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# Analytic and harmonic maps into a topological space

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**Abstract.** The relationship between the harmonicity and analyticity of a continuous map from the open unit disc to the underlying space of a real algebra is investigated.

**Keywords.** Analytic map; harmonic map; function algebra.

## 1. Introduction

Real function algebras were defined in [3] and certain aspects of the theory of real function algebras were developed along the lines of complex function algebras. In particular, sufficient conditions were given for the presence of an analytic structure in the carrier space of a real function algebra, [3]. These conditions automatically imply harmonic structure both in the carrier space as well as the maximal ideal space. A question was raised whether weaker conditions could be found which would imply harmonic structure in the maximal ideal space without necessarily implying analytic structure in the carrier space. This question acquires greater significance in view of the result that in a complex function algebra the existence of a harmonic structure implies that of an analytic structure or an anti-analytic structure, [4]. So, it is natural to inquire whether a similar relationship holds in a real function algebra.

In this paper, we prove the following two results which provide partial answers to the above question. In fact the first result has been proved in a more general setting than that of a real function algebra.

1. Let $X$ be a Hausdorff topological space and $A$, a real algebra of continuous complex-valued functions defined on $X$. Let $U$ be the open unit disc in the complex plane, $F: U \to X$ a continuous map such that Re $(f \circ F)$ is harmonic for all $f$ in $A$ and $Y$ be the set $Y = \{x \in X: \mathrm{Im} g(x) = 0 \text{ for all } g \text{ in } A\}$. Then $f \circ F$ is analytic for all $f$ in $A$ or $\overline{f \circ F}$ is analytic for all $f$ in $A$ on each connected component of $U - F^{-1}(Y)$.

2. If $X$ is a compact plane set whose interior is connected and which is symmetric with respect to the real axis, $\tau$ the complex conjugation and if $A$ is a real function algebra on $(X, \tau)$ such that Re$f$ is harmonic in $X^0 =$ interior of $X$ for all $f$ in $A$ then $f$ is analytic for all $f$ in $A$ or $f$ is anti-analytic for all $f$ in $A$ in $X^0$.

## 2. Basic definitions and lemmas

As usual **R** denotes the real line, **C** denotes the complex-plane and $U$ the open unit disc in **C**.

Let $K$ be a non-empty open subset of **C**, $f$ a complex-valued function on $K$ such that the real and imaginary parts of $f$ have continuous partial derivatives of order $k$. Such functions will be referred to as $C^k$-maps. We recall that $f$ is said to be anti-analytic if $\bar{f}$ is analytic.

LEMMA 1. Let $f$ be a complex-valued function defined on an open connected subset $K$ of **C** whose real and imaginary parts $u$ and $v$ are $C^2$-maps. Assume that $\mathrm{Re}f$, $\mathrm{Re}f^2$, $\mathrm{Re}f^3$ and $\mathrm{Re}f^4$ are all harmonic functions. Then, $f$ is an analytic function on $K$ or an anti-analytic function on $K$.

*Proof.* Refer [4], theorem 2.2.

LEMMA 2. Let $K$ be an open connected set in **C** and $A$ a subset of $C(K)$, the set of all continuous, complex-valued functions on $K$. Suppose $A$ is closed under multiplication, and every function in $A$ is analytic or anti-analytic in $K$. Then all functions in $A$ are analytic or all functions in $A$ are anti-analytic.

*Proof.* Proof is elementary and hence omitted. Readers can refer to theorem 3-1 of [4].

LEMMA 3. Let $V$ be an open set in $\mathbf{R}^n$, $A$ a ring of continuous complex-valued functions (under point-wise multiplication) on $V$ such that $\mathrm{Re}f$ is of class $C^k$ for each $f \in A$. Let $Y = \{x \in V : \mathrm{Im}g(x) = 0 \text{ for all } g \text{ in } A\}$. Then $\mathrm{Im}f$ is of class $C^k$ on $V-Y$ for each $f$ in $A$.

*Proof.* Let $f = u+iv \in A$ and $x \in V-Y$. Then there exists $g$ in $A$ with say $g = u_1+iv_1$ such that $v_1(x) \neq 0$.

Since $v_1$ is continuous in $V$, we can find a neighbourhood $D$ of $x$ such that $v_1(x) \neq 0$ for all $x$ in $D$.

By hypothesis, $u = \mathrm{Re}f$, $u_1 = \mathrm{Re}g$ and $\mathrm{Re}(fg) = uu_1 - vv_1$ are of class $C^k$. Thus $vv_1$ is of class $C^k$.

But in $D$, $v = vv_1/v_1$. Both $vv_1$ and $v_1$ are of class $C^k$ in $D$ and $v_1$ is non-vanishing in $D$. Hence $v$ is of class $C^k$ in $D$ and in particular at $x$. Thus, $v$ is a $C^k$-map in $V-Y$.

## 3. Main theorems

Let $X$ be a Hausdorff topological space and $A$ a real algebra of continuous, complex-valued functions on $X$. A continuous map $F: U \to X$ is called a harmonic map if $\mathrm{Re}(f \circ F)$ is harmonic on $U$ for all $f$ in $A$. $F$ is called an analytic map (respectively an anti-analytic map) if the function $f \circ F$ is analytic (respectively anti-analytic) on $U$ for all $f$ in $A$.

THEOREM 4. Let $X$ be a Hausdorff topological space and $A$ a real algebra of continuous complex-valued functions defined on $X$. Let $Y = \{x \in X : \mathrm{Im} g(x) = 0$ for all $g$ in $A\}$ and $F : U \to X$ a harmonic map. Then $F$ is an analytic map or an anti-analytic map on each connected component of $U - F^{-1}(Y)$.

*Proof.* Let $\tilde{f} = f \circ F$ for $f$ in $A$, and $\tilde{A} = \{\tilde{f} : f \in A\}$. Then, $\tilde{A}$ is a real algebra on $U$. Since $\mathrm{Re}\tilde{f} = \mathrm{Re}(f \circ F)$ is harmonic, it is of class $C^2$ on $U$ for each $\tilde{f} \in \tilde{A}$.
Hence by lemma (3), $\mathrm{Im}\tilde{f}$ is of class $C^2$ on $U - F^{-1}(Y)$ for all $\tilde{f}$ in $\tilde{A}$. Now let $K$ be a connected component of $U - F^{-1}(Y)$. Then it is easy to see that each $\tilde{f} \in \tilde{A}$ satisfies the hypotheses of lemma (1).
Hence $\tilde{f}$ is analytic or anti-analytic by lemma (2). Therefore, by lemma (3) all functions in $\tilde{A}$ are analytic on $K$ or all functions in $\tilde{A}$ are anti-analytic on $K$.

COROLLARY 5. If $U - F^{-1}(Y)$ is connected then $F$ is analytic or anti-analytic in $U - F^{-1}(Y)$. In particular, if $F^{-1}(Y)$ is the null set then $F$ is an analytic map or an anti-analytic map in $U$.

## 4. Real function algebras

Let $X$ be a compact, Hausdorff space. By $C(X)$ (respectively $C_{\mathbf{R}}(X)$) we denote the complex (respectively real) Banach algebra of all continuous, complex-valued (respectively real-valued) functions on $X$ with the supremum norm. A homeomorphism $\tau : X \to X$ with $\tau^2 = \tau \circ \tau =$ identity mapping on $X$ is called an involution on $X$ or an involutionary homeomorphism on $X$. Let

$$C(X, \tau) = \{f \in C(X) : f(\tau(x)) = \overline{f(x)} \text{ for all } x \text{ in } X\}.$$

Then $C(X, \tau)$ is a real commutative Banach algebra with the identity 1. Also $C(X, \tau)$ separates points on $X$, that is, for any $x_1$, $x_2$ in $X$ with $x_1 \neq x_2$, there exists $f \in C(X, \tau)$ such that $f(x_1) \neq f(x_2)$. A real function algebra on $(X, \tau)$ is a real subalgebra $A$ of $C(X, \tau)$ such that (i) $A$ is uniformly closed in $C(X, \tau)$; (ii) $A$ contains the real constants; (iii) $A$ separates points on $X$. For examples and other details on complex function algebras refer [1] and [2] and for details about real function algebras, see [3].

*Remark* 6. Let $A$ be a real function algebra on $(X, \tau)$ and $Y = \{x \in X : \tau(x) = x\}$ the set of fixed points of $\tau$. Since $A$ separates the points of $X$, it is easy to see that $Y = \{x \in X : \mathrm{Im} g(x) = 0$ for all $g$ in $A\}$. Thus, in view of theorem (4) if $F : U \to X$ is a continuous map such that $\mathrm{Re}(f \circ F)$ is harmonic for all $f$ in $A$ then $F$ is an analytic or anti-analytic map in the connected components of $U - F^{-1}(Y)$.

*Example* 7. Let $X$ be the annular region in $\mathbf{C}$ defined by $X = \{z \in \mathbf{C} : r \leq |z| \leq 1/r\}$ for some $r$ with $0 < r < 1$ and let the involutionary homeomorphism $\tau$ be given by the equation $\tau(z) = -1|\bar{z}$ for all $z$ in $X$. Let $A$ be a real function algebra on $(X, \tau)$. Note that $\tau$ has no fixed points in $X$. By corollary 5 if $F: U \to X$ is a harmonic map then $F$ must be an analytic map or an anti-analytic map on $U$.

*Note* 8. In the case of a complex function algebra every harmonic map is an analytic map or an anti-analytic map, (theorem 3.1 of [4]), whereas in the case of a real (or a real function) algebra we could only prove that a harmonic map is an analytic map or an anti-analytic map only in connected components of $U - F^{-1}(Y)$. However, we point out that under the hypotheses of theorem 4 the set $F^{-1}(Y)$ has an empty interior if $F$ is non-constant. (Note that if $F$ is constant, it is trivially an analytic map). For, if $F^{-1}(Y)$ contains a disc $D$ then for $f$ in $A$ and $\tilde{f} = f \circ F = u + iv$, $v = 0$ throughout $D$.

But $v^2$ is real analytic, hence is identically 0 in $U$. Thus $u$ and $u^2$ are harmonic in $U$, hence $\tilde{f} = u$ is constant, a contradiction.

THEOREM 9. Let $X$ be a subset of $\mathbf{C}$ which is compact with connected interior and which is symmetric with respect to the real-axis, that is $\bar{z} \in X$ whenever $z \in X$ and $\tau: X \to X$ be defined by $\tau(z) = \bar{z}$ for all $z$ in $X$. Suppose that $A$ is a real function algebra on $(X, \tau)$ such that Re$f$ is harmonic in $X^0$ = interior of $X$ for all $f$ in $A$. Then $f$ is analytic for all $f$ in $A$ or anti-analytic for all $f$ in $A$ in $X^0$.

*Proof.* Let $K = \{z \in X^0 : \operatorname{Im} z > 0\}$ and $f = u + iv$ be in $A$.

By hypotheses, Re$f$, Re$f^2$, Re$f^3$, and Re$f^4$ are all harmonic in $K$. Hence by lemma 3 $v = \operatorname{Im} f$ is of class $C^2$ and since $K$ is connected, lemma 1 shows that $f$ is analytic or anti-analytic in $K$.

Since $f(\bar{z}) = \overline{f(z)}$ for all $z$ in $X$, by Schwarz's reflection principle $f$ is analytic or anti-analytic in the whole of $X^0$. Now lemma 2 implies that $f$ is analytic for all $f$ in $A$ in $X^0$ or $f$ is anti-analytic for all $f$ in $A$ in $X^0$.

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