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On subnormal operators

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Abstract. Let S be a pure subnormal operator such that $C^*(S)$, the C^*-algebra generated by S, is generated by a unilateral shift U of multiplicity 1. We obtain conditions under which S is unitarily equivalent to $\alpha+\beta U$, α and β being scalars or S has C^*-spectral inclusion property. It is also proved that if in addition, S has C^*-spectral inclusion property, then so does its dual T and $C^*(T)$ is generated by a unilateral shift of multiplicity 1. Finally, a characterization of quasinormal operators among pure subnormal operators is obtained.

Keywords. Subnormal operators; self-dual subnormal operators; quasinormal operators; unilateral shifts; C^*-algebra; C^*-spectral inclusion property.

By an operator we mean a bounded linear operator on a fixed separable infinite dimensional Hilbert space H. For an operator A, let $\sigma_e(A)$, $\sigma_a(A)$ and $\sigma(A)$ denote respectively the essential spectrum, the approximate point spectrum and the spectrum of A and let $C^*(A)$ denote the C^*-algebra generated by A. Let S be a subnormal operator on H and

$$N=\begin{pmatrix} S & X \\ o & T^* \end{pmatrix} \ldots (*)$$

be its minimal normal extension on $\mathcal{K}=H\oplus H^{\perp}$. The subnormal operator T appearing in the representation $(*)$ is called the dual of S and has been studied by Conway [5]. Dual of S is unique up to unitary equivalence; and if S is pure, then N^* is the minimal normal extension of T. A subnormal operator S is said to be self-dual if it is unitarily equivalent to its dual.

A subnormal operator S is said to have C^*-spectral inclusion property (C^*-SIP) if $\sigma(L)\subset\sigma(T_L)$ for every operator $L\in C^*(N)$, where T_L is the compression of L to H. Keough [7] has shown that S has C^*-SIP if and only if $\sigma(N)=\sigma_a(S)$.

It is known that every pure quasinormal operator and so every unilateral shift of multiplicity 1 is self-dual and has C^*-SIP [5], [8]. In this paper, we investigate the properties of pure subnormal operators S having C^*-SIP and for which $C^*(S)$ is generated by a unilateral shift of multiplicity 1.

One may ask if there are self-dual pure subnormal operators S other than translations of scalar multiples of unilateral shift of multiplicity 1 (up to unitary equivalence) such that S has C^*-SIP and $C^*(S)$ is generated by a unilateral shift of multiplicity 1. For pure quasinormal operators, as we shall show below, the answer is no. We need the following result which appeared in [6].

THEOREM 1. Let T be a hyponormal operator. Then $C^*(T)$ is generated by a unilateral shift of multiplicity 1 if and only if T is unitarily equivalent to an operator A satisfying the following properties:

(a) A is irreducible;
(b) A^*A-AA^* is compact;
(c) $\sigma_e(A)$ is a simple closed curve γ;
(d) $\sigma(A)=\gamma \cup G$, where G is the bounded component of the complement of γ;
(e) for $\lambda \in G$, $\operatorname{ind}(A-\lambda)=-1$.

Let θ denote the class of all operators A for which A^*A commutes with $A+A^*$. Every quasinormal operator belongs to θ, and every hyponormal operators in θ is subnormal [3]. It is not known whether these subnormal operators are self-dual or have C^*SIP. However, we prove the following.

THEOREM 2. Let S be a self-dual pure subnormal operator in θ. If $C^*(S)$ is generated by a unilateral shift U of multiplicity 1, then S is unitarily equivalent to $\alpha+\beta\ U$ for some scalars α and β.

First we establish the following particular case which appears to be of independent interest.

PROPOSITION. Let S be a pure quasinormal operator. If $C^*(S)$ is generated by a unilateral shift of multiplicity 1, then S is unitarily equivalent to a constant multiple of U.

Proof. By Brown's characterization of quasinormal operators obtained in [1], there is a positive definite operator P on a Hilbert space M such that S is unitarily equivalent to

$$\begin{pmatrix} o & & & & & \\ P & o & & & & \\ o & P & o & & & \\ & o & P & o & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot \quad \cdot \end{pmatrix}$$

on $M \oplus M \oplus M \ldots\ldots$ Since $C^*(S)$ is generated by a unilateral shift of multiplicity 1, S satisfies the conditions (a), (b) and (c) of theorem 1. Therefore P must be compact and by corollary 2 of [11], $\sigma(P)=\{c\}$ for some constant $c>0$. The irreducibility of S now gives the desired result.

Proof of theorem 2. Since S is self-dual, by a result of Murphy [10], there exists a normal operator B such that $[S] = S^*S - SS^* = BB^*$ and $S^*B = BS$. Since $S \in \theta$, we have $S^*BB^* = BB^*S$. Therefore $B^*SB = S^*B^*B = BB^*S = B^*S^*B$ and so $[S]S[S]$ is self-adjoint. Now by theorem 4 of [2], $S = A + Q$ where A is a self-adjoint operator commuting with every operator in $C^*(S)$ and Q is a quasinormal operator. Since $C^*(S)$ is generated by the unilateral shift U of multiplicity 1, the self-adjoint operator A must be unitarily equivalent to an analytic Toeplitz operator on the H^2-space of unit circle [4, corollary 6.13]. Therefore $A = \alpha I$ for some scalar α, $S - \alpha I$ is pure quasinormal and $C^*(S - \alpha I)$ is generated by U. The above proposition now implies that there is a scalar β such that S is unitarily equivalent to $\alpha + \beta S$.

In [7], Keough has shown that if a pure subnormal operator S has C^*-SIP, then its minimal normal extension N has no isolated eigenvalues of finite multiplicity. In the converse direction, we have the following.

THEOREM 3. Suppose S is a self-dual subnormal operator and $C^*(S)$ is generated by a unilateral shift of multiplicity 1. If the minimal normal extension N has no isolated eigenvalues of finite multiplicity, then S has C^*-SIP.

Proof. Since $C^*(S)$ is generated by unilateral shift of multiplicity 1, S satisfies conditions (a) to (d) of theorem 1. The dual T of S is unitarily equivalent to S and so $\sigma(S) = \gamma \cup G$ is symmetric about the real axis [5, Proposition 2.1]. Therefore $\sigma_e(S) = \gamma$ is also symmetric about the real axis. Using theorem 13.9 of [4], we have $\sigma_e(N) = \sigma_e(S) \cup \sigma_e(T^*) = \sigma_e(S) \cup \sigma_e(S^*) = \sigma_e(S) = \sigma_a(S)$; and since N has no isolated eigenvalues of finite multiplicity, $\sigma(N) = \sigma_e(N)$. Therefore $\sigma(N) = \sigma_a(S)$ as required.

Our next result shows that a subnormal operator S with C^*-SIP for which $C^*(S)$ is generated by a unilateral shift of multiplicity 1 strives to be a self-dual.

THEOREM 4. If a subnormal operator S has C^*-SIP and $C^*(S)$ is generated by a unilateral shift of multiplicity 1, then its dual T has C^*-SIP and $C^*(T)$ is generated by a unilateral shift of multiplicity 1. If in addition, $\sigma_e(S)$ is symmetric about the real axis, then there exists a unitary operator V and a compact operator K such that $T = VSV^{-1} + K$.

Proof. The operator S satisfies conditions (a) to (e) of theorem 1. By theorem 13.9 of [4], corollary 1.8 and proposition 1.4 of [5] we have (i) $\sigma_a(S) = \sigma_e(S)$, (ii) T is irreducible, (iii) $T^*T - TT^*$ is compact, (iv) $\sigma_a(T) = \sigma_e(T)$ and (v) $\sigma(T) = \sigma(S^*)$.

We first show that $C^*(T)$ is generated by a unilateral shift of multiplicity 1. In view of the facts (ii), (iii) and (v) and theorem 1, it is sufficient to show that $\sigma_e(T) = \gamma^*$, the complex conjugate of γ; and ind $(T - \lambda) = -1$ for every $\lambda \in G^*$.

Since S has C^*-SIP, N has no isolated eigenvalues of finite multiplicity. Therefore $\sigma_e(N) = \sigma(N) = \sigma_a(S) = \sigma_e(S)$; and since $\sigma_e(N) = \sigma_e(S) \cup \sigma_e(T^*)$, we must have $\sigma_e(T^*) \subset \sigma_e(S) = \gamma$. Now for $\lambda \in G$, $0 = \text{ind}(N - \lambda) = \text{ind}(S - \lambda) + \text{ind}$

$(T^*-\lambda) = -1+\text{ind}(T^*-\lambda)$ so that $\text{ind}(T^*-\lambda) = 1$. Therefore $\sigma_e(T^*)$ must separate the plane and so we have $\sigma_e(T^*) = \gamma$, that is, $\sigma_e(T) = \gamma^*$. Now if $\lambda \in G^*$, then $\text{ind}(T-\lambda) = -\text{ind}(T^*-\lambda^*) = -1$.

Since $\sigma(N^*) = \gamma^* = \sigma_e(T) = \sigma_a(T)$, T has C^*-SIP.

The last conclusion of the theorem follows from theorem 11.1 of [9].

If S is a quasinormal operator with ker $S = \{0\}$ and if N is the minimal normal extension of S, then $H \subset \overline{N^*(H)}$ [8]. Since S^*S commutes with SS^*, this, when combined with the following result, characterizes quasinormal operators among pure subnormal operators.

THEOREM 5. If S is a pure subnormal operator for which S^*S commutes with SS^* and $H \subset \overline{N^*(H)}$, then S is quasinormal.

Proof. Consider the matrix representation $(*)$ of N. Let h be an arbitrary vector in H and let

$$k = \begin{pmatrix} S^*[S]h \\ X^*S^*Sh \end{pmatrix}.$$

Now $SS^*[S]h = [S]SS^*h = XX^*SS^*h$. Therefore $Nk \in H^\perp$, that is, $k \perp N^*(H)$. Since $H \subset \overline{N^*(H)}$, $k \in H^\perp$ and so $S^*[S]h = 0$. Therefore $S^*[S] = 0$ and S is quasinormal.

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