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Effect of aspect ratio on the meridional circulation of a rotating homogeneous fluid

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Abstract. The effect of aspect ratio on the meridional circulation of a homogeneous fluid is analyzed. Aspect ratio is allowed to range between zero and unity. Relationships between possible horizontal and vertical length scales are obtained by length scale analysis as well as by solving an idealized problem. It is found that when $E^{1/2} \ll Z \ll E^{1/2}/\delta$, where E is the Ekman number, the stream lines are closely packed near the sidewall within a thickness of $O(E^{1/2})$. The effect of stratification is unimportant within this depth range. In the depth range $E^{1/2}/\delta \ll Z \ll 1/E\delta$ the vertical boundary layer in which the streamlines are packed is of $O(EZ\delta)^{1/3}$. When $Z \gg 1/E\delta$ it is shown that the circulation decays algebraically with depth as $1/Z$.

Keywords. Aspect ratio; meridional circulation; homogeneous fluid.

List of symbols:

u, v, w	Velocity components
ψ	Stream function
p	Pressure
H, L	Characteristic dimensions in vertical and horizontal directions
δ	aspect ratio
A_v, A_h	Vertical and horizontal eddy coefficients of momentum
E_v, E_h	Vertical and horizontal Ekman numbers
h, l	Vertical and horizontal length scales
ξ	Vertical distance
m	Fourier transform variable
f	Coriolis parameter.

1. Introduction

Analysis of sidewall friction boundary layers has assumed considerable importance since these layers play a significant part in ocean circulation, spin-up problems etc.

Stewartson [6] considered steady linear axisymmetric flow of a homogeneous fluid in a rotating cylindrical container with unit aspect ratio and found that the meridional circulation of fluid driven by the Ekman layers is completed within two sidewall layers, now known as Stewartson layers of thickness $E^{1/4}$ and $E^{1/3}$ where $E(=\nu/\Omega L^2)$ is the Ekman number. Pedlosky [4] and Durance and Johnson [3] analyzed the upwelling boundary layers whose axial scale is $O(1)$ for a linear and homogeneous oceanic model with β-plane approximation. While Pedlosky's analysis spanned the range for the aspect ratio $\delta \ll E^{1/2}$, the analysis of Durance and Johnson was for the range $\delta \gg E^{1/2}$. The aim of this paper is to understand the role of various horizontal and vertical length scales associated with meridional circulation in a rotating homogeneous fluid with a constant Coriolis parameter for $0 \ll \delta < O(1)$.

The main motivation of this work is to elucidate the effect of aspect ratio on the various length scales and circulation pattern in a rotating hydrodynamic flow and to draw comparisons with other studies wherever possible. To keep the model more akin to an oceanic model, our mathematical analysis closely follows that of Blumsack [2] who had analyzed the transverse circulation near a coast in a rotating stratified fluid for $\delta^2 \ll S \ll 1$ where S is the stratification parameter. Since Blumsack's work was concerned with a stratified fluid and $S \gg \delta^2$ the effect of aspect ratio on the length scales and circulation pattern has not received considerable attention.

2. Governing equations

Following Blumsack [2], we assume that the perturbation state is independent of y, the alongshore coordinate and all dependent variables are functions of the offshore coordinate x and the vertical coordinate z. Then the linear steady state equations in nondimensional form (for example see [2,4]) are,

$$
\begin{aligned}
-v &= -\frac{\partial P}{\partial x} + \frac{E_V}{2}\frac{\partial^2 u}{\partial z^2} + \frac{E_H}{2}\frac{\partial^2 u}{\partial x^2}, \\
u &= \frac{E_V}{2}\frac{\partial^2 v}{\partial z^2} + \frac{E_H}{2}\frac{\partial^2 v}{\partial x^2}, \\
o &= -\frac{\partial P}{\partial z} + \delta^2\left[\frac{E_V}{2}\frac{\partial^2 w}{\partial z^2} + \frac{E_H}{2}\frac{\partial^2 w}{\partial x^2}\right], \\
o &= \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z},
\end{aligned}
\tag{1}
$$

Here δ = vertical length scale divided by horizontal length scale = H/L is an aspect ratio; u,v,w are velocity components, P is the pressure, $E_V = 2A_V/fH^2$ and $E_H = 2A_H/fL^2$ are the vertical and horizontal Ekman numbers where A_V and A_H are eddy coefficients of momentum and f is the Coriolis parameter.

Introducing stream function ψ so that $u=-\partial\psi/\partial z$ and $W=\partial\psi/\partial x$ and eliminating P and V from equations, we get

$$\left(\delta^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) \left(\frac{E_V}{2} \frac{\partial^2}{\partial z^2} + \frac{E_H}{2} \frac{\partial^2}{\partial x^2} \right)^2 \psi + \frac{\partial^2 \psi}{\partial z^2} = 0.$$

In oceanic models, since it is difficult to determine E_H and E_V, it is assumed that $E_V = E_H = E$ for simplicity of exposition. This assumption leads to $H = L(A_V/A_H)^{1/2}$. The above equation becomes,

$$\frac{\partial^2\psi}{\partial z^2} + \frac{E^2}{4} \nabla^4 \left(\frac{\partial^2}{\partial z^2} + \delta^2 \frac{\partial^2}{\partial x^2} \right) \psi = 0, \tag{2}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}, \quad E = 2A_H/fL^2 .$$

3. Length scale analysis

To find the length scale relationships, replace in the above equation $\partial/\partial x$ by l^{-1} and $\partial/\partial z$ by h^{-1} where l and h are the horizontal and vertical length scales respectively. Equation (2) then becomes,

$$h^4 l^6 + \frac{E^2}{4}(h^2+l^2)^2 (\delta^2 h^2 + l^2) = 0. \tag{3}$$

When $l \gg h$, equation (3) gives

$$h \approx E^{1/2} \text{ if } l \gg E^{1/2}.$$

This vertical length scale corresponds to the Ekman depth.

For $h \gg l$, (3) becomes

$$h^4 l^6 + \tfrac{1}{4}E^2 h^4 l^2 + \tfrac{1}{4}E^2 \delta^2 h^6 = 0. \tag{4}$$

From (4) we get

$$l \approx \delta h \text{ if } h \ll E^{1/2}/\delta, \tag{5a}$$

$$l \approx E^{1/2} \text{ if } E^{1/2} \ll h \ll E^{1/2}/\delta, \tag{5b}$$

$$l \approx (Eh\delta)^{1/3} \text{ if } h \gg E^{1/2}/\delta. \tag{5c}$$

For $h = O(1)$, these length scales agree with those discussed earlier (see [4] and [3]). It may be noted that the hydrostatic $E^{1/3}$ layer (see [4]) which arises solely due to β-effect does not appear in our case due to the assumption of constant coriolis parameter. For $h \gg E^{1/2}/\delta$, the nonhydrostatic horizontal length scale $(Eh\delta)^{1/3}$ increases with h. If $\delta = O(1)$, we have $l \approx (Eh)^{1/3}$ for $h \gg E^{1/2}$ and therefore its thickness will be $O(E^{1/3})$ when $h = O(1)$. If $h = O(1)$ and $\delta \gg E^{1/2}$ the thickness of this layer becomes $(E\delta)^{1/3}$. It may be guessed that this layer will be dynamically similar to Stewartson's $E^{1/3}$ layer, and it has been analyzed by Durance and Johnson [3].

From (5a,b,c), for $h \ll E^{1/2}/\delta$ we see that the $(Eh\,\delta)^{1/3}$ layer splits into $E^{1/2}$ and $h\delta$ layers. These length scales arise due to the constraint that $\delta \ll 1$. If $h = O(1)$, these vertical layers exist for $\delta \ll E^{1/2}$. If $\delta = O(1)$, we see from (5) that these length scales merge together and correspond to the corner region whose horizontal and vertical length scales are $O(E^{1/2})$.

In what follows, we shall consider an infinite depth fluid and see how the various length scales mentioned above manifest themselves in the circulation pattern. To achieve this, the integral Fourier transform technique will be used, and to suit this technique, the boundary conditions will be assumed suitably.

4. Structure of the boundary layers

We consider an infinite depth fluid and assume that there is a stress $F(x)$ acting only in the y-direction (along shore) on the fluid horizontal surface at $z = 0$. $F(x)$ is supposed to tend to zero as $x \to \infty$. The boundary conditions at $z = 0$ may then be written as

$$V_z(x,0) = F(x);$$

$$\psi(x,0) = \psi_{zz}(x,0) = 0.$$

The boundary conditions at $x = 0$ are all taken as homogeneous. These are,

$$V_z = \psi = \psi_{xx} = 0 \quad \text{at } x = 0.$$

These conditions are appropriate for a problem in which $F(x)$ is an odd function and we shall be describing a shear layer at $x = 0$ if $F(O) \neq 0$. However, as noted by Blumsack [2], these boundary conditions can also be extended to the case of a rigid wall (where the correct condition is $\psi_x(0,z) = 0$ instead of $\psi_{xx}(0,z) = 0$) without affecting the physics of the problem if the boundary condition on w is satisfied in a region too thin to affect the circulation pattern.

Fourier sine transform analysis of (2) subject to the above boundary conditions gives for values of $\xi(= -z)$ much larger than the Ekman depth $E^{1/2}$

$$\psi(x,\xi) = \frac{E}{\pi}\int_0^{\infty} \bar{F}(m)\frac{\sin mx}{1+\frac{E^2}{4}m^4}\exp(-\xi/Z_D)\,dm, \tag{6}$$

where

$$Z_D(m) = \left(1+\frac{E^2}{4}m^4\right)^{1/2} \Big/ \frac{E}{2}m^3\delta$$

and $\bar{F}(m)$ is the Fourier sine transform of $F(x)$ defined as

$$\bar{F}(m) = \int_0^\infty F(x) \sin mx \, dx$$

For $E^{1/2} \ll \xi \ll E^{1/2} / \delta$ we see that $Z_D(m) \gg \xi$ for $m \leq O(E^{-1/2})$.

Hence in this depth range, $\exp(-\xi/Z_D)$ in (6) may be replaced by unity. Assuming $F(x) = e^{-x}$, we get from (6)

$$\psi(x,\xi) = \frac{E}{2} e^{-x} \left[1 - \exp(-xE^{1/2}) \cos(xE^{-1/2}) \right]. \tag{7}$$

Thus there is a boundary layer of thickness $O(E^{1/2})$ at $x = 0$ within which there is an intense vertical flow. This layer serves to complete the meridional circulation induced due to Ekman pumping. (The nonhydrostatic thinner layer of thickness $O(\delta h)$ carries negligible mass flux.) We see from (7) that for $F(x)$ positive and $F'(x)$ (which gives the curl of the wind stress) negative, the Ekman layer pumps the fluid into the interior and the circulation is closed by an intense upward motion in the $E^{1/2}$ vertical layer. This result is physically obvious if we recall (see [5]) that the horizontal mass flux associated with the free surface Ekman layer friction velocity is completely perpendicular to the applied stress and the interior vertical velocity at the lower edge of the Ekman layer is given by the curl of the wind stress. Further this solution is the same as that of Blumsack [2] obtained for the stratified case. The effect of stratification is not felt in the above depth range.

Consider now depths in the range $E^{1/2}/\delta \ll \xi \ll 1/E\delta$. For $\xi \gg E^{1/2}/\delta$ it is easily seen from the expression for $Z_D(m)$ that only values of $m \ll E^{-1/2}$ contribute significantly to ψ in (6). Equation (6) becomes

$$\psi(x,\xi) = \frac{E}{\pi} \int_0^\infty \frac{m}{1+m^2} \sin(mx) \exp\left[-(m^3 \delta E)\xi \right] dm. \tag{8}$$

As such, it seems difficult to obtain a closed form solution for the above integral when $E\delta\xi \leq O(1)$. However, we see from the equation, that near $x = 0$, there is a region of width $(E\delta\xi)^{1/3}$ in which streamlines are packed. The width of the region varies from $E^{1/2}$ when $\xi = E^{1/2}/\delta$ to $O(1)$ when $\xi = 1/E\delta$. Thus the upwelling region whose radial extent remains constant upto a depth of $E^{1/2}/\delta$ spreads laterally for greater depths.

Finally, we investigate the circulation for $\xi \gg (E\delta)^{-1}$. Evaluating (8) by Laplace's method [1], we get,

$$\psi(x,\xi) \approx \frac{x}{3\pi\delta\xi}.$$

Hence, the transverse circulation decays with depth algebraically as $1/\xi$. It may be noted that in the stratified case [6], it decays like $1/\xi^3$. Since the stratification inhibits the vertical flow, the decay is faster in the stratified case than in the homogeneous case Also we see that, for homogeneous case, the upwelling layer extends down to a greater depth and it increases as the aspect ratio decreases. Further, since the vertical mass flux decreases with depth, none may enter the lower Ekman Layer. This justifies the assumption of infinite depth fluid.

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