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Some generating functions of Jacobi polynomials

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Abstract. In this paper, Weisner's group-theoretic method of obtaining generating functions is utilized in the study of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ by giving suitable interpretations to the index (n) and the parameter (β) to find out the elements for constructing a six-dimensional Lie algebra.

Keywords. Jacobi polynomials; generating function.

1. Introduction

It is known that the Jacobi polynomials, defined by [9],

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\,{}_2F_1\left[\begin{matrix}-n,\ 1+\alpha+\beta+n; & \dfrac{1-x}{2}\\ 1+\alpha; & \end{matrix}\right] \tag{1}$$

satisfies the following ordinary differential equation:

$$(1-x^2)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \{\beta-\alpha-(2+\alpha+\beta)x\}\frac{\mathrm{d}u}{\mathrm{d}x} + n(1+\alpha+\beta+n)u = 0. \tag{2}$$

The object of the present note is to derive some generating functions, which are believed to be new, of Jacobi polynomials as defined in (1) by suitably interpreting the index (n) and the parameter (β) simultaneously with the help of Weisner's group-theoretic method [10] (for previous work on $P_n^{(\alpha,\beta)}(x)$ by the same method see [1, 2, 4–8]). The main result (obtained by finding a set of infinitesimal operators $A_{ij}(i=1,2;\ j=1,2,3)$ constituting a Lie algebra) of our investigation is given in §3.

2. Group-theoretic method

Replacing $\mathrm{d}/\mathrm{d}x$ by $\partial/\partial x$, β by $y(\partial/\partial y)$, n by $z(\partial/\partial z)$ and $u(x,y,z)$ by $v(x,y,z)$ in (1) we get the following partial differential equation:

$$(1-x^2)\frac{\partial^2 v}{\partial x^2} + y(1-x)\frac{\partial^2 v}{\partial y\,\partial x} + yz\frac{\partial^2 v}{\partial z\,\partial y} + z^2\frac{\partial^2 v}{\partial z^2} - \{(1+x)\alpha+2x\}\frac{\partial v}{\partial x} + (\alpha+2)z\frac{\partial v}{\partial z} = 0. \tag{3}$$

Thus $v_1(x, y, z) = P_n^{(\alpha,\beta)}(x)y^\beta z^n$ is a solution of the differential equation (3) since $P_n^{(\alpha,\beta)}(x)$ is a solution of (2).

Now by using the following differential recurrence relations [1, 9]:

$$\frac{d}{dx} P_n^{(\alpha,\beta)}(x) = \frac{1}{x-1}[nP_n^{(\alpha,\beta)}(x) - (\alpha+n)P_{n-1}^{(\alpha,\beta+1)}(x)], \tag{4}$$

$$\frac{d}{dx} P_n^{(\alpha,\beta)}(x) = \frac{1}{1-x^2}[\{(1+\alpha+\beta+n)(x+1)-2\beta\} \times P_n^{(\alpha,\beta)}(x) - 2(n+1)P_{n+1}^{(\alpha,\beta-1)}(x)], \tag{5}$$

$$\frac{d}{dx} P_n^{(\alpha,\beta)}(x) = \frac{1}{1-x}[(1+\alpha+\beta+n)\{P_n^{(\alpha,\beta)}(x) - P_n^{(\alpha,\beta+1)}(x)], \tag{6}$$

$$\frac{d}{dx} P_n^{(\alpha,\beta)}(x) = \frac{1}{1-x^2}[-\{n(x+1)+2\beta\} \times P_n^{(\alpha,\beta)}(x) + 2(\beta+n)P_n^{(\alpha,\beta-1)}(x)]. \tag{7}$$

We define the infinitesimal operators $A_{ij}(i=1, 2;\ j=1, 2, 3)$ as follows:

$$\begin{aligned}
A_{11} &= y(\partial/\partial y),\\
A_{12} &= (x-1)y\,z^{-1}(\partial/\partial x) - y(\partial/\partial z)\\
A_{22} &= (1-x^2)y^{-1}z(\partial/\partial x) - z(x-1)(\partial/\partial y)\\
&\quad - (1+x)y^{-1}z^2(\partial/\partial z) - (1+\alpha)(1+x)y^{-1}z\\
A_{21} &= z(\partial/\partial z)\\
A_{13} &= (1-x^2)y^{-1}(\partial/\partial x) + 2(\partial/\partial y) + (1+x)y^{-1}z(\partial/\partial z),\\
A_{23} &= (1-x)y(\partial/\partial x) - y^2(\partial/\partial y) - yz(\partial/\partial z) - (1+\alpha)y,
\end{aligned}$$

such that

$$\begin{aligned}
A_{11}(P_n^{(\alpha,\beta)}(x)\,y^\beta z^n) &= \beta P_n^{(\alpha,\beta)}(x)\,y^\beta z^n,\\
A_{12}(P_n^{(\alpha,\beta)}(x)\,y^\beta z^n) &= -(\alpha+n)P_{n-1}^{(\alpha,\beta+1)}(x)\,y^{\beta+1}z^{n-1},\\
A_{22}(P_n^{(\alpha,\beta)}(x)\,y^\beta z^n) &= -2(n+1)P_{n+1}^{(\alpha,\beta-1)}(x)\,y^{\beta-1}z^{n+1},\\
A_{21}(P_n^{(\alpha,\beta)}(x)\,y^\beta z^n) &= n\,P_n^{(\alpha,\beta)}(x)\,y^\beta z^n,\\
A_{13}(P_n^{(\alpha,\beta)}(x)\,y^\beta z^n) &= 2(\beta+n)P_n^{(\alpha,\beta-1)}(x)\,y^{\beta-1}z^n,\\
A_{23}(P_n^{(\alpha,\beta)}(x)\,y^\beta z^n) &= -(1+\alpha+\beta+n)\,P_n^{(\alpha,\beta+1)}(x)\,y^{\beta+1}z^n.
\end{aligned}$$

Now we shall find the commutator relations. Using the notation:

$$[A, B]u = (AB - BA)u,$$

we have

$$
\begin{array}{ll}
[A_{11}, A_{12}] = A_{12}, & [A_{12}, A_{23}] = 0, \\
[A_{11}, A_{13}] = -A_{13}, & [A_{21}, A_{22}] = A_{22}, \\
[A_{11}, A_{21}] = 0, & [A_{21}, A_{23}] = 0, \\
[A_{11}, A_{22}] = -A_{22}, & [A_{13}, A_{21}] = 0, \\
[A_{11}, A_{23}] = A_{23}, & [A_{13}, A_{22}] = 0, \\
[A_{12}, A_{13}] = 0, & [A_{13}, A_{23}] = -2(2A_{11} + 2A_{21} + 1 + \alpha), \\
[A_{12}, A_{21}] = A_{12}, & [A_{22}, A_{23}] = 0. \\
[A_{12}, A_{22}] = 2(2A_{21} + (1 + \alpha)), &
\end{array} \tag{8}
$$

From the above commutator relations we state the following theorem:

THEOREM

The set of operators $\{1, A_{ij}\ (i = 1, 2;\ j = 1, 2, 3)\}$ where 1 stands for the identity operator, generates a Lie algebra.

It can be easily shown that the partial differential operator L, given by

$$
L = (1-x^2)\frac{\partial^2}{\partial x^2} + y(1-x)\frac{\partial^2}{\partial y \partial x} + yz\frac{\partial^2}{\partial z \partial y} + z^2\frac{\partial^2}{\partial z^2} - \Big\{(1+x)\,\alpha + 2x\Big\}\frac{\partial}{\partial x} + (\alpha+2)\,z\frac{\partial}{\partial z}
$$

can be expressed as follows:

$$
(x-1)\,Lu = (A_{22}\,A_{12} - 2A_{21}^2 - 2\alpha\,A_{21})u
$$

and
$$
(x-1)\,Lu = (-A_{13}\,A_{23} - 2(A_{11} + A_{21} + 1)(A_{11} + A_{21} + 1 + \alpha))u. \tag{9}
$$

From (8) and (9), one can easily verify that the operators A_{ij} $(i = 1, 2; j = 1, 2, 3)$ commute with $(x-1)L$, i.e.,

$$
\Big[(x-1)L, A_{ij}\Big] = 0, \tag{10}
$$

The extended form of the groups generated by A_{ij} $(i = 1, 2; j = 1, 2, 3)$ are given by

$$
\exp(a_{11}A_{11})\ u(x, y, z) = u(x, \exp(a_{11})\ y, z), \tag{11}
$$

$$
\exp(a_{21}A_{21})\ u(x, y, z) = u(x, y, \exp(a_{21})\ z), \tag{12}
$$

$$
\exp(a_{12}A_{12})\ u(x, y, z) = u\left(\frac{zx - a_{12}\,y}{z - a_{12}\,y}, y, z - a_{12}\,y\right), \tag{13}
$$

$$\exp(a_{22}A_{22})\ u(x,y,z) = u\left(\frac{y}{y+a_{22}(1+x)z}\right)^{\alpha+1} u\left(\frac{xy+a_{22}(1+x)z}{y+a_{22}(1+x)z},\right.$$

$$\left.\frac{y(y+2a_{22}z)}{y+a_{22}(1+x)z}, \frac{yz}{y+a_{22}(1+x)z}\right), \tag{14}$$

$$\exp(a_{13}A_{13})\ u(x,y,z) = u\left(\frac{xy+a_{13}(1+x)}{y+a_{13}(1+x)}, y+2a_{13},\right.$$

$$\left.\frac{z\{y+a_{13}(1+x)\}}{y}\right) \tag{15}$$

$$\exp(a_{23}A_{23})\ u(x,y,z) = (1+a_{23}y)^{-\alpha-1}$$

$$\times u\left(\frac{x+a_{23}y}{1+a_{23}y}, \frac{y}{1+a_{23}y}, \frac{z}{1+a_{23}y}\right) \tag{16}$$

where $a_{ij}(i=1,2;\ j=1,2,3)$ are arbitrary constants. Thus we have

$$\exp(a_{23}A_{23})\exp(a_{13}A_{13})\exp(a_{22}A_{22})\exp(a_{12}A_{12})\exp(a_{21}A_{21})$$
$$\times\exp(a_{11}A_{11})\ u(x,y,z) = [y/\{y(1+a_{23}y)+a_{22}z(1+x+2a_{23}y)\}]^{\alpha+1}$$
$$\times u(\xi,\eta,\zeta) \tag{17}$$

where

$$\xi = \frac{\begin{gathered}z[y\{y(x+a_{23}y)+a_{13}(1+a_{23}y)(1+x+2a_{23}y)\}+a_{22}z(1+x+2a_{23}y)\\ \{y+a_{13}(1+x+2a_{23}y)\}]-a_{12}[y\{y+2a_{13}(1+a_{23}y)\}\\ +2a_{22}z\{y+a_{13}(1+x+2a_{23}y)\}][y(1+a_{23}y)+a_{22}z(1+x2a_{23}y)]\end{gathered}}{\begin{gathered}[y(1+a_{23}y)+a_{22}z(1+x+2a_{23}y)][z\{y+a_{13}(1+x+2a_{23}y)\}\\ -a_{12}\{y(y+a_{13}(1+a_{23}y))+2a_{22}z(y+a_{13}(1+x+2a_{23}y))\}],\end{gathered}}$$

$$\eta = \exp(a_{11})\frac{y\{y+2a_{13}(1+a_{23}y)\}+2a_{22}z\{y+a_{13}(1+x+2a_{23}y)\}}{y(1+a_{23})+a_{22}z(1+x+2a_{23}y)} \tag{18}$$

and

$$\zeta = \exp(a_{21})\frac{\begin{gathered}z\{y+a_{13}(1+x+2a_{23}y)\}-a_{12}[y\{y+2a_{13}(1+a_{23}y)\}\\ +2a_{22}z\{y+a_{13}(1+x+2a_{23}y)\}]\end{gathered}}{y(1+a_{23}y)+a_{22}z(1+x+2a_{23}y)} \tag{19}$$

3. Generating functions

From (3) $v(x,y,z) = P_n^{(\alpha,\beta)}(x)y^{\beta}z^n$ is a solution of the system:

$$Lv=0; \qquad Lv=0; \qquad Lv=0;$$
$$(A_{11}-\beta)v=0; \quad (A_{21}-n)v=0; \quad (A_{11}+A_{21}-\beta-n)v=0.$$

From (13) we easily get

$$[(x-1)L]\,S(P_n^{(\alpha,\beta)}(x)y^{\beta}z^n)=S[(x-1)L]\,(P_n^{(\alpha,\beta)}(x)y^{\beta}z^n)=0,$$

where

$$S=\exp(a_{23}A_{23})\exp(a_{13}A_{13})\exp(a_{22}A_{22})\exp(a_{12}A_{12})\exp(a_{21})\exp(a_{11}).$$

Therefore the transformation $S(P_n^{(\alpha,\beta)}(x)y^{\beta}z^n)$ is also annulled by L.

Now by putting $a_{11}=a_{21}=0$ and replacing $u(x,y,z)$ by $P_n^{(\alpha,\beta)}(x)y^{\beta}z^n$ in (19) we get

$$\begin{aligned}&\exp(a_{23}A_{23})\exp(a_{13}A_{13})\exp(a_{22}A_{22})\exp(a_{12}A_{12})(P_n^{(\alpha,\beta)}(x)y^{\beta}z^n)\\&=y^{\alpha+1}[y(1+a_{23}y)+a_{22}z(1+x+2a_{23}y)]^{-1-\alpha-\beta-n}\\&\quad\times[y\{y+2a_{13}(1+a_{23}y)\}+2a_{22}z\{y+a_{13}(1+x+2a_{23}y)\}]^{\beta}\\&\quad\times(z\{y+a_{13}(1+x+2a_{23}y)\}-a_{12}[y\{y+2a_{13}(1+a_{23}y)\}\\&\quad+2a_{22}z\{y+a_{13}(1+x+2a_{23}y)\}])^n\,P_n^{(\alpha,\beta)}(\xi).\end{aligned}\tag{20}$$

But,

$$\begin{aligned}&\exp(a_{23}A_{23})\exp(a_{13}A_{13})\exp(a_{22}A_{22})\exp(a_{12}A_{12})\,(P_n^{(\alpha,\beta)}(x)y^{\beta}z^n)\\&=\sum_{s=0}^{\infty}\frac{a_{23}^s(-1)^s}{s!}(1+\alpha+\beta-r+n)_s\sum_{r=0}^{\infty}\frac{a_{13}^r}{r!}2^r(n+\beta-r+1)_r\\&\times\sum_{m=0}^{\infty}\frac{a_{22}^m}{m!}(-2)^m(n-k+1)_m\sum_{k=0}^{n+m}\frac{a_{12}^k}{k!}(-\alpha-n)_k\\&\times P_{n-k+m}^{(\alpha,\beta+k-m-r+s)}(x)\,y^{\beta+k-m-r+s}z^{n-k+m}\end{aligned}\tag{21}$$

Equating (20) and (21) we get

$$\begin{aligned}&[y/\{y(1+a_{23}y)+a_{22}z(1+x+2a_{23}y)\}]^{1+\alpha+\beta+n}\Big[y+2a_{13}(1+a_{23}y)+2a_{22}\frac{z}{y}\\&\times\{y+a_{13}(1+x+2a_{23}y)\}\Big]^{\beta}\Big(\frac{z}{y}\{y+a_{13}(1+x+2a_{23}y)\}\\&-a_{12}[y+2a_{13}(1+a_{23}y)\\&+2a_{22}\frac{z}{y}\{y+a_{13}(1+x+2a_{23}y)\}]\Big)^n P_n^{(\alpha,\beta)}(\xi)\\&=\sum_{s=0}^{\infty}\frac{(-a_{23})^s}{s!}(1+\alpha+\beta-r+n)_s\sum_{r=0}^{\infty}\frac{(2a_{13})^r}{r!}(n+\beta-r+1)_r\\&\times\sum_{m=0}^{\infty}\frac{(-2a_{22})^m}{m!}(n-k+1)_m\sum_{k=0}^{n+m}\frac{(a_{12})^k}{k!}(-\alpha-n)_k\\&\times P_{n-k+m}^{(\alpha,\beta+k-m-r+s)}(x)\,y^{\beta+k-m-r+s}z^{n-k+m},\end{aligned}\tag{22}$$

where

$$\xi = \frac{\begin{aligned} &z[y\{y(x+a_{23}y)+a_{13}(1+a_{23}y)(1+x+2a_{23}y)\}+a_{22}z(1+x+2a_{23}y) \\ &\{y+a_{13}(1+x+2a_{23}y)\}]-a_{12}[y\{y+2a_{13}(1+a_{23}y)\} \\ &+2a_{22}z\{y+a_{13}(1+x+2a_{23}y)\}]\,[y(1+a_{23}y)+a_{22}z(1+x+2a_{23}y)] \end{aligned}}{\begin{aligned} &[y(1+a_{23}y)+a_{22}z(1+x+2a_{23}y)][z\{y+a_{13}(1+x+2a_{23}y)\} \\ &-a_{12}\{y(y+2a_{13}(1+a_{23}y))+2a_{22}z(y+a_{13}(1+x+2a_{23}y))\}] \end{aligned}}$$

The above generating function does not appear before from which we can get a large number of particular generating relations by attributing different values to a_{12}, a_{22}, a_{13}, a_{23}.

4. Derivation of some known results

Case 1.

Putting $a_{13}=a_{23}=0$ in (22), we get

$$\begin{aligned} &\left(\frac{y}{y+a_{22}z(1+x)}\right)^{1+\alpha+n+\beta}(y+2a_{22}z)^{\beta}\{z-a_{12}(y+2a_{22}z)\}^{n} \\ &\times P_n^{(\alpha,\beta)}\left(\frac{z\{xy+a_{22}z(1+x)\}-a_{12}(y+2a_{22}z)\{y+a_{22}z(1+x)\}}{\{y+a_{22}z(1+x)\}\{z-a_{12}(y+2a_{22}z)\}}\right) \\ &=\sum_{m=0}^{\infty}\frac{(a_{22})^m}{m!}(-2)^m(n-k+1)_m\sum_{k=0}^{n+m}\frac{(a_{12})^k}{k!}(-\alpha-n)_k \\ &\times P_{n-k+m}^{(\alpha,\beta+k-m)}(x)\,y^{\beta+k-m}\,z^{n-k+m} \end{aligned} \tag{23}$$

which is the main result obtained by Chakraborti [1].

Case 2.

Putting $a_{12}=a_{13}=a_{23}=0$, $a_{22}=1$ and writing $z/y=-u/2$ we get

$$\begin{aligned} &\frac{(1-u)^{\beta}}{\{1-\frac{1}{2}u(1+x)\}^{1+\alpha+\beta+n}}P_n^{(\alpha,\beta)}\left(\frac{x-\frac{1}{2}u(1+x)}{1-\frac{1}{2}u(1+x)}\right) \\ &=\sum_{m=0}^{\infty}\frac{1}{m!}(n+1)_m P_{n+m}^{(\alpha,\beta-m)}(x)u^m. \end{aligned}$$

Finally putting $n=0$ we get

$$(1-u)^{\beta}(1-\tfrac{1}{2}u(1+x))^{-1-\alpha-\beta}=\sum_{m=0}^{\infty} P_m^{(\alpha,\beta-m)}(x)u^m, \tag{24}$$

which is a well-known formula due to Feldhim [3].

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References

[1] Chakraborti A B, group theoretic study of certain generating functions of Jacobi polynomials, *J. Indian Inst. Sci.* **B64** (1983) 97–103

[2] Chongdar A K, Group theoretic study for certain generating functions, *Bull. Calcutta Math. Soc.* **77** (1985) 151–157

[3] Feldhim E, Relations entre les polynomes de Jacobi, Laguerre et Hermite, *Acta Math.* **75** (1943) 117–198

[4] Ghosh B, Group theoretic study of certain generating functions of Jacobi polynomials, *Bull. Calcutta Math. Soc.* **75** (1983) 227–231

[5] Ghosh B, On a class of trilateral generating functions for Jacobi polynomials, *Ranchi Univ. Math. J.* **14** (1983) 11–16

[6] Ghosh B, Group theoretic origins of certain generating functions of Jacobi polynomials I, *Pure Math. Manuscript* **3** (1984) 139–144

[7] Ghosh B, Group theoretic origins of certain generating functions of Jacobi polynomials, II, (1986) communicated

[8] Ghosh B, Group theoretic origins of certain generating functions of Jacobi polynomials III, *Pure Math. manuscript* **3** (1984) (to appear)

[9] Rainville E D, *Special functions*, Chelsea Publishing Co. Brons, New York (1960)

[10] Weisner L, Group theoretic origin of certain generating functions, *Pacific J. Math.* **5** (1955) 1033–1039