0^z \frac{f(h)-f(xh)}{h-xh}\, dh, \ |x| \leqslant 1, \ x \neq 1$$

also belongs to $S^*(h)$.

Proof. This is immediate from the theorem since F_1 and F_2 are convolutions of f with ϕ_1 and ϕ_2 being two convex functions respectively, where

$$\phi_1(z) = \sum_1^\infty \left(\frac{c+1}{c+n}\right) z^n, \ \phi_2(z) = \sum_1^\infty \frac{1-x^n}{(1-x)^n} z^n.$$

Remark. For the choice $h(z) = (1+Az)/(1+Bz)$, $-1 \leqslant A < B \leqslant 1$ in the above theorem and corollary we deduce theorem 8 and its corollary in [11].

THEOREM 2.3. Let ϕ be a convex function in U with $\phi(0) = 0$. Then $\phi^* f \in B(1, h_1, h_2)$ whenever $f \in B(1, h_1, h_2)$.

Proof. Now consider,

$$\frac{z(\phi^* f)'(z)}{(\phi^* g)(z)} = \frac{(\phi^* zf')(z)}{(\phi^* g)(z)} = \frac{\left(\phi^* \frac{zf'}{g}, g\right)(z)}{(\phi^* g)(z)}.$$

Since $f \in B(1, h_1, h_2)$ there exists a $g \in S^*(h_1)$ such that $zf'(z)/g(z) \prec h_2(z)$ in U. Hence an application of Theorem C yields the result.

COROLLARY. If $f \in B(1, h_1, h_2)$ then

$$F_1(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)\, dt, \ \mathrm{Re}(c) > 0$$

$$F_2(z) = \int_0^z \frac{f(h)-f(xh)}{h-xh}\, dh, \ |x| \leqslant 1, \ x \neq 1$$

also belongs to $B(1, h_1, h_2)$.

Remark. When $h_1(z) = (1+Cz)/(1+Dz)$ $(-1 \leqslant D < C \leqslant 1)$ and $h_2(z) = (1+Az)/(1+Bz)$ $(-1 \leqslant B \leqslant A \leqslant 1)$, the above theorem reduces to theorem 4 in [12].

THEOREM 2.4. Let $\alpha > 0$, $\mathrm{Re}(c) > 0$ and let $f\, B(\alpha, h_1, h_2)$. Then $F(z)$ is defined by

$$F(z) = \left\{\frac{\alpha+c}{z^c}\int_0^z t^{c-1} f^\alpha(t)\, dt\right\}^{1/\alpha} \tag{1}$$

is also belongs to $B(\alpha, h_1, h_2)$.

Proof. Since f is a Bazilevic function of type $(\alpha, 0)$ by a result due to Ruscheweyh [9], $F(z)$ defined by [1] is also a Bazilevic function of type $(\alpha, 0)$ and hence is univalent in U. Thus $F(z) \neq 0$ in $U-\{0\}$. Now, (1) on differentiation yields:

$$\frac{1}{z^{\alpha}}\{\alpha z F^{\alpha-1}(z)+CF^{\alpha}(z)\} = \frac{(\alpha+c)f^{\alpha}(z)}{z^{\alpha}}.$$

Since $f \in B(\alpha, h_1, h_2)$, $\exists\, g \in S^*(h_1)$ such that

$$zf'(z)/f^{1-\alpha}(z)g^{\alpha}(z) \prec h_2(z).$$

Let

$$p(z) = zF'(z)/F^{1-\alpha}(z)G^{\alpha}(z),$$

where

$$G(z) = \left\{\frac{C+\alpha}{z^{c}}\int_0^z t^{C-1}\, g^{\alpha}(t)\, dt\right\}^{1/\alpha}.$$

Simplifying we get

$$\frac{zp'(z)}{\alpha\dfrac{zG'(z)}{G(z)}+C}+p(z) = \frac{zf'(z)}{f^{1-\alpha}(z)g^{\alpha}(z)} \prec h_2(z).$$

Also by Theorem 2.1, $G \in S^*(h_1)$. Now an application of Theorem B gives that $p(z) \prec h_2(z)$ when $\mathrm{Re}(c) > 0$, which means F is also in $B(\alpha, h_1, h_2)$ under the stated conditions of the theorem.

Remark 1. If

$$h_1(z) = \frac{1+(1-2\lambda)z}{1-z}$$

and

$$h_2(z) = \frac{1+(1-2\rho)z}{1-z}$$

in the above theorem, we get theorem 1 of [2].

Remark 2. If $\alpha = 1$ and $h_1(z) = h_2(z) = (1+Az)/(1+Bz)$ $(-1 \leqslant B < A \leqslant 1)$, in the above theorem we get a theorem due to Mehrok [6] as a particular case. Further if $A = 1$, $B = -1$ we get theorem 2 in [13].

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