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Sufficiency and strong commutants in quantum probability theory

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Abstract. A probability algebra $(\mathscr{A}, *, \omega)$ consisting of a *algebra $\mathscr{A}$ with a faithful state ω provides a framework for an unbounded noncommutative probability theory. A characterization of symmetric probability algebra is obtained in terms of an unbounded strong commutant of the left regular representation of $\mathscr{A}$. Existence of coarse-graining is established for states that are absolutely continuous or continuous in the induced topology. Sufficiency of a *subalgebra relative to a family of states is discussed in terms of noncommutative Radon-Nikodym derivatives (a form of Halmos-Savage theorem), and is applied to a couple of examples (including the canonical algebra of one degree of freedom for Heisenberg commutation relation) to obtain unbounded analogues of sufficiency results known in probability theory over a von Neumann algebra.

Keywords. Quantum probability; unbounded representations; commutants; conditional expectation; coarse-graining; sufficiency; Schrödinger representation.

1. Introduction

A probability algebra $(\mathscr{A}, *, \omega)$ consists of a unital linear associative algebra $\mathscr{A}$ over complex numbers, an involution $x \to x^*$ on $\mathscr{A}$ making $\mathscr{A}$ a *algebra and *a* state ω on $\mathscr{A}$ that is faithful in the sense that $\omega(x^*x) = 0$ implies that $x = 0$. An inner product is defined on $\mathscr{A}$ as $\langle x, y\rangle = \omega(y^*x)$ satisfying $\langle xy, z\rangle = \langle y, x^*z\rangle$. Let H be the Hilbert space completion of $\mathscr{A}$ to which ω can be uniquely extended as a continuous linear form. A hermitian representation [9] $(\pi, D(\pi), H)$ of $\mathscr{A}$ is defined as $\pi(x)y = xy$ with domain $D(\pi) = \mathscr{A}$. This gives an Op*-algebra $\pi(\mathscr{A})$ of operators, not necessarily bounded.

Taking this as a basis, an unbounded noncommutative probability theory is developed in [2] which also encompasses unbounded observables in contrast to a bounded noncommutative probability theory considered till then (e.g. [3], [7], [11], [12]). These notes are aimed at further clarifying this foundation and to improve some of the results in [2] and [8]. The contributions of these notes are briefly summarized below.

(a) Through a noncommutative Radon-Nikodym theorem, the commutants of $\pi(\mathscr{A})$ [5] naturally enters into this scheme. In [2], the role of the weak commutant $\pi(\mathscr{A})^c$ is discussed. Here we examine the strong commutant $\pi(\mathscr{A})^c_s$. This clarifies the inter-relations among various mathematical structures arising within the theory, e.g., a characterization of strong commutant is obtained analogous to the one for weak commutant [2, Theorem 6]. This implies that the probability algebra $\mathscr{A}$ is symmetric [2, §6] iff $\pi(\mathscr{A})^c_s$ is a *algebra. This bisects a result in [2, Theorem 6] that $\pi(\mathscr{A})^c$ is a *algebra iff $\mathscr{A}$ is symmetric and π is self-adjoint. We also discuss the probability algebra of simple operators associated with a noncommutative regular probability gauge space [13]. This is a noncommutative analogue of an example in [2, §2].

(b) In the discussion on conditional expectation and coarse-graining we establish the existence of a coarse-graining for a state that is absolutely continuous. This improves [2, Theorem 3(3)] wherein this has been shown for a dominated state. We also discuss the coarse-graining for states that are not absolutely continuous. This requires characterizing the linear forms on $\mathscr{A}$ that are continuous with respect to the induced topology [2, §6]. Information measures and coarse-graining relative to two different subalgebras are compared.

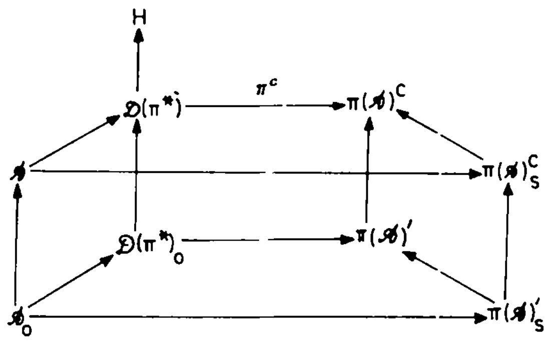
(c) Finally we discuss the sufficiency of *subalgebras of a probability algebra with respect to a family of state in terms of Radon-Nikodym derivatives. This is closely related with a noncommutative Halmos-Savage theorem on weak sufficiency (with a stronger notion of conditional expectation) obtained by Hiai *et al* [8]. Our sufficiency criterion leads to unbounded analogues of a number of results known in the context of probability theory over a von Neumann algebra. Let $\mathscr{A}$ be the canonical algebra of one degree of freedom for the Schrödinger representation of Heisenberg commutation relation. Let ω_β be the centred normal isotropic state [2, §7] of the parameter $\beta > 0$ on $\mathscr{A}$. Let G be the modular automorphism group associated with ω_β. Then an absolutely continuous state ν on $\mathscr{A}$ is G-invariant iff the centralizer of ω_β is sufficient for ν. This extends to the present framework, in the particular case of Schrödinger representation, a result for a von Neumann algebra with a faithful state obtained in [7, Theorem 2.2]. Likewise, we also obtain an analogue of [8, Example 1.3] for an abstract *algebra.

2. The strong commutants

Given a probability algebra $(\mathscr{A}, *, \omega)$ and the left regular representation $(\pi, D(\pi), H)$ the adjoint representation $(\pi^*, D(\pi^*), H)$ of $\mathscr{A}$ is $\pi^*(x) = \pi(x^*)^*|D(\pi^*)$ wherein the domain $D(\pi^*) = \cap\{D(\pi(x^*)^*|x \in \mathscr{A}\} \supset D(\pi)$ and $\pi \subset \pi^*$. The *right bounded part* of $\mathscr{A}$ is $\mathscr{A}_0 = \{x \in \mathscr{A} | \rho(x)$ is norm bounded$\}$, a subalgebra of $\mathscr{A}$. Here $\rho(x)y = yx$. The algebra $\mathscr{A}$ is *symmetric* if there exists an involution $y \to y^b$ on $\mathscr{A}$ such that $\omega(y^*x) = \omega(xy^b)$ $(x, y$ in $\mathscr{A})$. In this case, ρ is a ^{b}-antirepresentation of $\mathscr{A}$ and $\mathscr{A}_0$ is a ^{b}-subalgebra of $\mathscr{A}$.

Let $L(\mathscr{A})$ denote the set of all linear maps $\mathscr{A} \to H$. The *weak unbounded commutant of* π is $\pi(\mathscr{A})^c = \{C \in L(\mathscr{A}) | \langle C\pi(x)y, z\rangle = \langle Cy, \pi(x^*)z\rangle$ for all x, y, z in

$\mathscr{A}\}$; where as the *weak bounded commutant* is $\pi(\mathscr{A})' = \{C \in \pi(\mathscr{A})^c \mid C$ is norm bounded$\}$. These hermitian linear subspaces are not algebras. On the other hand, the *strong unbounded commutant* $\pi(\mathscr{A})^c_s = \{C \in \pi(\mathscr{A})^c \mid C\mathscr{A} \subset \mathscr{A}\}$ and the *strong bounded commutant* $\pi(\mathscr{A})'_s = \{C \in \pi(\mathscr{A})^c_s \mid C$ is norm bounded$\}$ are algebras but not closed under adjoint. Let $\pi^c\colon D(\pi^*) \to \pi(\mathscr{A})^c$ be $\pi^c(y)x = \pi^*(x)y$. It is shown in [2, Theorem 8] that π^c is a linear isomorphism of $D(\pi^*)$ onto $\pi(\mathscr{A})^c$ which induces a linear isomorphism of $D(\pi^*)_0$ onto $\pi(\mathscr{A})'$ where $D(\pi^*)_0 = \{y \in D(\pi^*) \mid x \to \pi^c(y)x$ is norm bounded on $\mathscr{A}\}$; and that $\pi(\mathscr{A})^c$ is a *algebra (operator adjoint as the involution) iff π is self-adjoint and $\mathscr{A}$ is symmetric. In fact, it so turns out that π^c can be used to describe the strong commutants which in turn give a characterization of symmetric probability algebras. The following schematically exhibit the relationships of various commutants with other related structures.



2.1 *Theorem*

Let $(\mathscr{A}, *, \omega)$ be a probability algebra
(a) π^c is a bijective anti-isomorphism of $\mathscr{A}_0$ onto $\pi(\mathscr{A})^c_s$ which induces an anti-isomorphism of $\mathscr{A}$ onto $\pi(\mathscr{A})'_s$.
(b) $\mathscr{A}$ is symmetric iff $\pi(\mathscr{A})^c_s$ is a *algebra. In this case, π^c is a ^{b}-anti-isomorphism of $\mathscr{A}$ onto $\pi(\mathscr{A})^c_s$. Similarly, $\pi(\mathscr{A})'_s$ is a *algebra iff there exists an involution b on $\mathscr{A}_0$ such that $\omega(y^*x) = \omega(xy^b)$ $(x, y$ in $\mathscr{A}_0)$. Then π^c is a ^{b}-anti-isomorphism of $\mathscr{A}_0$ onto $\pi(\mathscr{A})'_s$.

Proof. (a) For x, y in $\mathscr{A}$, $\pi^c(y)x = xy \in \mathscr{A}$. Thus $\pi^c(y)\mathscr{A} \subset \mathscr{A}$, $\pi^c(y) \in \pi(\mathscr{A})^c_s$. For a $C \in \pi(\mathscr{A})^c_s$, $C1 \in \mathscr{A}$, say $C1 = y$. Then $\pi^c(y) = C$. It is easily checked that for y_1, y_2 in $\mathscr{A}$ and $x \in D(\pi)$, $\pi^c(y_1y_2)\,x = \pi^c(y_2)\pi^c(y_1)x$. Further, $y \in \mathscr{A}_0$ iff $\pi^c(y)$ is a bounded operator.

(b) Let $\pi(\mathscr{A})^c_s$ be an Op*-algebra. Then for a $C \in \pi(\mathscr{A})^c_s$, $C^* = C^*|_{\mathscr{A}} \in \pi(\mathscr{A})^c_s$, C^* being the operator adjoint; and so $C^*\mathscr{A} \subset \mathscr{A}$. Now given

$y \in \mathscr{A}$, $\pi^c(y) \in \pi(\mathscr{A})^c_s$, and so does $\pi^c(y)^*$. Define $y^b = \pi^c(y)^* 1 \in D(\pi) = \mathscr{A}$. Then for all $x \in \mathscr{A}$,

$$\omega(xy^b) = \langle xy^b, 1\rangle = \langle y^b, x^*\rangle = \langle \pi^c(y)^* 1, x^*\rangle = \langle 1, \pi^c(y)x^*\rangle$$
$$= \langle 1, \pi^*(x^*)y\rangle = \langle 1, x^*y\rangle = \langle x, y\rangle = \omega(y^*x).$$

Hence by [2, Lemma 7], $y \to y^b$ is an involution on $\mathscr{A}$ and $\mathscr{A}$ is symmetric. Conversely, if $\mathscr{A}$ is symmetric with an involution b such that $\omega(y^*x) = \omega(xy^b)$ (x, y in $\mathscr{A}$); then for all x, y, z in $\mathscr{A}$,

$$\langle \pi^c(y^b)x, z\rangle = \langle xy^b, z\rangle = \langle x, zy\rangle = \langle \pi^c(y)^*x, z\rangle.$$

Hence $\pi^c(y)^* = \pi^c(y^b) \in \pi(\mathscr{A})^c_s$ in view of (a). Since $\pi^c(\mathscr{A}) = \pi(\mathscr{A})^c_s$, it follows that $\pi(\mathscr{A})^c_s$ is an Op*-algebra. Now it is clear that π^c is a ^{b}-anti-isomorphism. The remaining assertions can be analogously verified. This completes the proof.

We also note that $\pi(\mathscr{A})^c$ is closed under operator adjoint iff there exists a conjugate linear map $^b: D(\pi^*) \to D(\pi^*)$ such that $\langle x, y\rangle = \langle y^b, x\rangle$ for all $x \in \mathscr{A}$, $y \in D(\pi^*)$.

A state F: $\mathscr{A} \to \mathbb{C}$ is *absolutely continuous* (with respect to ω) $F < \omega$ if for any sequence (x_j) in $\mathscr{A}$, $\omega(x_j{}^* x_j) \to 0$ implies that $F(yx_j) \to 0$ for all $y \in \mathscr{A}$. F is *dominated* (by ω) $F < \omega$ if for some constant M, $F(x^*x) \leqslant M\omega(x^*x)$ ($x \in \mathscr{A}$). Clearly $F < \omega \Longrightarrow F < \omega$. A Radon-Nikodym theorem [2, Theorem 1] shows that $F < \omega$ iff there exists a $C \geqslant 0$ in $\pi(\mathscr{A})^c$ (called the *Radon Nikodym derivative* denoted by $\mathrm{d}F/\mathrm{d}\omega$) such that $F(x) = \omega(Cx)$ ($x \in \mathscr{A}$); and F is dominated iff $(\mathrm{d}F/\mathrm{d}\omega) \in \pi(\mathscr{A})'$. It follows from the above that for an $F < \omega$, $(\mathrm{d}F/\mathrm{d}\omega) \in \pi(\mathscr{A})^c_s$ iff $F(x) = \omega(z^*x)$ for a unique $z \in \mathscr{A}$, and $(\mathrm{d}F/\mathrm{d}\omega) \in \pi(\mathscr{A})'_s$ iff $z \in \mathscr{A}_0$.

We end this discussion with a noncommutative analogue of an instructive example (the probability algebra of the classical probability space) considered in [2], and characterize the absolutely continuous states and the dominated states.

2.2 *Example.* Let $\Gamma = \{H, \mathcal{M}, m\}$ be a regular probability gauge space [13] viz. $\mathcal{M}$ is a von Neumann algebra on a Hilbert space H with a faithful normal tracial state m. (In fact, we can take m to be a faithful normal semifinite trace). An operator T on H is simple [14] if $T = \Sigma\, \lambda_i E_i$, a finite sum with λ_i's in $\mathbb{C}$ and E_i's mutually disjoint projections in $\mathcal{M}$. Simple operators in $\mathcal{M}$ forms a probability algebra $\mathscr{A}$ with a faithful state $\omega(T) = \Sigma \lambda_i m(E_i)$. For $1 \leqslant p \leqslant \infty$, let $L^p(\mathcal{M})$ be the noncommutative L^p-spaces associated with $\mathcal{M}$ as in [14]. A version of Radon-Nikodym theorem in [14] implies (since $\mathscr{A}$ is ultraweakly dense in $\mathcal{M}$) that if f is an ultraweakly continuous linear form on $\mathscr{A}$, then there exists a unique $A \in L^1(\mathcal{M})$ (denoted by $A = D_\omega(f)$) such that $f(X) = m(D_\omega(f)X)$ ($X \in \mathscr{A}$). Now, using the noncommutative L^p-L^q duality and the corresponding Holder's inequality (see e.g. [13]), it can be shown that $f < \omega$ iff $D_\omega(f) \in L^2(\mathcal{M})$ and $f < \omega$ iff $D_\omega(f) \in L^\infty(\mathcal{M})$.

3. Conditional expectation and coarse graining

Let $(\mathscr{A}, *, \omega)$ be a probability algebra. Let $\mathscr{B}$ be a *subalgebra of $\mathscr{A}$ containing 1. Let $\pi|_{\mathscr{B}}$ be the restriction of π to $\mathscr{B}$ with domain $D(\pi|_{\mathscr{B}}) = \mathscr{B}$ considered as a representation on $\overline{\mathscr{B}}$, the norm closure of $\mathscr{B}$ in H. The *conditional expectation* of x in $\mathscr{A}$ given $\mathscr{B}$ is $E(x|\mathscr{B}) \in D((\pi|\mathscr{B})^*) \subset \overline{\mathscr{B}}$ such that for all $y \in \mathscr{B}$, $\omega(yx) = \omega[(\pi|\mathscr{B})^*(y)\, E(x|\mathscr{B})]$. It is shown in [2, Theorem 2] that $E(x|\mathscr{B})$ exists for all $x \in \mathscr{A}$ and $E(x|\mathscr{B}) = P_{\mathscr{B}}x = (\mathrm{d}\omega_{x^*}/\mathrm{d}\omega|\mathscr{B})_1$ where $P_{\mathscr{B}}: H \to \overline{\mathscr{B}}$ is the projection and $\omega_x(y) = \omega(xy)$.

Let ν be a linear form (state) on $\mathscr{B}$ such that $\nu < \omega|\mathscr{B}$. A linear form (state) ν_c on $\mathscr{A}$ is a $(\mathscr{B}, \omega)$ *coarse-graining* of ν if $\nu_c|\mathscr{B} = \nu$, $\nu_c < \omega$ and $\nu_c(x) = \nu_c[E(x|\mathscr{B})]$ for all $x \in \mathscr{A}$. It is shown in [2, Theorem 3] that a $(\mathscr{B}, \omega)$ coarse-graining ν_c of ν exists iff ν has an absolutely continuous extension ν_1 to $\mathscr{A}$ such that $(\mathrm{d}\nu_1/\mathrm{d}\omega)\, 1 \in \overline{\mathscr{B}}$. Further, if $\mathscr{B}$ is *positivity preserving* (in the sense that given $x \in \mathscr{A}$, there exists a sequence (y_i) in $\mathscr{B}$ such that $y_i^* y_i \to P_{\mathscr{B}}(x^*x)$), and if ν is dominated, then ν_c exists. We show that under the same condition, even if ν is absolutely continuous, it admits a coarse-graining.

3.1 *Theorem*

Let $\mathscr{B}$ be a *subalgebra of a probability algebra $(\mathscr{A}, *, \omega)$.

(a) Let $\mathscr{B}$ be positivity preserving and $\nu: \mathscr{B} \to \mathbb{C}$ be a state such that $\nu < \omega|\mathscr{B}$. Then a $(\mathscr{B}, \omega)$ coarse-graining ν_c of ν exists.

(b) Let $\mathscr{B}$ be positivity preserving in the strong sense and $(\pi|\mathscr{B})^*$ be a hermitian representation of $\mathscr{B}$. Let $\nu: \mathscr{B} \to \mathbb{C}$ be a state on $\mathscr{B}$ that is continuous with respect to the left induced topology $t^l_{\mathscr{B}}$ on $\mathscr{B}$ due to $\mathscr{B}$. Then a 'coarse-graining' $\nu_c: \mathscr{A} \to \mathbb{C}$ of ν exists satisfying the following:

(i) $\nu_c|\mathscr{B} = \nu$,

(ii) $\nu_c(x) = \nu(E(x|\mathscr{B}))$ for all $x \in \mathscr{A}$

(iii) ν_c is continuous in the left induced topology $t^l_{\mathscr{A}}$ on $\mathscr{A}$ due to $\mathscr{A}$

(iv) ν_c is a state.

The *left-induced topology* $t^l_{\mathscr{B}}$ on a *subalgebra $\mathscr{B}$ is the locally convex topology defined by the seminorms $x \in \mathscr{B} \to \|\pi(y)x\|$ $(y \in \mathscr{B})$. This is a natural topology [9] one considers on the domain of an unbounded representation, and it coincides with the norm topology if each $\pi(y)$ is bounded. We say that $\mathscr{B}$ is *positivity preserving in the strong sense* provided given $x \in \mathscr{A}$, there exists a sequence (y_i) in $\mathscr{B}$ such that for each $y \in \mathscr{B}$, $\|\pi(y)y_i^*y_i - (\pi|\mathscr{B})^*(y)\ E(x^*x|\mathscr{B})\| \to 0$. Note that if $\pi(\mathscr{B})$ (and, in particular $\pi(\mathscr{A})$) is an algebra of bounded operators, then positivity preserving and strong positivity preserving are identical concepts. Our proof of (a) is a modification of that of [2, Theorem 3], whereas that of (b) requires determining the linear forms on $\mathscr{A}$ that are continuous in the induced topology.

Proof. (a) ν is $\|\cdot\|$ continuous on $\mathscr{B}$, hence admits a unique continuous linear extension $\nu: \overline{\mathscr{B}} \to \mathbb{C}$. Since $E(x|\mathscr{B}) \in D((\pi|\mathscr{B})^*) \subset \overline{\mathscr{B}}$, we can define a linear map $\nu_c: \mathscr{A} \to \mathbb{C}$, $\nu_c(x) = \nu(E(x|\mathscr{B}))$. To show that $\nu_c < \omega$, take a sequence (x_i) in $\mathscr{A}$ such that $\omega(x_i^* x_i) \to 0$, let $y \in \mathscr{A}$. Then $E(yx_i|\mathscr{B}) \in \overline{\mathscr{B}}$ for all i. For a fixed x_i, choose a sequence (z_n) in $\mathscr{B}$ such that $z_n \to E(yx_i|\mathscr{B})$ in norm. Then

$$\nu_c(yx_i) = \nu(E(yx_i|\mathscr{B})) \lim_n \nu(z_n) = \lim_n \omega\left(\frac{d\nu}{d\omega|\mathscr{B}} z_n\right)$$

$$= \lim_n \left\langle z_n, \frac{d\nu}{d\omega|\mathscr{B}} 1\right\rangle = \left\langle P_{\mathscr{B}}(yx_i), \frac{d\nu}{d\omega|\mathscr{B}} 1\right\rangle$$

$$= \left\langle yx_i, \frac{d\nu}{d\omega|\mathscr{B}} 1\right\rangle \quad \text{since} \quad \frac{d\nu}{d\omega|\mathscr{B}} \in D((\pi|\mathscr{B})^*) \subset \overline{\mathscr{B}}$$

$$= \left\langle x_i, \pi^*(y^*) \frac{d\nu}{d\omega|\mathscr{B}} 1\right\rangle \text{ since } D((\pi|\mathscr{B})^*) \subset D(\pi^*)$$

$$\leq \left\|\pi^*(y) \frac{d\nu}{d\omega|\mathscr{B}} 1\right\| \omega(x_i^* x_i)^{1/2} \to 0 \text{ as } i \to \infty.$$

The positivity preserving condition implies that ν_c is a state if ν is a state.

(b) Since $\nu: \mathscr{B} \to \mathbb{C}$ is continuous in $t^l_{\mathscr{B}}$, there exist a constant C and an $a \in \mathscr{B}$ such that

$$|\nu(x)| \leq C\|\pi(a)x\| \leq C\langle \pi(y)x, x\rangle \ (y = 1 + a^*a).$$

Hence there exists a $z \in \overline{\mathscr{B}}$ such that $\nu(x) = \langle (\pi|\mathscr{B})^*(y)x, z\rangle$ for $x \in D((\pi|\mathscr{B})^*)$. ν_c on $\mathscr{A}$ is defined as

$$\nu_c(x) = \langle (\pi|\mathscr{B})^*(y) E(x|\mathscr{B}), z\rangle = \nu(E(x|\mathscr{B})).$$

Then for some sequence (z_n) in $\mathscr{B}$ with $\|z_n - z\| \to 0$,

$$\nu_c(x) = \lim_n \langle (\pi|\mathscr{B})^*(y) E(x|\mathscr{B}), z_n\rangle$$

$$= \lim_n \omega(z_n^*(\pi|\mathscr{B})^*(y) E(x|\mathscr{B}))$$

$$= \lim_n \omega((\pi|\mathscr{B})^*(z_n^* y) E(x|\mathscr{B})) = \lim_n \omega(z_n^* yx)$$

$$= \lim_n \langle yx, z_n\rangle = \langle yx, z\rangle = \langle \pi^*(y)x, z\rangle.$$

Thus ν_c is linear, continuous in the induced topology $t^l_{\mathscr{A}}$, hence extends uniquely to $D(\pi^*)$. Also given $x \in \mathscr{A}$, for some sequence (y_n) in $\mathscr{B}$,

$$\nu_c(x^*x) = \lim_n \langle ((\pi|\mathscr{B}))^*(y) y_n^* y_n, z\rangle = \lim_n \nu(y_n^* y_n) \geq 0$$

showing that ν_c is a state. This completes the proof. (It should be noted that the assumption of positivity or strongly positivity is used only to conclude that the extended functioned ν_c is positive if ν is positive.)

We compare the coarse-graining relative to two different subalgebras. Given a state ν on $\mathscr{A}$ that is absolutely continuous, the *information measure* $I_\omega(\nu)$ of ν is a number satisfying $I_\omega(\nu) \geq 0$ for all ν, $I_\omega(\omega) = 1$ and for any two absolutely continuous states ν_1 and ν_2 on $\mathscr{A}$ that are *mutually singular*

$$\left(\text{i.e. } \left\langle \frac{d\nu_1}{d\omega} 1, \frac{d\nu_2}{d\omega} 1 \right\rangle = 0\right),$$

$I_\omega(\nu_1 + \nu_2) = I_\omega(\nu_1) + I_\omega(\nu_2)$. It is shown in [2, Theorem 4] that

$$I_\omega(\nu) = \left\|\frac{d\nu}{d\omega} 1\right\|^2,$$

and for a state ν on $\mathscr{B}$, if a $(\mathscr{B}, \omega)$ coarse-graining ν_c of ν exists, then it is the unique absolutely continuous extension of ν with minimum information. This, together with Theorem 4 and Corollary on p. 4, both in [2], can be used to conclude the following. We omit the proof.

3.2 *Proposition*

Let $(\mathscr{A}, *, \omega)$ be a probability algebra. Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be *subalgebras of $\mathscr{A}$ such that $1 \in \mathscr{B}_1 \subset \mathscr{B}_2$. Let $\nu < \omega|\mathscr{B}_2$ be a state on $\mathscr{B}_2$. Let $\nu_c^2 = (\mathscr{B}_2, \omega)$ coarse-graining of ν and $\nu_c^1 = (\mathscr{B}_1, \omega)$ coarse-graining of ν. The following are equivalent.

(a) $\nu_c^1 = \nu_c^2$,

(b) $I_\omega(\nu_c^1) = I_\omega(\nu_c^2)$,

(c) $\nu_c^1|\mathscr{B}_2 = \nu$,

(d) $\dfrac{d\nu_c^2}{d\omega} 1 \in \overline{\mathscr{B}}_1$.

If ν is faithful on $\mathscr{B}_2$, then any of the above is equivalent to

(e) $\mathscr{B}_1 = \mathscr{B}_2$.

4. Sufficiency

Hiai *et al* [7] investigated sufficiency of states in a von Neumann algebra. In [8], they analyzed sufficiency and weak sufficiency in an abstract *algebra. However, the condition expectation considered in these papers is more restrictive. In fact, it may not exist at all [11]; whereas the one considered in the present framework always exists [2]. (We also refer to [4] for the discussion of relative merits of an analogous approach to condition expectation in von Neumann algebras.) In the present context, we define sufficiency as follows:

Let $(\mathscr{A}, *, \omega)$ be a probability algebra. Let $S_{\mathscr{A}}$ be the collection of all linear functionals (in particular, state) on $\mathscr{A}$ that are absolutely continuous. Let $S \subset S_{\mathscr{A}}$

and $\mathscr{B} \subset \mathscr{A}$ be a *subalgebra containing 1. Then $\mathscr{B}$ is *sufficient* for S if $\varphi(x) = \varphi(E(x|\mathscr{B}))$ for all $x \in \mathscr{A}$, $\varphi \in S$.

4.1 *Theorem*

The following are equivalent

(a) $\mathscr{B}$ is sufficient for S

(b) $\frac{d\varphi}{d\omega} \mathscr{B} \subset \overline{\mathscr{B}}$ for all $\varphi \in S$

(c) $\frac{d\varphi}{d\omega} \mathscr{B} \subset D((\pi|\mathscr{B})^*)$ for all $\varphi \in S$

(d) $\frac{d\varphi}{d\omega} 1 \in D((\pi|\mathscr{B})^*)$ for all $\varphi \in S$

(e) $\left.\frac{d\varphi}{d\omega}\right|_{\mathscr{B}} = \frac{d\varphi|\mathscr{B}}{d\omega|\mathscr{B}}$ for all $\varphi \in S$.

This is closely related with a kind of noncommutative Halmos-Savage theorem [8, Theorem 1] on weak sufficiency. However, the proof is transparent in view of [2, Theorem 3]. The objective here is to apply it to a couple of examples exhibiting sufficiency phenomena analogous to those discussed in the context of probability theory over a von Neumann algebra.

(A) Let $\mathscr{A}$ be a semifinite von Neumann algebra with a faithful normal semifinite trace τ. Given a normal state φ on $\mathscr{A}$, there exists a positive operator $T_\varphi = d\varphi/d\tau$ such that $\varphi(A) = \tau(T_\varphi A)$ $(A \in \mathscr{A})$. Example 1.3 in [7] implies that for any family S of normal states, the von Neumann algebra $\mathscr{B}$ generated by $\{(d\varphi/d\tau)|\varphi \in S\}$ is sufficient for S. The following extends this to the unbounded case.

4.2 *Example*

Let $(\mathscr{A}, *, \omega)$ be a probability algebra for which π is self-adjoint. Let $\mathscr{A}$ be symmetric admitting another involution $y \to y^b$ such that $\omega(y^*x) = \omega(xy^b)$ $(x, y$ in $\mathscr{A})$. Let S be a family of absolutely continuous states on $\mathscr{A}$. For $\varphi \in S$, $(d\varphi/d\omega) \in \pi(\mathscr{A})^c = \pi(\mathscr{A})_s^c = \pi^c(\mathscr{A})$ since π is self-adjoint, and so $(d\varphi/d\omega) = \pi^c(y_\varphi)$ for a unique $y_\varphi \in \mathscr{A}$. Let $\mathscr{B}$ be the ^{b}-subalgebra of $\mathscr{A}$ generated by $\{y_\varphi|\varphi \in S\}$.

Assertion. $\mathscr{B}$ is sufficient for S.

Proof. For $x \in \mathscr{A}$, $\varphi \in S$, choose (z_n) in $\mathscr{B}$ such that $z_n \to E(x|\mathscr{B})$. Then

$$\varphi(E(x|\mathscr{B}) = \lim_n \varphi(z_n) = \lim_n \omega\left(\frac{d\varphi}{d\omega} z_n\right) = \lim_n \omega(\pi^c(y_\varphi)z_n)$$

$$= \lim_n \omega(\pi^*(z_n)y_\varphi) = \lim_n \langle y_\varphi, z_n^*\rangle$$
$$= \lim_n \langle z_n, y_\varphi^b\rangle = \langle E(x|\mathcal{B}), y_\varphi^b\rangle = \langle E(x|\mathcal{B}), (\pi|\mathcal{B})y_\varphi^b\, 1\rangle$$
$$= \langle(\pi|\mathcal{B})\ (y_\varphi^b)^* E(x|\mathcal{B}), 1\rangle = \omega((\pi|\mathcal{B})\ (y_\varphi^b)^* E(x|\mathcal{B}))$$
$$= \omega((\pi(\mathcal{B})^*\ (y_\varphi^{b*}) E(x|\mathcal{B}))$$
$$= \omega(y_\varphi^{b*}x) = \langle x, y_\varphi^b\rangle = \langle y_\varphi, x^*\rangle = \langle \pi^*(x)y_\varphi, 1\rangle$$
$$= \langle \pi^c(y_\varphi)x, 1\rangle = \left\langle \frac{\mathrm{d}\varphi}{\mathrm{d}\omega} x, 1\right\rangle = \omega\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\omega} x\right).$$
$$= \varphi(x).$$

(B) Let σ_t^φ be the modular automorphism group associated with a faithful normal state φ on a von Neumann algebra $\mathcal{A}$. Let $I(\varphi)$ be the at of all σ_t^φ-invariant normal states on $\mathcal{A}$ and Z_φ be the centralizer of φ. It is shown in [7, Theorem 2.2] that a normal state $\psi \in I(\varphi)$ iff Z_φ is sufficient for $\{\varphi, \psi\}$. The following describes analogous phenomena for the canonical algebra of one degree of freedom for the Heisenberg commutation relation [2].

4.3 *Example*

Let $\mathcal{A}$ be the *algebra generated by hermitian elements p and q satisfying $pq-qp = -i\ 1$. The Schrödinger representation $(\pi_0, D(\pi_0), L^2(\mathbf{R}))$ of $\mathcal{A}$, with domain $D(\pi_0) = S(\mathbf{R})$ (Schwartz space) is defined as

$$\pi_0(p) = p_0,\ \pi_0(q) = q_0,\ p_0 f = -i\frac{df}{dt},\ q_0 f\ (t) = tf(t)\ (f \in S(\mathbf{R})).$$

For each

$$\beta > 0,\ \omega_\beta(x) = (1-e^{-\beta})^{-1} \sum_{n=0}^{\infty} e^{-\beta}\ \langle \pi_0(x)\ f_n, f_n\rangle\ (x \in \mathcal{A})$$

f_n being the normalized Hermite functions, be the centred normal isotropic state of β. Then $(\mathcal{A}, *, \omega)$ $(\omega = \omega_\beta)$ is a symmetric probability algebra (Note that, π_0 being self-adjoint, the considerations of example 4.1 can be applied to $\mathcal{A}$). For each $z \in \mathbb{C}$, $\Delta(z)p = \cosh(\beta z)p - i\sinh(\beta z)q$, $\Delta(z)q = i\sinh(\beta z)p + \cosh(\beta z)q$ define an automorphism group $G: t \in \mathbf{R} \to \Delta(it)$, called the modular automorphism group of $\mathcal{A}$. Now let $I(\omega) = \{\varphi: \mathcal{A} \to \mathbb{C}$ state$|\ \varphi < \omega,\ \varphi(\Delta(it)x) = \varphi(x)$ for all $x \in \mathcal{A}$, $t \in \mathbf{R}\}$ be the set of all absolutely continuous G-invariant states on $\mathcal{A}$. Let $Z_\omega = \{y \in \mathcal{A} | \omega(xy) = \omega(yx)$ for all $x \in \mathcal{A}\}$ be the centralizer of ω.

Assertions. (a) For $x \in \mathcal{A}$, $t \in \mathbf{R}$, $E(x|Z_\omega) = E(\Delta(it)x|Z_\omega)$.

(b) For any state $\nu < \omega$ on $\mathcal{A}$, $\nu \in I(\omega)$ iff Z_ω is sufficient for ν.

Proof. Observe that, using [2, §7], exactly as in [10, Lemma 15.8], $Z_\omega = \{y \in \mathscr{A} \mid \Delta(it)y = y \text{ for all } t \in \mathbf{R}\}$, the fixed point algebra of $\mathscr{A}$.
(a) For $x \in \mathscr{A}$ let $\omega_x \colon Z_\omega \to \mathbb{C}$ be $\omega_x(y) = \omega(xy)$. Then by [2, Theorem 2(3)],

$$\omega_x < \omega | Z_\omega,\ E(x|Z_\omega) = \frac{\mathrm{d}\omega_{x^*}}{\mathrm{d}\omega|Z_\omega}\, 1.$$

Now for all $y \in Z_\omega$, $t \in \mathbf{R}$,

$$\omega_{x^*}(y) = \omega_{x^*}(\Delta(it)y) = \langle \Delta(it)y,\ x\rangle = \langle y,\ \Delta(-it)x\rangle$$
$$= \omega_{\Delta(-it)x^*}(y).$$

Hence

$$E(x|Z_\omega) = \frac{\mathrm{d}\omega_{\Delta(-it)x^*}}{\mathrm{d}\omega|Z_\omega} 1 = E(\Delta(-it)x|Z_\omega).$$

(b) Let Z_ω be sufficient for ν. Then for all $x \in \mathscr{A}$, $\nu(x) = \nu(E(x|Z_\omega))$. By Theorem 4.1, $(\mathrm{d}\nu/\mathrm{d}\omega)\ 1 \in \bar{Z}_\omega$ (throughout, the closure and the orthogonal complements are with reference to Hilbert space completion of the inner product structure on $\mathscr{A}$ defined by ω). Choose a (y_n) in Z_ω such that $y_n \to (\mathrm{d}\nu/\mathrm{d}\omega)\ 1$. Then for all $x \in \mathscr{A}$, $t \in \mathbf{R}$, using [2, Theorem 20],

$$\nu(\Delta(it)x) = \left\langle \frac{\mathrm{d}\nu}{\mathrm{d}\omega} 1,\ (\Delta(it)x)^* \right\rangle = \left\langle \frac{\mathrm{d}\nu}{\mathrm{d}\omega} 1,\ \Delta(it)x^* \right\rangle$$
$$= \lim_n \langle y_n,\ \Delta(it)x^*\rangle = \lim_n \langle \Delta(-it)y_n,\ x^*\rangle$$
$$= \lim_n \langle y_n,\ x^*\rangle = \left\langle \frac{\mathrm{d}\nu}{\mathrm{d}\omega} 1,\ x^* \right\rangle = \omega\left(\frac{\mathrm{d}\nu}{\mathrm{d}\omega} x \right).$$

Thus $\nu \in I(\omega)$. Conversely, let $\nu \in I(\omega)$. Then for all $x \in \mathscr{A}$, $t \in \mathbf{R}$, $\nu(x) = \nu(\Delta(it)x)$. Hence

$$\left\langle \frac{\mathrm{d}\nu}{\mathrm{d}\omega} 1,\ x^* \right\rangle = \omega\left(\frac{\mathrm{d}\nu}{\mathrm{d}\omega} x \right) = \nu(x) = \nu(\Delta(it)x)$$
$$= \omega\left(\frac{\mathrm{d}\nu}{\mathrm{d}\omega} \Delta(it)x \right) = \left\langle \frac{\mathrm{d}\nu}{\mathrm{d}\omega} 1,\ \Delta(it)x^* \right\rangle.$$

Thus $(\mathrm{d}\nu/\mathrm{d}\omega)\ 1 \perp A$ where $A = \{x - \Delta(it)x \mid x \in \mathscr{A},\ t \in \mathbf{R}\}$. Now let $z \in \mathscr{A} \cap A^\perp$. Then for all $x \in \mathscr{A}$, $t \in \mathbf{R}$, $\langle z, x\rangle = \langle z, \Delta(it)x\rangle = \langle \Delta(-it)z, x\rangle$. Hence $z = \Delta(-it)z$, $z \in Z_\omega$. Thus $\mathscr{A} \cap A^\perp \subset Z_\omega$; and so $(\mathscr{A} \cap A^\perp)^- \subset \bar{Z}_\omega$. But $(\mathscr{A} \cap A^\perp)^- = (\mathscr{A} \cap A^\perp)^{\perp\perp} = (\mathscr{A}^\perp \vee A^{\perp\perp})^\perp = A^\perp$. Hence $A^\perp \subset \bar{Z}_\omega$. It follows that $(\mathrm{d}\nu/\mathrm{d}\omega)\ 1 \in \bar{Z}_\omega$; and by Theorem 4.1, Z_ω is sufficient for ν.

Remark. In general, it can be shown that if $\mathscr{B}$ is a *subalgebra of a probability algebra $\mathscr{A}$, then $\mathscr{B}$ is sufficient for $S = \{\omega_x | x \in \mathscr{B}\}$, $\omega_x(y) = \omega(xy)$ $(y \in \mathscr{A})$.

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