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Cohomology of the moduli of parabolic vector bundles

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Abstract. The purpose of this paper is to compute the Betti numbers of the moduli space of *parabolic vector bundles* on a curve (see Seshadri [7], [8] and Mehta & Seshadri [4]), in the case where every semi-stable parabolic bundle is necessarily stable. We do this by generalizing the method of Atiyah and Bott [1] in the case of moduli of ordinary vector bundles. Recall that (see Seshadri [7]) the underlying topological space of the moduli of parabolic vector bundles is the space of equivalence classes of certain unitary representations of a discrete subgroup Γ which is a lattice in PSL (2, **R**). (The lattice Γ need not necessarily be co-compact).

While the structure of the proof is essentially the same as that of Atiyah and Bott, there are some difficulties of a technical nature in the parabolic case. For instance the Harder–Narasimhan stratification has to be further refined in order to get the connected strata. These connected strata turn out to have different codimensions even when they are part of the same Harder–Narasimhan strata.

If in addition to 'stable = semistable' the rank and degree are coprime, then the moduli space turns out to be torsion-free in its cohomology.

The arrangement of the paper is as follows. In § 1 we prove the necessary basic results about algebraic families of parabolic bundles. These are generalizations of the corresponding results proved by Shatz [9]. Following this, in § 2 we generalize the analytical part of the argument of Atiyah and Bott (§ 14 of [1]). Finally in § 3 we show how to obtain an inductive formula for the Betti numbers of the moduli space. We illustrate our method by computing explicitly the Betti numbers in the special case of rank = 2, and one parabolic point.

Keywords. Cohomology; parabolic vector bundles; moduli space; Betti numbers; algebraic family; Sobolev spaces.

1. Algebraic families of parabolic bundles

For the basic definitions and properties of vector bundles with parabolic structures ('parabolic bundles' for short) see Seshadri [7] and [8], and Mehta and Seshadri [4]. For simplicity we assume throughout that the parabolic structures of the parabolic bundles considered are over only a single point P of the curve X. All our arguments generalize trivially to the case where there are more parabolic points on X.

As in [8], each parabolic bundle E on X has a unique (parabolic) Harder–Narasimhan filtration $0 \subset E_1 \subset \ldots \subset E_r = E$ where E_i are parabolic subbundles such that (i) each quotient parabolic bundle E_i/E_{i-1} is semi-stable (ii) the inequality par $\mu(E_i/E_{i-1}) >$ par $\mu(E_{i+1}/E_i)$ holds for each i, where par μ = (par deg/rank)

Hence to each parabolic bundle E, we may associate a convex polygon HNP (E) in $\mathbf{R}^2$ as defined by Shatz [9]. The vertices of HNP (E) (= the Harder–Narasimhan polygon of E) are the points $(0,0)$, (rank E_1, par deg E_1), ... , (rank E_r, par deg E_r). The polygons corresponding to parabolic bundles of a given rank and parabolic degree have a natural partial ordering as defined by Shatz [9], namely if λ_1 and λ_2 are two such polygons then $\lambda_1 \geq \lambda_2$ if all the vertices of λ_2 lie on or below the polygon λ_1. If $\lambda_1 \geq \lambda_2$, we say that λ_1 dominates λ_2. The polygon λ = HNP (E) is also called the *Harder–Narasimhan type* of the parabolic bundle E.

PROPOSITION 1.1. Let E be a parabolic bundle. Then any parabolic subbundle of E lies on or below the Harder–Narasimhan polygon of E. In particular, the polygon corresponding to any filtration of E by parabolic subbundles is dominated by the Harder–Narasimhan polygon HNP (E) of E.

Proof. If F_1 and F_2 are parabolic subbundles of E, and if the vector subbundles $F_1 \vee F_2$ and $F_1 \cap F_2$ of E (as defined in Langton [3]) are given the induced parabolic structure, then it is easy to see that par deg $(F_1 \vee F_2)$+par deg $(F_1 \cap F_2) \geq$ par deg (F_1)+par deg (F_2). Now the proof of proposition 2 follows from the proof of the theorem 2 of Shatz [9].

We want to study how the Harder–Narasimhan type changes within an algebraic family of parabolic bundles. For this, the following observation is basic.

Remark 1.2. Let E be a vector bundle on a scheme S and let F_1 and F_2 be subbundles. Let $\phi: F_1 \to E/F_2$ be the natural map. Then the function $s \mapsto \text{rank}_{k(s)} \phi(s)$ is lower-semicontinuous on S. (Equivalently, the function $s \mapsto \dim_{k(s)} F_1(s) \cap F_2(s)$ is upper semicontinuous).

Using this, it is easy to prove the following:

PROPOSITION 1.3. Let E_T be a family of parabolic bundles parametrized by the scheme T which is the spectrum of a discrete valuation ring. Let ξ and ξ_0 be the generic and special points of T. Let G be a coherent torsion-free quotient sheaf of E_T on $X \times T$ which is flat over T. Let G_ξ and G_{ξ_0} be the restrictions of G to $X \times \xi$ and $X \times \xi_0$. Let G'_{ξ_0} be the vector bundle on $X \times \xi_0$ generically generated by G_{ξ_0}. Let G_ξ and G'_{ξ_0} be given the induced parabolic structures, as quotients of E_ξ and E_{ξ_0} respectively. Then par $\deg(G_\xi) \geq$ par $\deg(G'_{\xi_0})$.

PROPOSITION 1.4. Let E_T be a family of parabolic bundles on X parametrized by T, where T is the spectrum of a discrete valuation ring. Let ξ, ξ_0 be the generic and special points of T. Then $\text{HNP}(E_{\xi_0}) \geq \text{HNP}(E_\xi)$.

Proof. Let $0 \subset E_{\xi,1} \subset \ldots \subset E_{\xi,r} = E_\xi$ be the Harder–Narasimhan filtration over the generic point $\xi \in T$. Then using completeness of appropriate quot schemes it follows (Shatz [9], proposition 9) that there exists a filtration $0 \subset E_1 \subset \ldots \subset E$ of the bundle $E \to T \times X$ by subbundles (i.e. torsion free, coherent subsheaves having a torsion free quotient) which induces the given filtration $0 \subset E_{\xi,1} \subset \ldots \subset E_{\xi,r} = E_\xi$ over the generic point. It follows from the proposition 1.3 that if $(E_i)'_{\xi_0}$ is the subbundle of E_0 generically generated by $(E_i)_{\xi_0}$, then par deg $(E_i)'_{\xi_0} \geq$ par deg $E_{\xi,i}$. Now the proposition follows from proposition 1.1.

Remark 1.5. Let k be an infinite field, let K/k be an extension field, let X be a curve over k, E a parabolic bundle on X, and E_K its pull back to X_K. Then it is proved by Seshadri [8] that the SCSS subbundle (i.e. β-subbundle) of the parabolic bundle E_K is the pull back of the SCSS subbundle of E. Hence the Harder–Narasimhan filtration of E_K is the pull back of that of E, and in particular E_K is semi-stable if and only if E is semi-stable.

From this remark and the proposition 1.4, we get the following result analogous to the theorem 3 of Shatz [9].

PROPOSITION 1.6. Let E_S be a family of parabolic bundles on X parametrized by a scheme S, and let $s, s_0 \in S$ such that s_0 lies in the closure of s. Then $\mathrm{HNP}(E_{s_0}) \geq \mathrm{HNP}(E_s)$.

Proof. There exists a morphism $T \to S$ of schemes where T is the spectrum of a *d.v.r.* sending the generic and special points ξ and ξ_0 respectively to s and s_0. Let E_T be the pulled back parabolic family. Then by Prop. 1.4 and remark 1.5,

$$\mathrm{HNP}(E_{s_0}) = \mathrm{HNP}(E_{\xi_0}) \geq \mathrm{HNP}(E_\xi) = \mathrm{HNP}(E_s).$$

PROPOSITION 1.7. Let E_S be a family of parabolic bundles on X parametrized by a scheme S. Then all points $s \in S$ such that E_s is semi-stable form an open subset of S.

Proof. Let $d = \deg(E_s)$, $r = \mathrm{rank}(E_s)$, which are constants. Then from the assumption that S is noetherian, the upper semi-continuity of dim $H^1(X_s, E_s)$, and the Riemann–Roch theorem it is easy to derive (see Narasimhan and Ramanathan [5]), that there exist an integer m such that for any coherent quotient sheaf E''_s of E_s for any $s \in S$ deg $(E''_s) \geq m$.

Now if a parabolic bundle E_s is not semi-stable, then there exists a quotient E''_s such that

$$\frac{\text{par deg } (E_s)}{\text{rank } (E_s)} \geq \frac{\text{par deg } (E''_s)}{\text{rank } (E''_s)}.$$

Now, par deg $(E''_s) \geq$ deg (E''_s). Hence we would have

$$\deg (E''_s) \leq \text{par deg } (E_s) \leq d + r \text{ where } r = \text{rank } E.$$

Hence in the family E_S, a parabolic bundle E_s is not semi-stable if and only if it has a quotient parabolic bundle E''_s with

(i) par $\mu(E) \geq$ par $\mu(E''_s)$,
(ii) $d + r \geq$ deg $(E''_s) \geq m$.

Let Q/S be the relative Quot scheme of quotients of the E_s of ranks $r_1 \leq r$, and degrees d_1 with $d+r \geq d_1 \geq m$. Then we know that Q is projective over S.

Let G be the universal sheaf over $X \times Q$. As G is coherent. and flat over Q, the points $t \in Q$ such that G_t is a torsion free sheaf on X_t form an open subset $Y \subseteq Q$ as follows from EGA IV_3, theorem 12.2.1 (iii) or (iv).

Let π: $Q \rightarrow S$ be the structure morphism. Then for each $y \in Y$, G_y is a quotient bundle of $(\pi^* E)_y$, and hence acquires a parabolic structure. Note that as G is a flat quotient, the function $y \mapsto \deg(G_y)$ is a locally constant function. It now easily follows from remark 1.2 that $y \mapsto$ par deg (G_y) is a lower semi-continuous function on Y. The function $y \mapsto$ rank (G_y) is locally constant. Hence, the function $y \mapsto$ par $\mu(G_y)$ is also a lower semi-continuous function on Y. Hence

$$Z = \{y \in Y \,|\, \text{par } \mu(E) \geq \text{par } \mu(G_y)\} \text{ is a closed subset of } Y.$$

Note that E_s is unstable (= non semi-stable) if and only if there exists some $y \in Z$ such that $\pi(y) = s$. Let S_{unstable} be the set of all $s \in S$ such that E_s is not semi-stable. Hence $\pi(Z) = S_{\text{unstable}}$. We wish to prove that S_{unstable} is closed in S. For this, let $\tilde{Z}$ be the closure of Z in Q. As $\pi: Q \rightarrow S$ is projective, $\pi(\tilde{Z})$ is closed in S. We claim that $\pi(\tilde{Z}) \subseteq S_{\text{unstable}}$ which would complete the proof. So let $y_0 \in \tilde{Z}$ Then as Z is a locally closed subscheme of Q, there exists some $y \in Z$ such that y_0 lies in the closure of y. Hence $\pi(y_0)$ lies in the closure of $\pi(y) \in S_{\text{unstable}}$. Hence by Prop. 1.6, $E_{\pi(y_0)}$ is unstable, which completes the proof.

Proposition 1.8. Let E_S be a family of parabolic bundles parametrized by an irreducible scheme S. Then there exists an open subset $U \subseteq S$ such that

(i) for all $s \in U$, $\mathrm{HNP}(E_s)$ = constant.

(ii) there exists an algebraic filtration on the restricted family E_U which specializes to the Harder–Narasimhan filtration at each point of U.

Proof. Let ξ be the generic point of S. Let $0 \subset E_{1,\xi} \subset \ldots \subset E_{r,\xi} = E_\xi$ be the Harder–Narasimhan filtration of the parabolic bundle E on X. As ξ is the generic point of S, there exists an open subset $V \subseteq S$, and a filtration $0 \subset E_1 \subset \ldots \subset E_r = E_V$ of E_V by vector subbundles which restricts to the Harder–Narasimhan filtration at the generic point ξ. Give each $E_{i,s}$, for $s \in V$, the induced parabolic structure. Then it is easy to see that there exists an open subset $V' \subseteq V$ such that par deg $(E_{i,s})$ = par deg $(E_{i,\xi})$ for all s in V'. Now, each quotient $E_{i,\xi}/E_{i-1,\xi}$ is semi-stable. Hence by proposition 1.7, there exists an open subset $U \subseteq V$ such that each $E_{i,s}/E_{i-1,s}$ is semi-stable for $s \in U$. Further, as $\mu(E_{i,\xi}/E_{i-1,\xi}) > \mu(E_{i+1,\xi}/E_{i,\xi})$, $\mu(E_{i,s}/E_{i-1,s} > \mu(E_{i+1,s}/E_{i,s})$ for each $s \in U$. Hence $0 \subset E_{1,s} \subset \ldots \subset E_{r,s}$ is the Harder–Narasimhan filtration of E_s. This proves the proposition.

Proposition 1.9. For any family E_S of parabolic bundles, the function $s \rightarrow \mathrm{HNP}(E_s)$ is upper semi-continuous.

Proof. Let η be the generic point of an irreducible component S_1 of S. Then by proposition 1.7 there is an open subset U of S_1 such that $\mathrm{HNP}(E_s)$ is constant for $s \in U$. Note that $\dim (S_1 - U) < \dim S_1$. If $\eta_0 \in S_1 - U$ then by Prop. 1.6, $\mathrm{HNP}(E\eta_0) \geq \mathrm{HNP}(E_\eta)$. Now the result follows by noetherian induction.

PROPOSITION 1.10. Let the base field be **C**. Let E_S be a family of parabolic bundles of a constant Harder–Narasimhan type parametrized by a variety S. Then there exists a filtration of E_S which is continuous in the complex topology and which specializes to the Harder–Narasimhan filtrations at all points of S.

Proof. By proposition 1.7 there exists a Zariski open subvariety $U \subseteq S$ such that the Harder–Narasimhan filtration varies algebraically over U. Hence to prove the continuity of the filtration over all of S, it is enough to prove that it is continuous over any locally closed curve in S which intersects U. Hence we may assume that $\dim S = 1$. There is no essential difference if S is replaced by its desingularization, so we may assume that S is non-singular. Let U be an open subset of S and $0 \subset E_1 \subset \ldots \subset E_r$ a filtration of E_U which restricts to the Harder–Narasimhan filtration of E_s for any $s \in U$. By the properness over base of the appropriate Grassman bundles, the filtration $0 \subset E_1 \subset \ldots \subset E_r$ of E_U extends to a filtration of E_S over all of S which is an algebraic filtration. Since this filtration has the right Harder–Narasimhan polygon, it must restrict to the Harder–Narasimhan filtration at each $s \in S$.

Next we introduce the notion of the *compound type* (λ, I) of a parabolic bundle. The reason we introduce this is as follows. Given a fixed parabolic data and a fixed parabolic Harder–Narasimhan type λ, there can exist two parabolic bundles E and F which have the given parabolic data and the given Harder–Narasimhan type λ, but such that the parabolic data for the corresponding quotients E_i/E_{i-1} and F_i/F_{i-1} is distinct for some i. It is easy to see that the parabolic data for E_i/E_{i-1} will be identical to that for F_i/F_{i-1} all i if and only if the bundles E and F have the same *intersection matrix* I defined as follows.

DEFINITION 1.11. Let E be a parabolic bundle over X and let $E_P = F_1 \supset \ldots \supset F_m \supset 0$ be the parabolic flag over the parabolic point $P_\varepsilon X$. Let $0 \subset G_1 \subset \ldots \subset G_r = E_P$ be the filtration of the fibre E_P induced by the Harder–Narasimhan filtration of E. Then the *intersection matrix* of E is the $r \times m$ matrix I with integral entries defined by the formula

$$I_{1,m} = \dim\ (G_1 \cap F_m)$$

$$I_{p,q} = \dim\ (G_p \cap F_q) - \sum_{\substack{i \leq p \\ j \geq q}} I_{i,j} \qquad (i,j) \neq (p,q)$$

The pair (λ, I) will be called the compound type of E.

PROPOSITION 1.12. Let E_S be a family of parabolic bundles of a constant Harder–Narasimhan type. Then the subset S_I of S corresponding to any particular intersection matrix I is closed in S.

Proof. The proof follows easily from the proposition 1.8 and arguments similar to the ones used for earlier propositions.

The infinitesimal deformations of a parabolic bundle E are parametrized by the vector space $H^1(X, \text{Par End } E)$ where Par End E is the sheaf of germs of endomorphisms of E which preserve the parabolic structure. The following generalization of the lemma 15.5 of Atiyah and Bott [1], says that there exist 'sufficiently large' families of parabolic bundles.

PROPOSITION 1.13. For any parabolic bundle E_0 on X, there exists a family E_V of parabolic bundles parametrized by a non-singular variety V, and a closed point $x_0 \in V$, such that

(i) $E_{x_0} \cong E_0$ and

(ii) the infinitesimal deformation map from $T_{x_0}(V)$ to $H^1(X, \text{Par End } E_0)$ is an isomorphism.

Proof. By lemma 15.5 in Atiyah and Bott [1], there exists a family E_S of ordinary vector bundles parametrized by a non-singular variety S, and a closed point $s_0 \in S$, such that $E_{s_0} \cong E_0$, and the deformation map $T_{s_0}(S) \to H^1(X, \text{End } E_0)$ is surjective. Let $P \in X$ be the parabolic point, and let $\{E_{0,p} = F_1 \supset F_2 \supset \ldots \supset F_r \supset 0\} = F$ be the parabolic flag in the fibre $E_{0,p}$. Let $\mathcal{F}$ be the flag manifold of all flags of the above type in $E_{0,p}$. Let $E_{S\times\mathcal{F}}$ be the vector bundle on X parametrized by $S\times\mathcal{F}$ which is the pull-back of E_S. Then $E_{S\times\mathcal{F}}$ has a natural flag structure over $P \in X$ making it a family of parabolic bundles on X parametrized by $S\times\mathcal{F}$, such that $E_{(s_0, F)} \cong E_0$ as a parabolic bundle. It is easy to see that the infinitesimal deformation map

$$T_{(s_0, F)}(S\times\mathcal{F}) \to H^1(X, \text{Par End } E_0)$$

for this family at $(s_0, F) \in S\times\mathcal{F}$ is surjective. Hence we may choose an appropriate non-singular locally closed subvariety $V \subseteq S\times\mathcal{F}$, which contains the point $x_0 = (s_0, F)$, such that the infinitesimal deformation map $T_{x_0}(V) \to H^1(X, \text{Par End } E_0)$ for the induced family E_V is an isomorphism.

Let E be a parabolic bundle, and let Par End$'$ E be the sheaf of germs of all endomorphisms of E which preserve both the parabolic filtration and the Harder–Narasimhan filtration of E. Let Par End$''$ E be the cokernel of the inclusion Par End$'$ $E \hookrightarrow$ Par End E.

PROPOSITION 1.14. For any parabolic bundle E, $H^0(X, \text{Par End}'' E) = 0$.

Proof. The successive quotients of the Harder–Narasimhan filtration are semi-stable parabolic bundles with decreasing par μ. Now if D_1 and D_2 are two semi-stable parabolic bundles with par $\mu(D_1) >$ par $\mu(D_2)$ then $\mathrm{Hom}(D_1, D_2) = 0$ by the proposition 9, Part 3 of Seshadri [8]. From this our desired result follows using induction over the length of the Harder–Narasimhan filtration.

Let E_S be a family of parabolic bundles parametrized by a variety S. For any Harder–Narasimhan type λ, let S_λ be the corresponding subset of S. Then by Prop. 1.8, S_λ is locally closed subvariety of S. Let $s_0 \in S_\lambda$ and let $\phi: T_{s_0}(S) \to H^1(X, \text{Par End } E_{s_0})$ be the infinitesimal deformation map at s_0. Then the following is easy to prove and we omit the proof.

PROPOSITION 1.15. The tangent space $T_{s_0}(S_\lambda)$ to the subvariety S_λ at s_0 is the inverse image of $H^1(X, \text{Par End}' E_{s_0}) \subseteq H^1(X, \text{Par End } E_{s_0})$ under the infinitesimal deformation map $\phi: T_{s_0}(S) \to H^1(X, \text{Par End } E_{s_0})$.

COROLLARY 1.16. If ϕ above is surjective, then the codimension of S_λ in S at s_0 equals the dimension of $H^1(X, \text{Par End}'' E_{s_0})$.

Proof. This is clear from Prop. 1.13 by which $H^0(X, \text{Par End}'' E_{s_0}) = 0$.

PROPOSITION 1.17. For a parabolic bundle E, the dimension of $H^1(X, \text{Par End}'' E)$ is given by the following formula

$$h^1(X, \text{Par End}'' E) = \sum_{i \le j} (m_j d_i - m_i d_j) + g(X)\,((\text{rank } E)^2 - \sum_{i \le j} m_i m_j) - \sum_{i \le j} n_i n_j + \sum_{\substack{p \ge k \\ q \le l}} l_{pq} l_{kl},$$

where m_i and d_i are the rank and (ordinary) degree of the ith quotient of the Harder–Narasimhan filtration, and n_i is the multiplicity of the ith weight in the parabolic structure.

Proof. Let $\text{End}' E$ be the sheaf of germs of endomorphisms of E which preserve the filtration $0 \subset E_1 \subset \ldots \subset E_r$, but may or may not preserve the parabolic structure. Let $\text{End}'' E = \text{End } E/\text{End}' E$. Then there is an exact sequence $0 \to \text{Par End}'' E \to \text{End}'' E \to \mathcal{S} \to 0$ where $\mathcal{S}$ is some skyscraper sheaf. As $h^0(X, \text{Par End}'' E) = 0$ by Prop. 1.13, $h^1(X, \text{Par End}'' E) = -\chi(\text{Par End}'' E) = -\chi(\text{End}'' E) + \chi(\mathcal{S})$. The Riemann–Roch theorem shows that

$$-\chi(\text{End}''\ E) = \sum_{i \le j} (m_j d_i - m_i d_j) + (g(X) - 1)\ ((\text{rank } E)^2 - \sum_{i \le j} m_i m_j),$$

while it is easy to see explicitly that $h^0(X, \mathcal{S})$ equals

$$(\text{rank } E)^2 - \sum_{i \le j} n_i n_j + \sum_{i \le j} n_i n_j + \sum_{\substack{i \le j \\ p \ge q}} l_{ip} l_{jq}$$

As $\mathcal{S}$ is a skyscraper sheaf, $h^0(X, \mathcal{S}) = \chi(\mathcal{S})$, which completes the proof.

2. Sobolev spaces

Let E be a fixed C^∞ parabolic bundle on the compact Riemann surface X, of rank n, degree d, parabolic filtration $E_P = F_1 \supset \ldots \supset F_m \supset 0$ over the parabolic point $P \in X$, and weights $0 \leq \alpha_1 < \ldots < \alpha_m < 1$. Let E be given a fixed Hermitian metric, and X be given a fixed Riemannian metric.

As shown by Atiyah and Bott [1], all the holomorphic structures on E form an affine space $\mathscr{C}(E)$ (or simply $\mathscr{C}$) whose tangent space is the infinite dimensional vector space $\Omega^{0,1}$ (End E) of all C^∞ global endomorphisms of E. The points of $\mathscr{C}$ are the operators $d'': \Omega^0(E) \to \Omega^{0,1}(E)$ which satisfy (i) d'' is $\mathbb{C}$-linear and (ii) $d''(fv) = \bar{\partial} f \otimes v + f d'' v$ where f is a local C^∞ function and v a local C^∞ section of E. The holomorphic bundle corresponding to an operator d'' will be denoted by $E_{d''}$.

Let Aut(E) denote the group of all C^∞-automorphisms of E, and let $\mathscr{P}$ = Par Aut(E) be the subgroup which preserves the parabolic filtration. Then Par Aut(E) acts on $\mathscr{C}$, and the orbits are the isomorphism classes of holomorphic parabolic bundles on X of the given rank, degree and parabolic structure.

We want to analyze the action of $\mathscr{P}$ on $\mathscr{C}$. For this, following § 14 of Atiyah and Bott we first replace $\mathscr{C}$ by the Sobolev space $\mathscr{C}^{k-1}$ (denoted by $\mathscr{A}^{k-1}$ in [1]) of all d''-operators of class $\mathscr{H}^{k-1}$, where k is some large positive integer. Consider the Sobolev space $\mathscr{H}^k$ (End E) of all global sections of End E of class $\mathscr{H}^k$. If ϕ is a C^∞ endomorphism of E, then we can restrict ϕ to the parabolic point $P \in X$ to get a continuous linear map from Ω^0 (End E) to End (E_p). Even though $\{P\} \subset X$ is a set of measure zero, we can extend this map to the Sobolev space $\mathscr{H}^k$ (End E) (which contains Ω^0 (End E) as a subspace) provided k is large enough. This extension is the well known *trace map* $\mathscr{H}^k$ (End E) $\to$ End (E_p) (see e.g. Triebel [10] §2.7.2). As k is assumed to be large enough, the trace map $\mathscr{H}^k$ (End E) $\to$ End (E_P) is defined and continuous where E_P is the fibre of E over the parabolic point $P \in X$. The kernel of this map is the Hilbert space Par $\mathscr{H}^k$ (End E) of all parabolic endomorphisms of E of class $\mathscr{H}^k$. This is then the Lie algebra of the Hilbert Lie groups $\mathscr{P}^k$ of all parabolic automorphisms of E of class $\mathscr{H}^k$. Note that $\mathscr{P}^k$ is definable as a closed subgroup of the Hilbert Lie group Aut$(E)^k$ of all automorphisms of E of class k by using the trace map which is defined and continuous as k is large.

As shown by Atiyah and Bott Aut$(E)^k$ acts smoothly on $\mathscr{C}^{k-1}$ which extends the action of Aut(E) on $\mathscr{C}$. It follows that $\mathscr{P}^k$ also acts smoothly on $\mathscr{C}^{k-1}$.

For any $A \in \mathscr{C}^{k-1}$, let F: Aut$(E)^k \to \mathscr{C}^{k-1}$ be the map given by the action on A i.e. $F(g) = g(A)$. Then lemma 14.6 of Atiyah and Bott says that the differential dF at identity is a Fredholm operator. As the Lie algebra Par $\mathscr{H}^k$ End E of $\mathscr{P}^k$ is of finite codimension in the Lie algebra $\mathscr{H}^k$ End E of Aut$(E)^k$, it follows that for the restricted map $\mathscr{P}^k \to \mathscr{C}^{k-1}$, the differential at identity is a Fredholm operator. Applying the smooth group action of $\mathscr{P}^k$ on $\mathscr{C}^{k-1}$, it follows that the differential is a Fredholm operator (of constant kernel and cokernel dimensions) at all points of the orbit of A under $\mathscr{P}^k$. The implicit function theorem for Banach manifolds then implies the following:

PROPOSITION 2.1. For neighbourhoods U of the identity in $\mathcal{P}^k$ and V of A in $\mathcal{C}^{k-1}$, the image $U(A)$ is a closed Banach submanifold of V of finite codimension.

Now the proof of the lemma 14.8 of Atiyah and Bott applies to prove the following:

PROPOSITION 2.2. C^∞ points are dense in every $\mathcal{P}^k$ orbit in $\mathcal{C}^{k-1}$.

The next proposition follows at once from lemma 14.9 of Atiyah and Bott [1].

PROPOSITION 2.3. Let $A, B \in \mathcal{C}$ and $g \in \mathcal{P}^k$ with $B = g(A)$. Then $g \in \mathcal{P}$, i.e. g is C^∞.

PROPOSITION 2.4. Let $A \in \mathcal{C} \subset \mathcal{C}^{k-1}$. Then the fibre of the normal bundle to the $\mathcal{P}^k$-orbit of A at A is the vector space $H^1(X, \text{Par End } E)$ where E is given the holomorphic structure corresponding to A.

Proof. An infinitesimal parabolic endomorphism $\phi \in \text{Par } \mathcal{H}^k \text{ End } E$ alters any element A of $\mathcal{C}^{k-1}$ by the addition of $d''_A\phi$, where d''_A is the operator from $\mathcal{H}^k$ End E to $\mathcal{H}^{k-1}\Omega^{0,1}$ End E corresponding to A. Here $\mathcal{H}^{k-1}\Omega^{0,1}$ End E denotes $(0,1)$-forms of class $\mathcal{H}^{k-1}$ with coefficients in End E. It is easy to see that if E is given a holomorphic structure corresponding to $A \in \mathcal{C}$, then the sequence of sheaves $0 \to \text{Par End } E \to \textit{Par } \mathcal{H}^k \textit{ End } E \xrightarrow{d''_A} \mathcal{H}^{k-1}\Omega^{0,1} \textit{ End } E \to 0$ is exact where *Par* $\mathcal{H}^k$ *End E* is the sheaf of germs of parabolic endomorphisms of E of class $\mathcal{H}^k$ and $\mathcal{H}^{k-1}\Omega^{0,1}$ *End E* has an analogous meaning. Note that the sheaves *Par* $\mathcal{H}^k$ *End E* and $\mathcal{H}^{k-1}\Omega^{0,1}$ *End E* are fine sheaves. Hence $H^1(X, \text{Par End } E)$ is the cokernel of d''_A: Par $\mathcal{H}^k$ End $E \to \mathcal{H}^{k-1}\Omega^{0,1}$ End E, which is just the normal to the $\mathcal{P}^k$-orbit at A.

The infinitesimal deformation map of an algebraic family E_S of parabolic bundles now has the following interpretation (see the proof of lemma 15.5 of Atiyah and Bott). Let $s_0 \in S$ be a smooth point, and let U be a small (euclidean) neighbourhood of s_0 in S such that the family when restricted to U is C^∞-trivial. By fixing a C^∞ isomorphism of the family E_U with the trivial C^∞ family $E_{s_0} \times U \to X \times U$, we get a C^∞ map ψ of U into $\mathcal{C} = \mathcal{C}(E_{s_0})$. The differential of ψ at s_0 is then a map of $T_{s_0}(S)$ into the tangent space to $\mathcal{C}$ at $\psi(s_0)$. Projecting onto the normal to the $\mathcal{P}$-orbit (which is $H^1(X, \text{Par End } E_{s_0})$ as seen above) then gives the infinitesimal deformation map.

The existence of 'large enough' algebraic families of parabolic bundles (see Prop. 1.13, §1) now means the following.

PROPOSITION 2.5. For any C^∞-point $A \in \mathcal{C}^{k-1}$ the $\mathcal{P}^k$-orbit of A has as a local transversal at A a finite dimensional manifold which represents an algebraic family of parabolic bundles parametrized by a non-singular variety.

Let $\mathcal{C}_{\lambda,I}$ be the subset of $\mathcal{C}$ corresponding to a compound type (λ, I), where λ is the Harder–Narasimhan type and I the intersection matrix. Define the subset $\mathcal{C}^{k-1}_{\lambda,I}$ of $\mathcal{C}^{k-1}$ to be the set of all elements of $\mathcal{C}^{k-1}$ which are $\mathcal{P}^k$-equivalent to an element of $\mathscr{C}_{\lambda, I}$. By Prop. 2.2 and Prop. 2.3, we see that the $\mathscr{C}_{\lambda, I}^{k-1}$ are well-defined, mutually disjoint and their union is all of $\mathscr{C}^{k-1}$.

Remark 2.6. Given any compound type (λ, I) (such that λ and I are compatible and λ is a convex polygon), there exists at least one parabolic bundle E on X of compound type (λ, I). This is so because given any rank, degree, and parabolic structure, there exists (see [4], [8], [9]) at least one semi-stable parabolic bundle of this rank, degree and parabolic structure. Hence for any compound type (λ, I) parabolic bundles corresponding to the successive quotient types of (λ, I), and then their direct sum is a parabolic bundle of compound type (λ, I). In particular, each $\mathscr{C}_{\lambda, I}$ is non-empty and hence each $\mathscr{C}_{\lambda, I}^{k-1}$ is non-empty.

Let the partial ordering on the Harder–Narasimhan types λ be extended to compound types (λ, I) by defining $(\lambda_1, I_1) < (\lambda_2, I_2)$ iff $\lambda_1 < \lambda_2$. Note that if $\lambda_1 = \lambda_2$, then unless $I_1 = I_2$, these two compound types are incomparable. Then propositions 1.9 and 1.12 imply that the compound type is upper semi-continuous in an algebraic family of parabolic bundles.

Proposition 2.7. The compound type is an upper semi-continuous function on $\mathscr{C}^{k-1}$. Further, each $\mathscr{C}_{\lambda, I}^{k-1}$ is a locally closed submanifold of $\mathscr{C}^{k-1}$ codimension given by Prop. 1.17, and $\mathscr{C}_{\lambda, I}$ is also a locally closed submanifold of $\mathscr{C}$ of the above codimension.

Proof. Let $A \in \mathscr{C} \subset \mathscr{C}^{k-1}$. Then the $\mathscr{P}^k$-orbit has a local transversal S at A which represents an algebraic family such that the infinitesimal deformation map is an isomorphism at A (though it may not be injective at surrounding points). Let $G \subset \mathscr{P}^k$ be the isotropy group at A, which is just the group of holomorphic parabolic automorphisms of E_A. Since G is finite dimensional, it has a closed complement W in a neighbourhood of identity in $\mathscr{P}^k$. Then W is a Banach manifold and the differential of the map $\alpha: W \times S \to \mathscr{C}^{k-1}$ (given by the group action) is an isomorphism at (e, A) where $e \in \mathscr{P}^k$ is the identity. Hence by the inverse function theorem for Banach manifolds, we may assume, after shrinking W and S suitably, that the map $\alpha: W \times S \to \mathscr{C}^{k-1}$ is an isomorphism of $W \times S$ with some open neighbourhood $W(S)$ of A in $\mathscr{C}^{k-1}$.

Now, the compound type is an upper semi-continuous function on S. Hence it is upper semi-continuous on $W(S)$. Further, as the infinitesimal deformation map for S is everywhere surjective it follows by 1.15, 1.16 and 1.17 that each $S_{\lambda, I}$ is a non-singular locally closed subvariety of S of codimension given by Prop. 1.17. Hence each $\mathscr{C}_{\lambda, I}^{k-1}$ is a locally closed submanifold in $W(S)$ of the correct codimension. At any C^∞ point $B \in \mathscr{C}_{\lambda, I}^{k-1}$, there exists a C^∞ complement. Hence within $W(S)$, $\mathscr{C}_{\lambda, I}$ is a locally closed submanifold of $\mathscr{C}$ of the correct codimension. As open sets of the form $W(S)$ cover $\mathscr{C}^{k-1}$, proposition 2.7 is proved.

Proposition 2.8. The semi-stable stratum $\mathscr{C}_{ss}$ is connected.

Proof. Each C^∞ point $A \in \mathscr{C}^{k-1}$ has an open neighbourhood of the form $W(S)$ as in the proof of Prop. 2.7. Let $W^\infty = W \cap \mathscr{P}$ be the set of C^∞ points of W. Then by Prop. 2.3, $W^\infty(S)$ is equal to $W(S) \cap \mathscr{C}$. By suitably shrinking W and S, we may assume that both W^∞ and S are connected. We know that the semi-stable part of S is open in the Zariski topology, hence connected and dense provided it is non-empty. Hence the semi-stable subset of $W^\infty(S)$ is connected and dense provided it is non-empty. Hence each point $A \in \mathscr{C}$ has a neighbourhood V such that $V \cap \mathscr{C}_{ss}$ is connected and dense provided it is non-empty. From this it follows at once that $\mathscr{C}_{ss}$ is connected since $\mathscr{C}$ is connected.

Let $\mathscr{F}_{\lambda,I}$ be the space of all C^∞ filtrations of E of compound type (λ, I). Fix any one such filtration, and let $\mathscr{P}_{\lambda,I} = \text{Par Aut}\,(E_{\lambda,I})$ denote the subgroup of Par Aut$\,(E)$ which preserves this filtration. Then $\mathscr{F}_{\lambda,I}$ is by definition the homogeneous space $\mathscr{P}/\mathscr{P}_{\lambda,I}$. We have a canonical map $f: \mathscr{C}_{\lambda,I} \to \mathscr{F}_{\lambda,I}$ which associates to any holomorphic structure on E of compound type (λ, I) the associated Harder–Narasimhan filtration. Let $\mathscr{B}_{\lambda,I}$ be the fibre of f over the chosen base point in $\mathscr{F}_{\lambda,I}$. Note that $\mathscr{B}_{\lambda,I}$ is carried into itself by the action of $\mathscr{P}_{\lambda,I}$. We prove in the next section that $\mathscr{B}_{\lambda,I}$ is a manifold isomorphic to $\prod_i \mathscr{C}_{ss}(D_i) \times \prod_{i<j} \Omega^0$ Hom (D_j, D_i) where D_i are the successive quotients of the chosen filtration of E. An analogous statement holds for the space $\mathscr{B}^{k-1}_{\lambda,I}$ used in the proof of the next proposition.

PROPOSITION 2.9. The equivariant cohomology of the pair $(\mathscr{C}_{\lambda,I}, \mathscr{P})$ is isomorphic to the equivariant cohomology of $(\mathscr{B}_{\lambda,I}, \mathscr{P}_{\lambda,I})$.

Proof. The proof is the exact analogue of the proof for the corresponding statement given in Atiyah and Bott, § 14. For the sake of completeness, we sketch the steps involved. Let $\mathscr{P}^k_{\lambda,I}$ be the subgroup of $\mathscr{P}^k$ which preserves the chosen filtration of E, and let $\mathscr{F}^k_{\lambda,I} = \mathscr{P}^k/\mathscr{P}^k_{\lambda,I}$. The map $f: \mathscr{C}_{\lambda,I} \to \mathscr{F}_{\lambda,I}$ then extends to give a $\mathscr{P}^k$-equivariant map $f^k: \mathscr{C}^{k-1}_{\lambda,I} \to \mathscr{F}^k_{\lambda,I}$. The continuity of f^k follows from Prop. 1.10 which says that the Harder–Narasimhan filtration varies continuously in an algebraic family, together with the earlier results in this section about the action of $\mathscr{P}^k$ on $\mathscr{C}^{k-1}$. Now $\mathscr{P}^k_{\lambda,I}$ is a closed subgroup of the Hilbert Lie group $\mathscr{P}^k$, and hence the fibration $\mathscr{P}^k \to \mathscr{F}^k_{\lambda,I}$ is locally trivial and hence is a principal $\mathscr{P}^k_{\lambda,I}$ bundle. Then it is shown that $f^k: \mathscr{C}^{k-1}_{\lambda,I} \to \mathscr{F}^k_{\lambda,I}$ is $\mathscr{P}^k$-equivariantly isomorphic to the associated bundle to $\mathscr{P}^k \to \mathscr{F}^k_{\lambda,I}$ with fibre $\mathscr{B}^{k-1}_{\lambda,I}$, where $\mathscr{B}^{k-1}_{\lambda,I}$ is the fibre of f^k over the chosen base point in $\mathscr{F}_{\lambda,I}$. Hence from § 13 of Atiyah and Bott it follows that the equivariant cohomologies of the pairs $(\mathscr{C}^{k-1}_{\lambda,I}, \mathscr{P}^k)$ and $(\mathscr{B}^{k-1}_{\lambda,I}, \mathscr{P}^k_{\lambda,I})$ are isomorphic. Finally, an appeal to the 'standard approximation theorems' which say that the homotopy properties of the various function spaces are all independent of k (see Palais [6], theorem 13.14) shows that for equivariant cohomology the pairs $(\mathscr{C}_{\lambda,I}, \mathscr{P})$ and $(\mathscr{C}^{k-1}_{\lambda,I}, \mathscr{P}^k)$ are equivalent and $(\mathscr{B}_{\lambda,I}, \mathscr{P}_{\lambda,I})$ and $(\mathscr{B}^{k-1}_{\lambda,I}, \mathscr{P}^{k-1}_{\lambda,I})$ are equivalent. Hence the pairs $(\mathscr{C}_{\lambda,I}, \mathscr{P})$ and $(\mathscr{B}_{\lambda,I}, \mathscr{P}_{\lambda,I})$ are equivalent for equivariant cohomology.

3. Betti numbers of the moduli space

3.1. Atiyah and Bott prove the following result which is basic for our computation (see §1 of [1]). Let M be a (possibly infinite dimensional) connected manifold together with a given action of a group G. Let $M = \bigcup_{j \in J} M_j$ be a G-invariant stratification of M by countably many locally closed submanifolds M_j which are connected and have finite codimensions c_j. For each integer q let there be only finitely many M_j such that $c_j < q$. Let the indexing set J of the stratification have a partial ordering such that (i) the indexing function $M \to J$ is upper semi-continuous (ii) J has a minimum $j_0 \in J$ (iii) for every finite subset $I \subseteq J$ there exist a finite number of minimal elements of the complement of I such that any other element of the complement is greater than at least one of them. Let the normal N_j of any stratum M_j be orientable. Let the equivariant Euler class of N_j not be a zero-divisor in the equivariant cohomology $H^*_G(M_j, \mathbf{Z})$. Then the stratification is equivariantly perfect over $\mathbf{Z}$. In particular the various equivariant Poincaré series are related by

$$\bar{P}_t(M) = \Sigma t^{c_j} \bar{P}_t(M_j).$$

We now proceed to apply the above to our stratification $\mathscr{C} = \bigcup \mathscr{C}_{\lambda, I}$ to deduce the equivariant cohomology of the semi-stable stratum. We begin with the equivariant cohomology of ($\mathscr{C}$, Par Aut (E)).

PROPOSITION 3.2. Let $\mathscr{F}$ be the flag variety of flags in $\mathbb{C}^n$ of the type specified by the parabolic structure, and let B Aut (E) be the classifying space of the group Aut (E). Then the Par Aut (E)-equivariant cohomology of $\mathscr{C}$ is given by

$$H^*_{\text{Par Aut}(E)}(\mathscr{C}) \approx H^*(B \text{ Aut}(E)) \otimes H^*(\mathscr{F}).$$

Proof. Since $\mathscr{F} \approx \text{Aut}(E)/\text{Par Aut}(E)$, we have a fibration

$$\mathscr{F} \to B \text{ Par Aut}(E) \to B \text{ Aut}(E).$$

Let P be the parabolic subgroup of $GL(n, \mathbb{C})$, where $n = \operatorname{rank} E$, which preserves a flag of the given type. Then the above fibration is a pull-back of the fibration

$$\mathscr{F} \to BP \to BGL(n, \mathbb{C}).$$

Now using theorem 20.6 of Borel [2] it follows easily that the latter fibration is cohomologically trivial (with $\mathbf{Z}$-coefficients). Hence the first fibration is also cohomologically trivial. Hence we get

$$H^*(B \text{ Par Aut}(E)) \approx H^*(B \text{ Aut } E) \otimes H^*(\mathscr{F}).$$

As $\mathscr{C}$ is contractible, the equivariant cohomology of $\mathscr{C}$ is the cohomology of B Par Aut (E). Hence

$$H^*_{\text{Par Aut}(E)}(\mathscr{C}) \approx H^*(B \text{ Aut } E) \otimes H^*(\mathscr{F}).$$

Remark 3.3. Atiyah and Bott have determined the cohomology of B Aut (E) (see theorem 2.15 in [1]). In particular, they prove that B Aut (E) is torsion-free. Hence it follows from Prop. 3.1 that $H^*_{\mathrm{Par\ Aut}\,(E)}(\mathscr{C})$ has no torsion.

We next reduce the equivariant cohomology of any non semi-stable stratum $\mathscr{C}_{\lambda,I}$ to the tensor product of the equivariant cohomologies of semi-stable strata for some parabolic bundles of lower ranks. This is the basic inductive step.

PROPOSITION 3.4. Let $0 \subset E_1 \subset E_2 \subset \ldots \subset E_r = E$ be a fixed C^∞-filtration of E of type (λ, I). Then

$$H^*_{\mathrm{Par\ Aut}\,(E)}(\mathscr{C}_{\lambda,I}) \approx \bigotimes_{i=1}^{r} H^*_{\mathrm{Par\ Aut}\,(D_i)}(\mathscr{C}_{ss}(D_i)),$$

where $D_i = E_i/E_{i-1}$ with the induced parabolic structure.

Proof. As shown in §2, the pairs $(\mathscr{C}_{\lambda,I}$, Par Aut $(E))$ and $(\mathscr{B}_{\lambda,I}$, Par Aut $(E_{\lambda,I}))$ are equivalent for equivariant cohomology. Now fix a C^∞ parabolic splitting $E \approx D_1 \oplus \ldots \oplus D_r$ of the chosen filtration $0 \subset E_1 \subset E_2 \subset \ldots \subset E_r$ of type (λ, I), so that $E_i = D_1 \oplus \ldots \oplus D_i$. Let Par Aut $(E^0_{\lambda,I})$ and $\mathscr{B}^0_{\lambda,I}$ be the automorphisms and complex structures in $\mathscr{B}_{\lambda,I}$ compatible with this decomposition. Then we have

$$\text{Par Aut}\,(E^0_{\lambda,I}) \approx \Pi\ \text{Par Aut}\,(D_i)$$

$$\mathscr{B}^0_{\lambda,I} \approx \Pi\,\mathscr{C}_{ss}(D_i).$$

(Note that the parabolic structure on D_i here is by definition the parabolic structure of E_i/E_{i-1}). On the other hand the natural homomorphism

$$\text{Par Aut}\,(E_{\lambda,I}) \to \text{Par Aut}\,(E^0_{\lambda,I})$$

is a homotopy equivalence, and the fibration $\mathscr{B}_{\lambda,I} \to \mathscr{B}^0_{\lambda,I}$ is the globally trivial fibration with fibre the infinite dimensional vector space $\bigoplus_{i<j}$ Hom (D_j, D_i) and is compatible with the group action. It follows that, for purposes of equivariant cohomology, the pairs $(\mathscr{C}_{\lambda,I}$, Par Aut $(E))$ and $(\mathscr{B}_{\lambda,I}$, Par Aut $(E^0_{\lambda,I}))$ are equivalent. Hence we see that the equivariant cohomology of the stratum $\mathscr{C}_{\lambda,I}(E)$ is isomorphic to the tensor product of the equivariant cohomology of the semi-stable strata for the quotients D_i. This is the exact analogue of Prop. 7.12 of Atiyah and Bott.

PROPOSITION 3.5. Each stratum $\mathscr{C}_{\lambda,I}$ is connected.

Proof. We have already proved that the semi-stable stratum is connected (see §2). Hence $\mathscr{B}^0_{\lambda,I} \approx \Pi\,\mathscr{C}_{ss}(D_i)$ is connected. Hence $\mathscr{B}_{\lambda,I}$ is connected as it is a fibration over $\mathscr{B}^0_{\lambda,I}$ with a vector space as the fibre. Now we have a fibration $\mathscr{C}_{\lambda,I} \to \mathscr{F}_{\lambda,I}$ with fibre $\mathscr{B}_{\lambda,I}$. Hence it remains to show that $\mathscr{F}_{\lambda,I}$ is connected. For this we need the following which is easy to prove.

LEMMA. Let G be a topological group, let G', H be closed subgroups, and let $H' = G' \cap H$. Let G/G' be simply connected and let G/H and H/H' be connected. Then G'/H' is connected.

Now take $G' = \text{Aut}(E)$, $G' = \text{Par Aut}(E)$, and H to be the subgroup of G which preserves the filtration $0 \subset E_1 \subset E_2 \subset \ldots \subset E_r$ but not necessarily the parabolic structure. Atiyah and Bott show in §9 of [1] that G/H is connected. Now, $H' = \text{Par Aut}(E_{\lambda,l})$, and both G/G' and H/H' are homomorphic to the flag variety $GL(n)/P$ which is simply connected. It follows that $\mathcal{F}_{\lambda,l} = G'/H'$ is connected.

PROPOSITION 3.6. The equivariant Euler class of the normal bundle $N_{\lambda,l}$ to a stratum $\mathcal{C}_{\lambda,l}$ is not a zero divisor in the equivariant cohomology ring $H^*{}_{\text{Par Aut }(E)}(\mathcal{C}_{\lambda,l})$ with integral coefficients.

Proof. The equivariant Euler class of $N_{\lambda,l}$ is by definition the usual Euler class of the vector bundle $EG \times_G N_{\lambda,l}$ on the space $EG \times_G \mathcal{C}_{\lambda,l}$ where G denotes the group Par Aut(E). Exactly the same reduction as in the proof of Prop. 3.4 shows that we can replace the triple. (Par Aut (E), $\mathcal{C}_{\lambda,l}$, $N_{\lambda,l}$) by the triple (Par Aut $(E^0_{\lambda,l})$, $\mathcal{B}^0_{\lambda,l}$, $N^0_{\lambda,l}$) where $N^0_{\lambda,l}$ is the restriction of $N_{\lambda,l}$ to $\mathcal{B}^0_{\lambda,l}$. Now as Par Aut $(E^0_{\lambda,l}) \approx \Pi$ Par Aut (D_i), it contains the r-dimensional torus T^r, which acts trivially on $\mathcal{B}^0_{\lambda,l}$. Now at a point of $\mathcal{B}^0_{\lambda,l}$ our bundle E is a holomorphic direct sum of the D_i, and hence Par End$''$ $E \approx \bigoplus_{i<j}$ Par Hom (D_i, D_j). Now on Par Hom (D_i, D_j) the element $(t_1, \ldots, t_r) \in T^r$ acts by $t_i^{-1}t_j$ and so it acts by the same character on H^1 $(X$, Par Hom $(D_i, D_j))$. Since the fibre of the normal bundle N is H^1 $(X$, Par End$''$ $E)$, it follows that the representation of T^r on a fibre of N is 'primitive', i.e., its one-dimensional components in a direct sum decomposition are indivisible elements of the character group of T^r. Hence by Prop. 13.4 of Atiyah and Bott it follows that the equivariant Euler class of $N_{\lambda,l}$ is not a zero-divisor in the equivariant cohomology of $\mathcal{C}_{\lambda,l}$.

With this we see that all the conditions stated in §3.1 are satisfied by the stratification $\mathcal{C} = \cup \mathcal{C}_{\lambda,l}$. Hence we have the following formula relating the various equivariant Poincaré series

$$\tilde{P}_t(\mathcal{C}) = \Sigma t^{C_{\lambda,l}} \tilde{P}_t(\mathcal{C}_{\lambda,l}). \tag{3.7}$$

Substituting the result for $\tilde{P}_t(\mathcal{C})$ from Prop. 3.2, we get the following formula in which the summation on the right hand side is taken only over all non semi-stable strata.

PROPOSITION 3.8. The equivariant Poincaré series for the semi-stable stratum is as given by the following inductive formula

$$\tilde{P}_t(\mathcal{C}_{ss}) = P_t(\mathcal{F})P_t(B \text{ Aut } E) - \Sigma t^{2d_{\lambda,l}} \tilde{P}_t(\mathcal{C}_{\lambda,l})$$

where $P_t(\mathcal{F})$ is the Poincaré series for the flag variety $\mathcal{F}$ (which is well known), $P_t(B \text{ Aut } E)$ is the Poincaré series for B Aut E (which is known by theorem 2.15 of Atiyah and Bott [1]), $d_{\lambda,l}$ is the complex codimension of $\mathcal{C}_{\lambda,l}$ in $\mathcal{C}$ which is as given by Prop. 1.17, and $\bar{P}_t(\mathscr{C}_{\lambda,l})$ is the equivariant Poincaré series for the stratum $\mathscr{C}_{\lambda,l}$ which is in turn expressible in terms of the Poincaré series for the semi-stable strata for parabolic bundles of lower ranks using Prop. 3.4.

PROPOSITION 3.9. Let S be the moduli space of *stable* parabolic bundles of the given rank, degree and parabolic data. Let $f: \mathscr{C}_s \to S$ be the canonical set-theoretic map from the stable part of $\mathscr{C}$ to S. Then f induces a homeomorphism

$$\bar{f}: \mathscr{C}_s/\text{Par Aut}(E) \xrightarrow{\sim} S.$$

Proof. The universal property of S implies that the restriction of f to any algebraic family contained in $\mathscr{C}_s$ is continuous. From the discussion in §2 it now follows that f is itself continuous. Since the fibres of f are precisely the orbits of Par Aut (E) in $\mathscr{C}_s$, we get a continuous bijection $\bar{f}: \mathscr{C}_s/\text{Par Aut}(E) \to S$. Now, S is a non-singular quasi-projective variety, and for any local algebraic family V in $\mathscr{C}_s$ which is transversal to the orbits, the map $f: V \to S$ is of maximal rank and hence open. From this it follows that f is open and hence $\bar{f}$ is a homeomorphism.

THEOREM 3.10. The Poincaré series for the moduli space S of parabolic bundles over a curve in the case 'stable = semi-stable' is as follows.

$$P_t(S) = (1-t^2)\cdot\bar{P}_t(\mathscr{C}_{ss}),$$

where $\bar{P}_t(\mathscr{C}_{ss})$ is given by Prop. 3.8. Further, if the rank and degree of the parabolic bundles are coprime, then the integral cohomology of S is torsion free.

Proof. Let $\bar{\mathscr{P}}$ be the quotient group of $\mathscr{P} = \text{Par Aut}(E)$ by the subgroup $\mathbb{C}^*$. Let $\overline{\text{Aut}(E)}$ be the quotient of Aut (E) by $\mathbb{C}^*$. Atiyah and Bott show that the fibration

$$B\mathbb{C}^* \to B\,\text{Aut}(E) \to B\overline{\text{Aut}(E)}$$

is trivial in rational cohomology (and if in addition, (rank, degree) = 1, then it is trivial in integral cohomology). Hence by base change, the fibration

$$B\mathbb{C}^* \to B\,\text{Par Aut}(E) \to B\bar{\mathscr{P}}$$

is trivial in rational cohomology and also trivial in integral cohomology when (rank, degree) = 1. Hence it follows that for rational coefficients,

$$H^*_{\text{Par Aut}(E)}(\mathscr{C}_{ss}) \approx H^*_{\bar{\mathscr{P}}}(\mathscr{C}_{ss}) \otimes H^*(B\mathbb{C}^*)$$

which also holds for integral coefficients whenever (rank, degree) = 1. Now, every stable bundle is simple (see Seshadri [8]). Hence $\bar{\mathscr{P}}$ acts freely on $\mathscr{C}_{ss}$ and hence $H^*_{\bar{\mathscr{P}}}(\mathscr{C}_{ss})$ is just the singular cohomology of the moduli space. Since $P_t(B\mathbb{C}^*) = 1/(1-t^2)$ we get

$$P_t(S) = (1-t^2)\ \bar{P}_t(\mathscr{C}_{ss}).$$

Note that Prop. 3.8 gives $\bar{P}_t(\mathscr{C}_{ss})$ inductively in terms of the $\bar{P}_t(\mathscr{C}_{ss})$ for lower ranks. To begin the process we need to know $H^*_{\text{Par Aut}(L)}(\mathscr{C}_{ss}(L))$ for a parabolic line bundle L. But in this case, the moduli space is just the Jacobian J and the rank and degree of L are coprime. Hence

$$H^*_{\text{Par Aut}(L)}(\mathscr{C}_{ss}(L)) \approx H^*(J) \otimes H(B\mathbb{C}^*)$$

with integral coefficients. Hence $H^*_{\text{Par Aut}(L)}(\mathscr{C}_{ss}(L))$ is torsion free and

$$\tilde{P}_t(\mathscr{C}_{ss}(L)) = \frac{(1+t)^{2g}}{(1-t^2)}.$$

Since the stratification $\mathscr{C} = \cup \mathscr{C}_{\lambda, I}$ is equivariantly perfect over $\mathbf{Z}$, it now follows inductively that $H^*_{\text{Par Aut}(E)}(\mathscr{C}_{ss}(E))$ is torsion free for any E. Hence when (rank, deg) $= 1$, it follows from

$$H^*_{\text{Par Aut}(E)}(\mathscr{C}_{ss}, \mathbf{Z}) \approx H^*(S, \mathbf{Z}) \otimes H^*(B\mathbb{C}, \mathbf{Z})$$

that $H^*(S, \mathbf{Z})$ is also torsion-free.

Remark 3.11. Following exactly a similar argument used by Atiyah and Bott, one may show that the Poincaré series $P_t(S_0)$ of the moduli $S_0 \subset S$ of parabolic bundles of a fixed determinant in the 'stable = semi-stable' case is given by the formula

$$P_t(S_0) = \frac{P_t(S)}{(1+t)^{2g}}.$$

Finally, as an example we determine the Poincaré polynomial of the moduli space when rank $= 2$. Let $P \in X$ be the parabolic point and $0 \leq \alpha_1 < \alpha_2 < 1$ be the weights for the parabolic flag in E_p of the type $E_P = F_1 \supset F_2 \supset 0$ where dim $F_2 = 1$. Let d be the fixed degree. It is easy to see that any semi-stable bundle of this type is automatically stable.

PROPOSITION 3.11. In the above case, the Poincaré polynomial of the moduli space is

$$P_t(S) = \frac{(1+t)^{2g-2}((1+t^3)^{2g} - t^{2g}(1+t)^{2g})}{(1-t)^2}.$$

(Note: It is easy to see directly that this rational function is indeed a polynomial).

Proof. Let E be a non-semi-stable bundle of the above type and let $0 \subset E_1 \subset E$ be its parabolic Harder–Narasimhan filtration. Let

$$k = \deg F_1 \text{ and } e = \dim\ ((E_1)_p \cap F_2).$$

The pair (k, e) characterizes the compound type (λ, I) of E as can be seen immediately from the definition of compound type. Also note that Par $\mu(E_1) >$ Par $\mu(E)$ if and only if $2k - d + e \geq 1$. Hence we may write the stratification of $\mathscr{C}$ as

$$\mathscr{C} = \mathscr{C}_{ss} \cup (\bigcup_{(k,e)} \mathscr{C}_{k,e})$$

where $2k-d+e \geq 1$, $k \in \mathbf{Z}$, $e = 0, 1$. From Prop. 1.17 it follows that the complex codimension of the stratum $\mathscr{C}_{k,e}$ is $d_{k,e} = 2k-d+e-1+g$ where $g = \text{genus}(X)$. Also, it follows from Prop. 3.4 and our knowledge of $\bar{P}_t(\mathscr{C}_{ss}(L))$ for a line bundle L that

$$\bar{P}_t(\mathscr{C}_{k,e}) = \bar{P}_t(\mathscr{C}_{ss}(L))^2 = \frac{(1+t)^{4g}}{(1-t^2)^2}.$$

In our case, the flag variety $\mathscr{F}$ is just $\mathbf{P}^1$, so $P_t(\mathscr{F}) = 1+t^2$. Moreover, by theorem 2.15 of [1],

$$P_t(B\ \text{Aut}\ E) = \frac{(1+t)^{2g}(1+t^3)^{2g}}{(1-t^2)^2(1-t^4)}.$$

Hence Prop. 3.8 gives, after some simplification,

$$\bar{P}_t(\mathscr{C}_{ss}) = \frac{(1+t)^{2g-2}((1+t^3)^{2g} - t^{2g}(1+t)^{2g})}{(1-t^2)(1-t)^2}$$

An application of theorem 3.10 now completes the proof.

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