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Computation of $g(1,5;\ 10)$

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Abstract. It is established that the exact covering number $g(1,5;\ 10)$ is 102. It is further shown that this configuration is unique. It can be obtained from the unique Steiner system $S(5,\ 6,\ 12)$.

Keywords. Exact covering number; Steiner system.

1. Introduction

A $(\lambda, \mu; v)$ design is an arrangement of v varieties (also called points or elements) into blocks in such a way that every set of μ varieties occurs in exactly λ blocks $(\lambda \geqslant 1, 1 \leqslant \mu < v)$. The blocks must be incomplete (length $v-1$ or less), but need not all have the same length. The $\lambda - \mu$ problem is to determine $g(\lambda, \mu; v)$, the minimum number of blocks in any $(\lambda, \mu; v)$, design. A $(\lambda, \mu; v)$ design with $g(\lambda,\ \mu; v)$ blocks is called *minimal*. A Steiner system $S(t,\ k,\ v)$ is arrangement of v treatments into blocks of size k such that every t-tuple occurs exactly once.

In this paper we show that $g(1,\ 5;\ 10) = 102$ and that there is a unique configuration achieving this bound. It is known [1] that Steiner system $S(5,\ 6,\ 12)$ exists and is unique. Naturally the unique minimal $(1, 5; 10)$ design is the two-point deletion of $S(5,\ 6,\ 12)$.

2. Block sizes and upper bound on $g(1,\ 5;\ 10)$

Obviously for a minimal $(\lambda, \mu; v)$ design we consider blocks of size at least μ. We first state a useful result due to Woodall [2].

THEOREM A (Woodall). If $(1, \mu; v)$ design contains a block of length k then it contains at least $1 + h_1\ (\mu, k,\ v)$ blocks where

$$h_1(\mu,\ k,\ v) = (v-k)\binom{k}{\mu-1}\left\{1 - \frac{v-k-1}{2(k-\mu+2)}\right\}.$$

It is known [1] that there is a (unique) Steiner system $S(5,\ 6,\ 12)$. Deleting two points form $S(5,\ 6,\ 12)$ we get a $(1,\ 5;\ 10)$ design with 102 blocks. Thus $g(1,\ 5;$

10) $\leqslant$ 102. Using this information along with theorem A we get the following:

PROPOSITION 2.1. A minimal (1, 5; 10) design cannot contain a block of size greater than 7.

3. The blocks of size 7 in a minimal (1, 5; 10) design

The following result is an immediate consequence of the definition of a (1, 5; 10) design.

PROPOSITION 3.1. There can be at most 3 blocks of size 7 in a (1, 5; 10) design.

We next successively rule out the existence of blocks of size 7 in a minimal (1, 5; 10) design. We use the following notation: n_i denotes the number of blocks of size i in a minimal $(\lambda, \mu; v)$ design, $\mu \leqslant i \leqslant v-1$. Clearly in a minimal (1, 5; 10) design, where now the admissible block sizes are 5, 6 and 7, we must have

$$n_5 + \binom{6}{5} n_6 + \binom{7}{5} n_7 = \binom{10}{5} = 252. \quad (1)$$

PROPOSITION 3.2. In a minimal (1, 5; 10) design there cannot be 3 blocks of size 7.

Proof. Let, without loss of generality, the three blocks of size 7 be (1, 2, 3; 4, 5, 6, 7), (1, 2, 3, 4, 8, 9, 10) and (4, 5, 6, 7, 8, 9, 10). Now a block of size 6 is of the form (0, 0, 0, 8, 9, 10) or of the form $(0, 0, 0, 0, x, x)$ where $0 \in \{1, 2, 3, 4, 5, 6, 7\}$ and $x \in \{8, 9, 10\}$. It can be directly seen that a block of size 6 cannot be of the form (0, 0, 0, 8, 9, 10). It can also be directly seen that $n_6 \leqslant 3 \times 3 = 9$. Now, using (1) we see that with $n_7 = 3$, $n_6 \leqslant 5$ we get $n_5 \geqslant 135$. Since $g(1, 5; 10) \leqslant 102$, the result follows.

PROPOSITION 3.3. In a minimal (1, 5; 10) design there cannot be two blocks of size 7.

Proof. As before, without loss of generality, let the two blocks of size 7 be (1, 2, 3, 4, 5, 6, 7) and (1, 2, 3, 4, 8, 9, 10). Let m (respectively t) be the number of blocks of size 6 of the form (0, 0, 0, 8, 9, 10) (respectively $(0, 0, 0, 0, x, x)$ where $0 \in \{1, 2, 3, 4, 5, 6, 7\}$ and $x \in \{8, 9, 10\}$. As before it can be seen that $m \leqslant 3$ and $t \leqslant 3 \times 6 = 18$. So $n_6 \leqslant 21$. However, from (1) and the fact that $g(1, 5; 10) \leqslant 102$, we must have $n_6 \geqslant 22$, a contradiction.

PROPOSITION 3.4. In a minimal (1, 5; 10) design there cannot be exactly 1 block of size 7.

Proof. Suppose there is a unique block of size 7. Take it as (1, 2, 3, 4, 5, 6, 7). We have the following picture of blocks of size 6 and size 5.

l	m
0	0
0	0
0	0
x	0
x	x
x	x
n_6	

a	b	c
0	0	0
0	0	0
x	0	0
x	x	0
x	x	x
n_5		

Here l, m, a, b, c denote the number of blocks of respective type. We call any block among these l blocks an L-type block. M-type block is similarly defined. Note that 5-tuples not covered by the unique block of size 7 are of three types: $(0, 0, x, x, x)$, $(0, 0, 0, x, x)$ and $(0, 0, 0, 0, x)$. Counting them in two ways we get respectively,

(i) $3l + a = \binom{7}{2} = 21$,

(ii) $3l + 4m + b = \binom{7}{3} \times \binom{3}{2} = 105$,

(iii) $2m + c = \binom{7}{4} \times \binom{3}{1} = 105$.

Thus

$$6l + 6m + a + b + c = 231 = 6n_6 + n_5. \tag{2}$$

Observe that a given 3-tuple from 7 points is contained in four 4-tuples on those 7 points. We also note that if (i, j, k, x, x, x) is one of the L-type blocks then the triple i, j, k cannot be a part of any M-type block. To be precise, if (i, j, k, x, x, x) is a block then it rules out blocks (i, j, k, t, x, x) for any $t \in \{1, 2, \ldots, 7\} - \{i, j, k\}$. Thus

$$m \leqslant \binom{7}{4} - 4l = 35 - 4l. \tag{3}$$

Let $D_k(n)$ denote the maximum number of k-subsets of an n-set such that no $(k-1)$-tuple is repeated. It is easy to see that $D_4(7) = 7$. Thus

$$m \leqslant D_4(7) \times 3 = 21. \tag{4}$$

Now, let $m = 35 - 4l - i$, $0 \leqslant i \leqslant 35 - 4l$. The total number of blocks is

$$1 + l + m + a + b + c = 1 + l + (35 - 4l - i) + 231 - 6l - 6(35 - 4l - i). \text{using (2)}$$

$$= 57 + 15l + 5i. \tag{5}$$

If $l \geqslant 4$ then from (5) we see that the total number of blocks exceeds 102. Next, for $l \leqslant 3$ take $m = 21 - i$ in view of (4). The total number of blocks is then

$$1 + l + m + a + b + c = 1 + l + (21 - i) + 231 - 6l - 6(21 - i)$$

$$= 127 - 5l + 5i. \tag{6}$$

This rules out $l \leqslant 3$ for a minimal (1, 5; 10) design. The proof of Proposition 3.4 is now complete.

Combining the above three results we get

THEOREM 3.5. In a minimal (1, 5; 10) design the block sizes are 6 and 5.

4. Minimal (1, 5; 10) designs with block sizes 6 and 5

We now have

$$6n_6 + n_5 = \binom{10}{5} = 252. \tag{7}$$

Since $n_5 + n_6 \leq 102$ we get

$$n_6 \geq 30. \tag{8}$$

Now let $(1, 2, 3, 4, 5, 6) = B_0$ be a block of size 6. The other blocks of size 6 fall into three types L, M and T as picturised below:

l	m	t
0	0	0
0	0	0
7	0	0
8	x	0
9	x	x
10	x	x

where l, m, t indicate the number of blocks of type L, M and T respectively, $0 \in \{1, 2, 3, 4, 5, 6\}$ and $x \in \{7, 8, 9, 10\}$. Clearly

$$l \leq 3. \tag{9}$$

Consider the following finer picture for the blocks of type M and type T.

m_1	m_2	m_3	m_4	t_1	t_2	t_3	t_4	t_5	t_6
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
7	7	7	8	0	0	0	0	0	0
8	8	9	9	7	7	7	8	8	9
9	10	10	10	8	9	10	9	10	10

It is easy to see that $D_3(6) = 4$. Hence we have

$$0 \leq m_i \leq 4 \ \forall i, \ 1 \leq i \leq 4. \tag{10}$$

It is easy to see that $D_4(6) = 3$, and therefore

$$0 \leq t_i \leq 3 \ \forall i, \ 1 \leq i \leq 6. \tag{11}$$

It can easily be seen that

$$\begin{aligned}
m_1 &= 4 \Rightarrow t_1+t_2+t_4 \leq 3,\\
m_1 &= 3 \Rightarrow t_1+t_2+t_4 \leq 6,\\
m_2 &= 4 \Rightarrow t_1+t_3+t_5 \leq 3,\\
m_2 &= 3 \Rightarrow t_1+t_3+t_5 \leq 6,\\
m_3 &= 4 \Rightarrow t_2+t_3+t_6 \leq 3,\\
m_3 &= 3 \Rightarrow t_2+t_3+t_6 \leq 6,\\
m_4 &= 4 \Rightarrow t_4+t_5+t_6 \leq 3,\\
m_4 &= 3 \Rightarrow t_4+t_5+t_6 \leq 6.
\end{aligned} \tag{12}$$

We are now in a position to prove the following result which leads to the unique minimal (1, 5; 10) design.

THEOREM 4.1. $m+t \leq 26$. Further, the equality holds if and only if $m=8$ and $t=18$ with $m_i=2$, $t_j=3$, $\forall_i$, $1 \leq i \leq 4$, and for all j, $1 \leq j \leq 6$.

Proof. We consider two cases.

Case 1. Suppose some $m_i=4$.

If at least two m_i's are 4, then it follows from (11) and (12) that $t=\sum_{i=1}^{6} t_i \leq 3+3+3=9$ and then $m+t \leq 16+9=25$.

If precisely one m_i is 4 then $m \leq 4+3+3+3=13$ and $t=\sum_{i=1}^{6} t_i \leq 3+3\times 3=12$ giving $m+t \leq 25$.

Case 2. No m_i is 4.

If all the m_i's are 3, then $m=12$ and $2\sum_{i=1}^{6} t_i \leq 24$ giving $t \leq 12$ and hence $m+t \leq 24$.

If precisely three m_i's are 3, let without loss of generality, $m_1=m_2=m_3=3$. Then $m \leq 3+3+3+2=11$ and $2(t_1+t_2+t_3)+(t_4+t_5+t_6) \leq 6+6+6=18$. If $t=(t_1+t_2+t_3)+(t_4+t_5+t_6) \geq 15$, we get $t_1+t_2+t_3 \leq 3$. This implies that $t_4+t_5+t_6 \geq 12$, a contradiction to (10). Thus $t \leq 14$ and hence $m+t \leq 11+14=25$.

If the number of m_i's which equal 3 is one or two, then $m \leq 3+3+2+2=10$ and $t \leq 6+3\times 3=15$ giving $m+t \leq 25$.

If no m_i is 3, then $m \leq 8$ and $t \leq 18$ giving $m+t \leq 26$.

It is clear from the above proof that $m+t=26$ if and only if $m=8$ and $t=18$ with each $m_i=2$ and each $t_j=3$, $1 \leq i \leq 4$, $1 \leq j \leq 6$.

We now give our final result of this section.

THEOREM 4.2. In a minimal $g(1, 5; 10)$ design $n_6=1+l+m+t=1+3+8+18=30$ and $n_5=72$. Consequently $g(1, 5; 10)=102$.

Proof. From theorem 3.5 we know that a minimal (1, 5; 10) design has blocks of sizes 6 and 5 only. From theorem 4.1 we know that $n_6 = 1+l+(m+t) \leqslant 1+3+26 = 30$. From (8) we have $n_6 \geqslant 30$. Thus $n_6 = 30$ (and $l = 3$). Consequently from (7) we get $n_5 = 72$ and it now follows that $g(1, 5; 10) = n_6 + n_5 = 30 + 72 = 102$.

5. Construction of the unique minimal (1, 5; 10) design

Start with the block $B_0 = (1, 2, 3, 4, 5, 6)$. We know that $l = 3$, $m = 8$, $t = 18$ with $m_i = 2\ \forall i$, $t_j = 3\ \forall j, 1 \leqslant l \leqslant 4, 1 \leqslant j \leqslant 6$. Therefore, without loss of generality, we have the following partial set-up for the $n_6 = 30$ blocks of size 6 in a minimal (1, 5; 10) design. We have written blocks as columns.

B_0						m					
1	1	3	5	1	2	1	2	1	2	1	2
2	2	4	6	3	4	3	4	4	3	4	3
3	7	7	7	5	6	6	5	5	6	6	5
4	8	8	8	7	7	7	7	7	7	8	8
5	9	9	9	8	8	8	8	9	9	9	9
6	10	10	10	9	9	10	10	10	10	10	10

(13)

								t									
0																	
0																	
0																	
0																	
7	7	7	7	7	7	7	7	7	8	8	8	8	8	8	9	9	9
8	8	8	9	9	9	10	10	10	9	9	9	10	10	10	10	10	10

(14)

where the blank spaces under t are to be filled with suitable choices of 4-tuples of 0's, $0 \in \{1, 2, 3, 4, 5, 6\}$. Remembering all the time that no 5-tuple is to be repeated, it can directly be seen that the blank spaces under t can be filled in a unique way. Without loss of generality, the unique completion of the blanks under t which give us blocks of type T is as follows:

1	1	2	1	1	2	3	1	1	3	1	1	1	1	2	1	2	1
2	4	3	2	3	3	4	2	2	4	2	2	2	3	3	2	4	3
3	5	5	5	4	4	5	3	4	5	3	4	5	4	4	3	5	5
4	6	6	6	6	5	6	5	6	6	6	5	6	5	6	4	6	6
7	7	7	7	7	7	7	7	7	8	8	8	8	8	8	9	9	9
8	8	8	9	9	9	10	10	10	9	9	9	10	10	10	10	10	10

The $n_6 = 30$ blocks given by (12) and (14) together completely describe the unique minimal (1, 5; 10) design. The $n_5 = 72$ blocks of size 5 are precisely those 5-tuples which are not covered by the 30 blocks of size 6.

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