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Convective instability of a rotating layer of ferromagnetic fluid

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Abstract. The instability of a hot horizontal layer of ferromagnetic fluid rotating about a vertical axis has been investigated when the Prandtl number $P < 1$. Earlier it was shown that for $P > 1$ the overstability cannot occur. In this paper the convective and overstable marginal states have been investigated separately for $P < 1$ and it is found that though convective marginal state is possible for all a, the non-dimensional wave number, and N the Taylor number, the overstability is possible only if $N > (1+P)\pi^4/(1-P)$ and in case the condition is satisfied, overstability is possible for all those values of a which satisfy $a^2 < [N(1-P)\pi^2/(1+P)]^{1/3} - \pi^2$. If $R_c^{(\text{con})}$ and $R_c^{(\text{o.s})}$ are the critical values of the convective and the overstable marginal states respectively, then it is also found that $R_c^{(\text{con})} < R_c^{(\text{o.s})}$ provided N is not sufficiently large.

Keywords. Stability; convective marginal state; overstability; ferromagnetic fluid.

1. Introduction

Ferromagnetic fluids are formed by suspending submicron sized magnetic particles in a medium like kerosene or water with the addition of some anticoagulating liquid like Oleic acid. It is found that the continuum theory is applicable to study the flow behaviour of such a fluid. Due to certain important applications of the ferromagnetic fluid in energy conversion devices etc., there is a great deal of interest in studying the flow characteristics of the fluid in recent years.

Neuringer and Rosenweig [3] investigated the heat transfer processes in ferromagnetic fluid. Finlayson [2] investigated the stability of a hot layer of ferromagnetic fluid and showed that the principle of exchange of stabilities was valid in case the boundaries were free. Das Gupta and Gupta [1] studied the instability of a hot layer of ferromagnetic fluid rotating about a vertical axis and showed that the overstability could not occur if the Prandtl number $P > 1$. Since it appeared that there was the possibility of having overstability if $P < 1$, we have investigated in this paper the conditions under which the critical marginal state is convective or overstable when $P < 1$. To shorten our discussion, we have referred to the non-dimensional equations incorporating the normal mode analysis and the boundary conditions for free boundaries as were used by Das Gupta and Gupta [1] in §2. The critical values of the Rayleigh numbers $R_c^{(\text{con})}$ and $R_c^{(\text{o.s})}$ for the convective and the overstable marginal states have been calculated separately in

§3. In §4, we have presented the numerical results in two tables, and the plots $R_c^{(\mathrm{con})}$ and $R_c^{(\mathrm{o.s})}$ vs a^2 followed by a discussion.

2. Nondimensional equations and the boundary conditions

We consider a horizontal layer of incompressible ferromagnetic fluid heated from below and rotating about a vertical axis. Following the linearization of the basic equations, non-dimensionalization and using the normal mode analysis as were done by Das Gupta and Gupta [3], we present

$$\frac{\partial}{\partial t}(D^2 - a^2)W = -N^{1/2}D\zeta + aR^{1/2}[M_1 D\phi - (1 + M_1)T] + (D^2 - a^2)^2 W, \tag{1}$$

$$\frac{\partial \zeta}{\partial t} = (D^2 - a^2)\zeta + N^{1/2}DW, \tag{2}$$

$$P\frac{\partial T}{\partial t} = (D^2 - a^2)T + aR^{1/2}W, \tag{3}$$

$$D^2\phi - a^2 M_3 \phi - DT = 0, \tag{4}$$

and the boundary conditions

$$W = D^2 W = T = D\phi = 0 \quad \text{at} \quad z = \pm 1/2. \tag{5}$$

We have used the same notations as used by Das Gupta and Gupta [1] and we have taken N to represent the Taylor number $4\Omega^2 d^4/r^2$. For simplification, we have dropped M_2 in the above equations as $M_2 \simeq 10^{-6}$ ([2] and [1]).

Since the solutions of (1)–(5) can be separated into even and odd modes, we take them as

$$W = A\exp(\sigma_n t) \begin{matrix}\cos\\ \sin\end{matrix}\, r_n z, \quad T = B\exp(\sigma_n t) \begin{matrix}\cos\\ \sin\end{matrix}\, r_n z,$$

$$\phi = \frac{c}{r_n}\exp(\sigma_n t) \begin{matrix}\sin\\ \cos\end{matrix}\, r_n z, \quad \zeta = E\exp(\sigma_n t) \begin{matrix}\sin\\ \cos\end{matrix}\, r_n z, \tag{6}$$

where the upper terms signify even mode for which $r_n = (2n-1)\pi$ $(n = 1, 2, 3, \ldots)$ and the lower terms, for which $r_n = 2n\pi$ $(n = 1, 2, 3, \ldots)$, signify odd mode of the perturbation.

Substituting (6) into (1)–(4) and eliminating A, B, C and E, we get two characteristic equations for even and odd modes, which we combine into one, as

$$U\sigma_n^3 + V\sigma_n^2 + W\sigma_n + X = 0, \tag{7}$$

where

$$U = P(n^2\pi^2 + a^2)\,(n^2\pi^2 + a^2 M_3), \tag{8}$$

$$V = (2P+1)\ (n^2\pi^2 + a^2)^2\ (n^2\pi^2 + a^2 M_3), \tag{9}$$

$$W = (P+2)\ (n^2\pi^2 + a^2)^3\ (n^2\pi^2 + a^2 M_3) + NPn^2\pi^2(n^2\pi^2 + a^2 M_3) - a^2 R[n^2\pi^2 + a^2 M_3(1+M_1)], \tag{10}$$

$$X = (n^2\pi^2 + a^2)\ (n^2\pi^2 + a^2 M_3) + Nn^2\pi^2(n^2\pi^2 + a^2)\ (n^2\pi^2 + a^2 M_3) - a^2 R(n^2\pi^2 + a^2)\ [n^2\pi^2 + a^2 M_3(1+M_1)], \tag{11}$$

where n is a positive integer, even in the case of an odd mode and odd in the case of an even mode of perturbation.

We set $\sigma_n = \sigma_n^{(r)} + i\sigma_n^{(i)}$; $\sigma_n^{(r)}$ and $\sigma_n^{(i)}$ being the real and the imaginary parts of σ_n and obtain from (7), after separating the real and imaginary parts

$$U\sigma_n^{(r)}\ [\{\sigma_n^{(r)}\}^2 - 3\{\sigma_n^{(i)}\}^2] + V[\{\sigma_n^{(r)}\}^2 - \{\sigma_n^{(i)}\}^2] + W\sigma_n^{(r)} + X = 0 \tag{12}$$

and

$$U\sigma_n^{(i)}\ [3\{\sigma_n^{(r)}\}^2 - \{\sigma_n^{(i)}\}^2] + \sigma_n^{(i)}[2V\sigma_n^{(r)} + W] = 0 \tag{13}$$

If the marginal state is convective, we obtain from (12) after setting $\sigma_n^{(i)} = 0$, as

$$U\{\sigma_n^{(r)}\}^3 + V\{\sigma_n^{(r)}\}^2 + W\sigma_n^{(r)} + X = 0. \tag{14}$$

We note that in the convective marginal state when at least one root of (14) is zero, other real roots must be non-positive. Thus, in the case of the convective marginal state, we must have

$$X = 0, \quad W \geq 0 \tag{15}$$

as $U \geq 0$, and $V \geq 0$.

Again for overstable marginal state, after eliminating $\sigma_n^{(i)}$ from (12) and (13), under the assumption $\sigma_n^{(i)} \neq 0$, we have

$$8U^2\{\sigma_n^{(r)}\}^3 + 8UV\{\sigma_n^{(r)}\}^2 + 2(WU + V^2)\ \{\sigma_n^{(r)}\} + \{WV - UX\} = 0. \tag{16}$$

In the case of the overstability when at least one of the roots of $\sigma_n^{(r)}$ in (16) is zero, other real roots must be non-positive and moreover at least one $\sigma_n^{(i)}$ obtained from (13) must be real. Noting that U and V are positive, we obtain from (16) and (13) that in the overstable marginal state

$$WV - UX = 0, \quad WU + V^2 \geq 0 \tag{17}$$

and

$$(\sigma_n^{(i)})^2 = W/U > 0. \tag{18}$$

2.1 Evaluation of $R_c^{(\mathrm{con})}$

We use the conditions (15) to get the value of $R_c^{(\mathrm{con})}$. Taking $X = 0$, we have, after setting $R = R_n^{(\mathrm{con})}$,

$$R_n^{(\mathrm{con})} = \frac{[(n^2\pi^2 + a^2)^3 + Nn^2\pi^2]\ (n^2\pi^2 + a^2 M_3)}{a^2[n^2\pi^2 + a^2 M_3(1+M_1)]}, \tag{19}$$

where we observe

$$R_n^{(\mathrm{con})} < R_{n+1}^{(\mathrm{con})}\,. \tag{20}$$

Again the condition $W \geq 0$ in (15) gives after using $R = R_n^{(\mathrm{con})}$ as given by (19)

$$(n^2\pi^2 + a^2)^3 + N\frac{(P-1)}{(P+1)}\,n^2\pi^2 \geq 0. \tag{21}$$

For $P > 1$, the condition (21) is satisfied for all n and after noting (20) we get $R_c^{(\mathrm{con})}$, the critical value of the Rayleigh number for the convective marginal state as

$$R_c^{(\mathrm{con})} = R_1^{(\mathrm{con})} = \frac{[(\pi^2 + a^2)^3 + N\pi^2]\,(\pi^2 + a^2 M_3)}{a^2[\pi^2 + a^2 M_3(1 + M_1)]}\,. \tag{22}$$

The above result had been obtained by Das Gupta and Gupta [4]. We now consider the case $P < 1$ and determine the correct value of n from (21). We write $x_n = n^2\pi^2 + a^2$ and express (21) as

$$(x_n - \alpha_1)\;(x_n - \alpha_2)\;(x_n - \alpha_3) \geq 0, \tag{23}$$

where

$$\alpha_1 = -\left\{\frac{N(1-P)}{2(1+P)}\right\}^{1/3}\,[(a^2+\beta)^{1/3} + (a^2-\beta)^{1/3}], \tag{24}$$

$$\alpha_2 = -\left\{\frac{N(1-P)}{2(1+P)}\right\}^{1/3}\,[\omega(a^2+\beta)^{1/3} + \omega^2(a^2-\beta)^{1/3}], \tag{25}$$

$$\alpha_3 = -\left\{\frac{N(1-P)}{2(1+P)}\right\}^{1/3}\,[\omega^2(a^2+\beta)^{1/3} + \omega(a^2-\beta)^{1/3}], \tag{26}$$

and

$$\beta = \left\{a^4 - \frac{4N(1-P)}{27(1+P)}\right\}^{1/2}, \quad \omega = -\frac{1}{2}\,(1 + \sqrt{3}i). \tag{27}$$

From the above, we find that if

$$a \geq \left\{\frac{4N(1-P)}{27(1+P)}\right\}^{1/4} = a^* \quad \text{(say)}, \tag{28}$$

then β is real, $0 \leq \beta \leq a^2$, α_3 is the complex conjugate of α_2 and α_1 is real and negative. We find that for all n, x_n satisfies (23). Hence, due to (20), we have $R_c^{(\mathrm{con})} = R_1^{(\mathrm{con})}$ as given in (22).

In case $a < a^*$, we put $\beta = i\beta_1$, where β_1 is obtained from (27), as

$$\beta_1 = \left\{\frac{4N(1-P)}{27(1+P)} - a^4\right\}^{1/2}. \tag{29}$$

Substituting $\beta = i\beta_1$ and expressing

$$a^2+\beta=a^2+i\beta_1=\left\{\frac{4N(1-P)}{27(1+P)}\right\}^{1/2}\exp(i\psi), \tag{30}$$

$$a^2-\beta=\left\{\frac{4N(1-P)}{27(1+P)}\right\}^{1/2}\exp(-i\psi), \tag{31}$$

where $0<\psi<\pi/2$, we get from (24)–(26)

$$\alpha_1=-\left[\frac{4N(1-P)}{3(1+P)}\right]^{1/2}\cos\psi/3, \tag{32}$$

$$\alpha_2=\left[\frac{4N(1-P)}{3(1+P)}\right]^{1/2}\cos(\psi+\pi)/3, \tag{33}$$

$$\alpha_3=\left[\frac{4N(1-P)}{3(1+P)}\right]^{1/2}\cos(\psi-\pi)/3, \tag{34}$$

and note that $\alpha_1<0$ and $0<\alpha_2<\alpha_3$. Therefore, from (23) we get the condition that

$$x_n\leq\alpha_2 \quad \text{or} \quad x_n\geq\alpha_3. \tag{35}$$

If $x_1=\pi^2+a^2\leq\alpha_2$ obviously $R_c^{(\text{con})}$ is given by $R_1^{(\text{con})}$ which is the same as (22). If $x_1=\pi^2+a^2>\alpha_2$ we have to consider all those n for which $x_n\geq\alpha_3$. Hence due to (20), we find that

$$R_c^{(\text{con})}=R_{m_0}^{(\text{con})}, \tag{36}$$

where

$$m_0=\text{Min } m\ \{n\colon x_n=n^2\pi^2+a^2\geq\alpha_3\}. \tag{37}$$

Obviously, $m_0=1$ if $\pi^2>\alpha_3$ and $R_c^{(\text{con})}$ is given by the same expression as (22). Hence we find that when $P<1$, $a<a^*$ and $\alpha_2-\pi^2<a^2<\alpha_3-\pi^2$ (if $\pi^2<\alpha_3$) the convective marginal state is exhibited by a perturbation with mode higher than the lowest one. In all other cases, the $R_c^{(\text{con})}$ is given by $R_1^{(\text{con})}$.

2.2 *Evaluation of* $R_c^{(\text{o.s})}$

Using the conditions (17) we get

$$R_n^{(\text{o.s})}=\frac{2(P+1)}{a^2}\,\frac{\left[(n^2\pi^2+a^2)^3+\dfrac{NP^2n^2\pi^2}{(P+1)^2}\right](n^2\pi^2+a^2M_3)}{[n^2\pi^2+a^2M_3(1+M_1)]} \tag{38}$$

with

$$R_n^{(\text{o.s})}<R_{n+1}^{(\text{o.s})}, \tag{39}$$

and after using (38), we have (as $WU+V^2\geq 0$)

$$(n^2\pi^2 + a^2)^3 + \frac{NP^2(1-P)n^2\pi^2}{(1+3P)\,(1+P)^2} \geq 0. \tag{40}$$

Moreover from (18), we have

$$\{\sigma_n^{(i)}\}^2 = \frac{Nn^2\pi^2(1-P)}{(n^2\pi^2 + a^2)\,(1+P)} - (n^2\pi^2 + a^2)^2 > 0. \tag{41}$$

For $P < 1$, we find from (41) that the overstability cannot occur if

$$N < \frac{(1+P)\pi^4}{(1-P)} = N^* \quad \text{(say)}$$

and if $N > N^*$, it can occur only for those values of a for which

$$a^2 < \left[\frac{N\pi^2(1-P)}{(1+P)}\right]^{1/3} - \pi^2$$

and $R_c^{(\mathrm{o.s})}$ is given by

$$R_c^{(\mathrm{o.s})} = R_1^{(\mathrm{o.s})} =$$

$$\frac{2(P+1)}{a^2}\,\frac{\left[(\pi^2+a^2)^3 + \dfrac{NP^2\pi^2}{(P+1)^2}\right](\pi^2 + a^2M_3)}{[\pi^2 + a^2M_3(1+M_1)]} \tag{42}$$

3. Numerical results and discussion

From the above, we observe that if $P < 1$, the convective marginal state exists and the overstability can occur only if N is sufficiently large. We have observed that in case the overstability occurs the perturbation in the marginal state must necessarily be in the lowest mode. In the case of convective marginal state, we have seen that though it can occur for all values of a, there exists a situation when the perturbation in the marginal state need not be in the lowest mode. The nature of the critical marginal state can only be determined by calculating $R_c^{(\mathrm{con})}$ and $R_c^{(\mathrm{o.s})}$ and finding out the smaller of the two.

In table 1, we have presented $R_c^{(\mathrm{con})}$ for $4 \leq a^2 \leq 11.0$, $P = 0.01$, $M_1 = 10$, $M_3 = 5$, $N = 10$ and 100. For these values of the parameters overstability cannot occur and the perturbation in the marginal state is found to correspond to the lowest mode. In table 2, we have presented $R_c^{(\mathrm{o.s})}$ and $R_c^{(\mathrm{con})}$ for $N = 1000$ and

Table 1. Values of $R_c^{(con)}$ for $M_1 = 10$, $M_3 = 5$, for various values of a ($4.0 \le a^2 \le 11.0$) when $N = 10$ and 100, $P = 0.01$.

	$M_1 = 10,\ M_3 = 5$	
	$N = 10$	$N = 100$
a^2	$R_c^{(con)}$	$R_c^{(con)}$
4.0	89.87825	118.7332
4.5	85.88653	110.5127
5.0	82.90372	104.6493
5.5	81.11916	100.4401
6.0	80.07015	97.43691
6.5	79.58203	95.34284
7.0	79.53391	93.95316
7.5	79.83957	93.12292
8.0	80.43295	92.74441
8.5	81.27645	92.74075
9.0	82.32481	93.05116
9.5	83.55358	93.62953
10.0	84.94081	94.43945
10.5	86.46923	95.45143
11.0	88.12475	96.64412

Table 2. Values of $R_c^{(con)}$ and $R_c^{(o.s)}$ for $M_1 = 10$, $M_3 = 5$, for various values of a ($4.0 \le a^2 \le 11.0$) when $N = 1000$ and $N = 10{,}000$, $P = 0.01$.

	$M_1 = 10,\ M_3 = 5$			
	$N = 1000$		$N = 10{,}000$	
a^2	$R_c^{(o.s)}$	$R_c^{(con)}$	$R_c^{(o.s)}$	$R_c^{(con)}$
4.0	175.1411	6972.308	175.7124	685437.38
4.5	167.5692	5976.910	168.0650	579830.75
5.0	162.6326	5227.156	163.0634	498824.06
5.5	159.5664	4646.128	159.9490	435142.56
6.0	157.8820	4185.261	158.2258	384052.94
6.5	157.2527	3812.573	157.5647	342357.81
7.0	157.4338	3506.237	157.7392	066013.00
7.5	158.3237	3250.919	158.5867	059873.31
8.0	159.7453	3035.566	169.9889	054691.41
8.5	161.6305	2852.029	161.8574	050271.96
9.0	163.9121	2694.183	164.1244	046467.67
9.5	166.5387	2557.342	166.7382	043165.75
10.0	169.4693	2437.897	169.6574	040278.50
10.5	172.6713	2332.961	172.8492	037737.10
11.0	176.1184	2240.274	176.2871	035486.25

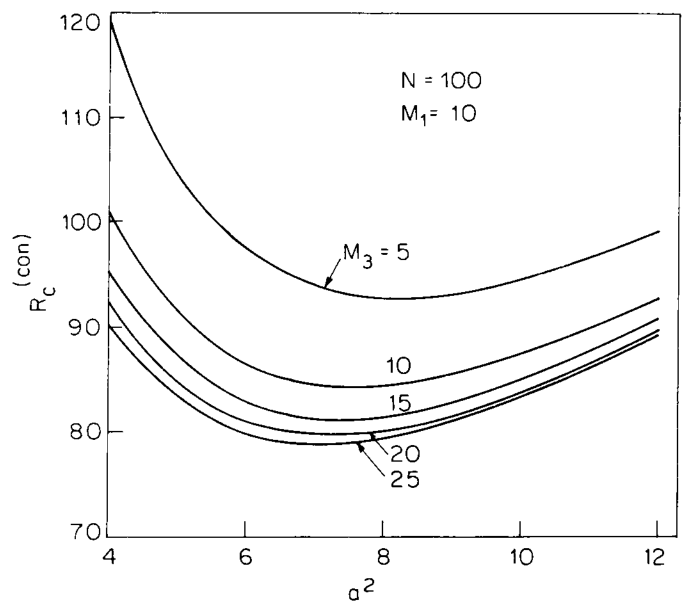

Figure 1. Plots of $R_c^{(con)}$ vs a^2 ($4.0 \le a^2 \le 12.0$) for $M_1 = 10$, $N = 100$ and $M_3 = 5$, 10, 15, 20 and 25, $P = 0.01$.

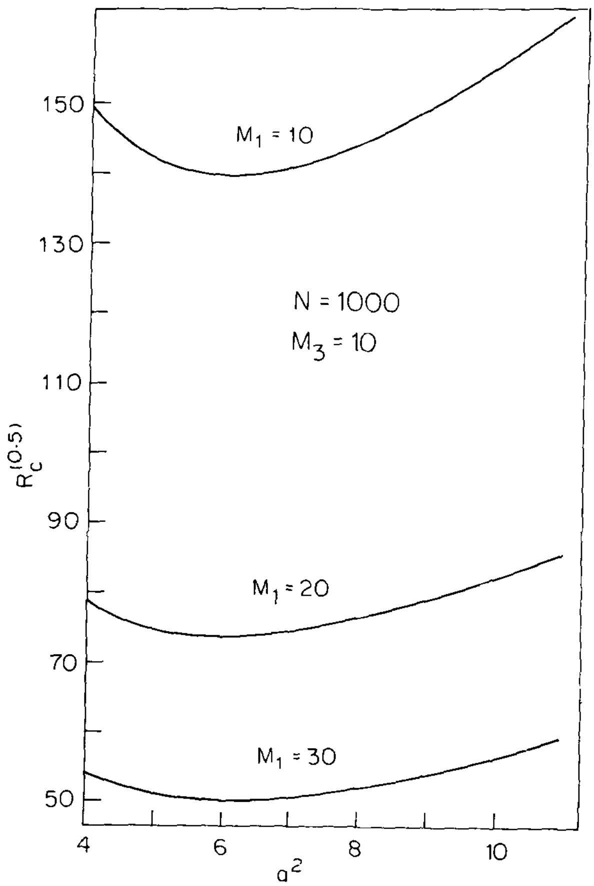

Figure 2. Plots of $R_c^{(o.s)}$ vs a^2 ($4.0 \leq a^2 \leq 11.0$) for $M_3 = 10$, $N = 1000$ and $M_1 = 10, 20$ and 30, $P = 0.01$.

10000, other parameters being the same. We find $R_c^{(o.s)} < R_c^{(con)}$ so that the nature of the marginal state is overstable. For $R_c^{(con)}$, the perturbation has been found to correspond to higher mode. In all the above cases, the critical Rayleigh numbers increase as N increases and hence N is found to have a stabilizing effect.

We have presented the plots of $R_c^{(con)}$ vs a^2 for $P = 0.01$, $M_1 = 10$, $M_3 = 5, 10, 15, 20$ and 25 and $N = 100$ in figure 1. The plots show that M_3 has a destabilizing effect. The plots of $R_c^{(o.s)}$ vs a^2 for $P = 0.01$, $M_3 = 10$, $N = 1000$, $M_1 = 10, 20$ and 30 have been presented in figure 2. M_1 is also found to have a destabilizing effect.

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