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Instability of a planar liquid layer in an alternating longitudinal magnetic field with non-zero mean

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Abstract. The conducting liquid interface is found to undulate in an alternating magnetic field. It was shown earlier that if $M = |B_0|^2/\mu\eta\omega$, B_0, ω, μ and η being the amplitude (complex) of the alternating longitudinal magnetic field imposed at the interface, the angular frequency of the field, the magnetic permeability and the viscosity respectively, and if M_c was the critical value of M then the planar layer was stable or unstable according as $M < M_c$ or $M > M_c$. In this paper we have determined the stability criterion when in addition to the alternating longitudinal field there acts a uniform field in the same direction. After comparing our results with those obtained earlier, in the absence of the uniform field, we find that the additional uniform field has a significant destabilizing effect.

Keywords. Stability; alternating magnetic field; interface; geomagnetic field.

1. Introduction

In certain cases, where magnetic fields are used for levitation, melting of metals and shaping of the interface to form materials like glass, it has been found that the alternating field causes undulations [3, 8, 4, 5]. Such undulation takes place even in the case of hydromagnetic surface waves [7].

In order to determine the possible causes of the undulations, an analysis based on a 'lumped parameter' model [4] taking thermal effects into account and also a linear stability analysis based on usual MHD model neglecting thermal effects [5] was carried out. It was suggested that electromechanical motion driven by eddy current heating and the cooling from the upper surface could cause such undulations. The effect of such additional uniform field such as the geomagnetic field, acting in the layer along with the alternating field has not been studied earlier. We have recently investigated the stability of a conducting planar liquid layer in a longitudinal alternating magnetic field imposed at the interface with a uniform field acting in the transverse direction [2]. It is found that such an additional uniform field has considerable destabilizing effect. In order to make a comparative study, we have investigated the stability when the uniform magnetic field acts in the same direction as the longitudinal alternating field. In this paper, we have presented in §2 the linear stability analysis and the numerical results.

2. Linear stability analysis and the numerical results

We take a rectangular cartesian system of axes $oxyz$ with o as the origin on the interface, ox axis along the outward normal to the layer and oz axis in the direction of the imposed longitudinal field real $B_0 \exp(j\omega t)$, where B_0 is the amplitude (complex) of the field, ω the angular frequency, t the time and $j = \sqrt{-1}$. We assume the uniform field B_1 to be acting in the liquid in the same direction so that if $\delta = (2/\omega\mu\sigma)^{1/2}$, where σ is the conductivity, then the magnetic field $\mathbf{B}$ in the liquid in the absence of any motion is given by

$$\begin{aligned} \mathbf{B} &= \mathbf{B}_e + \mathbf{B}_1, \ \mathbf{B}_1 = B_1 \mathbf{i}_z \ (B_1 \text{ real}), \\ \mathbf{B}_e &= B_0 \exp(1+j)\frac{x}{\delta} \exp(j\omega t)\mathbf{i}_z. \end{aligned} \tag{1}$$

The unit vectors in the direction of x, y and z axes are being denoted by $\mathbf{i}_x$, $\mathbf{j}_y$ and $\mathbf{i}_z$ respectively.

In the static state, we take p_0 as the hydromagnetic pressure, whereas in the disturbed state we take $p_0 + p$ as the pressure, $\mathbf{v}$ as the velocity and $\mathbf{B} + \mathbf{b}$ as the magnetic field where p, $\mathbf{v}$ and $\mathbf{b}$ are assumed to be small in magnitude. The linearized MHD equations are

$$\begin{aligned} \rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \eta \nabla^2 \mathbf{v} + \frac{1}{\mu}(\mathbf{B}_e + \mathbf{B}_1)\cdot\nabla\mathbf{b} \\ + \frac{1}{\mu}(\mathbf{b}\,.\,\nabla)\mathbf{B}_e, \end{aligned} \tag{2}$$

$$\frac{\partial \mathbf{b}}{\partial t} = \frac{1}{\mu\sigma}\nabla^2\mathbf{b} + (\mathbf{B}_0 + \mathbf{B}_1)\cdot\nabla\mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{B}_e \tag{3}$$

$$\nabla\cdot\mathbf{v} = \nabla\cdot\mathbf{b} = 0, \tag{4}$$

where ρ and η are density and viscosity respectively.

We seek the solution of (2)–(4) in the following approximate forms

$$v = \text{real } v(x) \exp(st - jk_y y - jk_z z), \tag{5}$$

$$p = \text{real } p(x) \exp(st - jk_y y - jk_z z), \tag{6}$$

$$\begin{aligned} b = \text{real}[&b^+(x)\exp(st + j\omega t - jk_y y - jk_z z), \\ &+ b^-(x)\exp(st - j\omega t - jk_y y - jk_z z), \\ &+ c(x)\exp(st - jk_y y - jk_z z)]. \end{aligned} \tag{7}$$

Here s has been taken as complex. It may be noted that due to the presence of $\mathbf{B}_e$ as given by (1) in (2) and (3), the solution can be expressed as a series of terms, $\exp(st - jk_y y - jk_z z)$, $\exp(st \pm j\omega t - jk_y y - jk_z z)$, $\exp(st \pm 2j\omega t - jk_y y - jk_z z)$ etc. Assuming the viscous damping and inertia to inhibit the growth of terms with frequencies higher than that given by Im(s) (Im(s) being the imaginary part of s).

We have retained terms involving only $\exp(st - jk_y y - jk_z z)$ in (5) and (6) as approximate solutions.

We observe from physical considerations that since the velocity field and the magnetic field in a conducting fluid cannot be considered to be decoupled, the terms $b^{\pm}(x)\exp(st \pm j\omega t - jk_y y - jk_z z)$ and also $c(x)\exp(st - jk_y y - jk_z z)$ in (7) must be retained to balance the equations (2) and (3) to the lowest order.

Here we present certain considerations similar to those presented by McHale and Melcher [5] in the introduction and followed in the subsequent calculations. We assume $\omega \simeq 2$ kHz and $|\text{Im}(s)|$ to be small due to viscosity. As $|\text{real}(s)|$ may be assumed to be sufficiently small if we are interested in the study of the state when it is close to the marginal state, we can take $|s| \ll \omega$ in the following analysis.

We substitute (5), (6), (7) in (2), (3), (4) eliminating p and introducing the following non-dimensional variables denoted by dashes

$$(x, y, z) \equiv \delta(x', y', z),\ (k, k_z) \equiv \delta^{-1}(k', k'_z),$$

$$(v_x, v_y, v_z) \equiv (\mu\sigma\delta)^{-1}(v'_x, v'_y, v'_z),$$

$$(b_x^+, b_y^+, b_z^+) \equiv B_0(b_x^{+\prime}, b_y^{+\prime}, b_z^{+\prime}),$$

$$(b_x^-, b_y^-, b_z^-) \equiv B_0(b_x^{-\prime}, b_y^{-\prime}, b_z^{-\prime}),$$

$$(c_x, c_y, c_z) \equiv |B_0|(c'_x, c'_y, c'_z),$$

also the following non-dimensional parameters

$$P_m = \mu\sigma\eta/2\rho,\ S' = S/\omega P_m,\ B'_1 = B_1/|B_0|,\ M = |B_0|^2/\mu\eta\omega$$

and obtain the following equations in non-dimensional forms after neglecting $S'P_m$ due to the considerations stated above. We have

$$[(D^2 - k^2)(D^2 - k^2 - s) + Mk_z^2(2B_1^2 + \exp(2x))]v_x + 4Mk_z\{b_x^+\exp((1-j)x) - b_x^-\exp((1+j)x)\} = 0, \tag{8}$$

$$(D^2 - k^2 - 2j)b_x^+ = \tfrac{1}{2}(jk_z)\exp((1+j)x)v_x, \tag{9}$$

$$(D^2 - k^2 + 2j)b_x^- = \tfrac{1}{2}(jk_z)\exp((1-j)x)v_x, \tag{10}$$

$$(D^2 - k^2)c_x = jk_z B_1 v_x, \tag{11}$$

$$D = \mathrm{d}/\mathrm{d}x.$$

In the above equations dashes have been dropped for simplicity.

We assume the layer to be thick and the liquid air interface rigid and perfectly conducting. We take the boundary conditions as

$$v_x = Dv_x = b_x^+ = b_x^- = c_x = 0 \ \text{ at } \ x = 0 \tag{12}$$

together with the conditions: v_x, Dv_x etc. tend to zero as $x \to -\infty$.

3. Validity of the principle of exchange of stability

After multiplying (8) by v_x^*, the complex conjugate of v_x and integrating with respect to x from $-\infty$ to 0, we obtain, after using (9) and (10)

$$\int_{-\infty}^{0} [|D^2 v_x|^2 + (2k^2+s)|Dv_x|^2 + k^2(k^2+s)|v_x|^2$$
$$+ Mk_z^2\{2B_1^2 + \exp(2x)\}|v_x|^2]\mathrm{d}x,$$
$$-8Mj \int_{-\infty}^{0} [|Db_x^+|^2 - |Db_x^-|^2 + k^2\{|b_x^+|^2 - |b_x^-|^2\}$$
$$-2j\{|b_x^+|^2 + |b_x^-|^2\}]\mathrm{d}x = 0. \quad (13)$$

From (9) and (10) we get

$$\left.\begin{aligned} (D^2 - k^2 - 2j)b_x^+ &= \frac{jk_z}{2}\exp((1+j)x)v_x \\ (D^2 - k^2 - 2j)(b_x^-)^* &= -\frac{jk_z}{2}\exp((1+j)x)v_x^* \end{aligned}\right\}, \quad (14)$$

where $(b_x^-)^*$ is the complex conjugate of b_x^-. We find from (5) that $v(x)$ must be either real or purely imaginary, since we are interested in finding the criterion under which a single mode of perturbation grows or decays. Hence from (14), we get $b_x^- = -(b_x^+)^*$ when v_x is real and $b_x^- = (b_x^+)^*$ when v_x is purely imaginary. In both cases we have $|Db_x^+| = |Db_x^-|$ and $|b_x^+| = |b_x^-|$ so that (13) finally reduces to

$$\int_{-\infty}^{0} [|D^2 v_x|^2 + (2k^2+s)|Dv_x|^2 + k^2(k^2+s)|v_x|^2$$
$$+ Mk_z^2\, 2B_1^2 + \exp(2x)|v_x|^2]\mathrm{d}x,$$
$$-32M \int_{-\infty}^{0} |b_x^+|^2\,\mathrm{d}x = 0. \quad (15)$$

If we take the imaginary part of the above equation, we get

$$\mathrm{Im}(s) \int_{-\infty}^{0} \{|Dv_x|^2 + k^2|v_x|^2\}\mathrm{d}x = 0 \quad (16)$$

and hence $\mathrm{Im}(s) = 0$ for any mode of disturbance. Thus the principle of exchange of stabilities is valid indicating that the marginal state must be convective.

To solve equations (8)–(11) subject to the boundary condition (12) with the physical consideration that all disturbances should remain bounded in magnitude at an infinitely large distance away from the surface, we follow a method presented in [1] and suitably extended in [5]. We observe that if we set $\xi = \exp(2x)$ in (8)–(11) the differential equations can be transformed into equations for which $\xi = 0$ appears to be a regular singularity. We seek solutions of (8)–(11) in series of powers of $\xi = \exp(2x)$ and write formally

$$V_x = A_0 \exp(r_n x) \sum_{m=0}^{\infty} D_m \exp(2mx), \tag{17a}$$

$$b_x^+ = A_0 \exp\{(r_n + 1 + j)x\} \sum_{m=0}^{\infty} E_m^+ \exp(2mx), \tag{17b}$$

$$b_x^- = A_0 \exp\{(r_n + 1 - j)x\} \sum_{m=0}^{\infty} E_m^- \exp(2mx), \tag{17c}$$

$$c_x = A_0 \exp(r_n x) \sum_{m=0}^{\infty} F_m \exp(2mx) \tag{17d}$$

Since the set of equations (8)–(11) constitutes a differential equation of order ten, the success of the method lies in determining ten linearly independent solutions of (8)–(11), which should correspond to ten roots of r_n all of which need not be distinct.

Substituting (17a–d) in (8)–(11) and equating the coefficients of the same powers of $\exp(2x)$ we have the following set of equations

$$\{(r_n^2 - k^2)(r_n^2 - k^2 - s) + 2B_1^2 M k_z^2\} D_0 = 0, \tag{18a}$$

$$[\{r_n + (1+j)\}^2 - k^2 - 2j] E_0^+ = \tfrac{1}{2}(jk_2) D_0, \tag{18b}$$

$$[\{r_n + (1-j)\}^2 - k^2 + 2j] E_0^- = \tfrac{1}{2}(jk_z) D_0, \tag{18c}$$

$$(r_n^2 - k^2) F_0 = jk_z B_1 D_0, \tag{18d}$$

and for $m \geq 1$

$$[\{(r_n + 2m)^2 - k^2\}\{(r_n + 2m)^2 - k^2 - s\} + 2B_1^2 M k_z^2] D_m$$
$$+ M k_z^2 D_{m-1} + 4 M k_z (E_{m-1}^+ - E_{m-1}^-) = 0, \tag{19a}$$

$$[\{r_n + (1+j) + 2m\}^2 - k^2 - 2j] E_m^+ = \tfrac{1}{2}(jk_z) D_m, \tag{19b}$$

$$[\{r_n + (1-j) + 2m\}^2 - k^2 + 2j] E_m^- = \tfrac{1}{2}(jk_z) D_m, \tag{19c}$$

$$[(r_n + 2m)^2 - k^2] F_m = jk_z B_1 D_m. \tag{19d}$$

From (18a–d) we find that after eliminating D_0, E_0^+, D_0^- and F_0 we get an algebraic equation of degree ten in r_n. Hence for an arbitrary choice of D_0, E_0^+, E_0^- and F_0 we can calculate D_m, E_m^+, E_m^- and F_m for all $m \geq 1$ after taking one of the roots of r_n, giving finally one particular solution. In this way all the ten linearly independent solutions can be obtained though there may arise a difficulty in case of two or more roots of r_n happen to be equal in which case we may have to use the method of Frobenius with necessary modifications.

We can simplify the above procedure if we consider that our solution must satisfy the condition of boundedness as $x \to -\infty$ and therefore we need consider only those particular solutions for which real $(r_n) > 0$. Moreover we find that instead of considering a single set of values of D_0, E_0^+, E_0^- and F_0 we can consider different sets of values of D_0, E_0^+, E_0^- and F_0 to obtain particular solutions, provided such solutions are linearly independent. We therefore present the following scheme.

Solutions 1, 2.

$$D_0 = 1;\ (r_n^2 - k^2)(r_n^2 - k^2 - s) \times 2B_1^2 M k_z^2 = 0,$$

$$r_n = r_1,\ r_2,\ \text{real}(r_1, r_2) > 0.$$

Solution 3.

$$D_0 = E_0^- = F_0 = 0;\ E_0^+ = 1,\ (r_n + 1 + j)^2 = k^2 + 2j,$$

$$r_n = r_3;\ \text{real}(r_3) > 0.$$

Solution 4.

$$D_0 = E_0^+ = F_0 = 0;\ E_0^- = 1,\ (r_n + 1 - j)^2 = k^2 - 2j,$$

$$r_n = r_4;\ \text{real}(r_4) > 0.$$

Solution 5.

$$D_0 = E_0^+ = E_0^- = 0;\ F_0 = 1;\ r_n^2 - k^2 = 0,$$

$$r_n = r_5;\ \text{real}(r_5) > 0.$$

In case we get five distinct values of r_n we find that we can express the general solution as

$$\begin{aligned} V_x = {} & A_1 \exp(r_1 x)[1 + D_1^{(1)} \exp(2x) + D_2^{(1)} \exp(4x) + \dots] \\ & + A_2 \exp(r_2 x)[1 + D_1^{(2)} \exp(2x) + D_2^{(2)} \exp(4x) + \dots] \\ & + A_3 \exp(r_3 x)[D_1^{(3)} \exp(2x) + D_2^{(3)} \exp(4x) + \dots] \\ & + A_4 \exp(r_4 x)[D_1^{(4)} \exp(2x) + D_2^{(4)} \exp(4x) + \dots] \\ & + A_5 \exp(r_5 x)[D_1^{(5)} \exp(2x) + D_2^{(5)} \exp(4x) + \dots] \end{aligned}$$

$$\begin{aligned} b_x^+ = {} & A_1 \exp\{(r_1 + 1 + j)x\}[E_0^{+(1)} + E_1^{+(1)} \exp(2x) \\ & + E_2^{+(1)} \exp(4x) + \dots] \\ & + A_2 \exp\{(r_2 + 1 + j)x\}[E_0^{+(2)} + E_1^{+(2)} \exp(2x) \\ & + E_2^{+(2)} \exp(4x) + \dots] \\ & + A_3 \exp\{(r_3 + 1 + j)x\}[1 + E_1^{+(3)} \exp(2x) + E_2^{+(3)} \exp(4x) + \dots] \\ & + A_4 \exp[(r_4 + 1 + j)x][E_1^{+(4)} + \ \exp(2x) + E_2^{+(4)} \exp(4x) + \dots] \\ & + A_5 \exp[(r_5 + 1 + j)x][E_1^{+(5)} + \ \exp(2x) + E_2^{+(5)} \exp(4x) + \dots] \end{aligned}$$

and similarly b_x^- and c_x can be expressed as series where all the quantities $D_1^{(1)}$, $D_2^{(1)}$ etc are obtained from (18a–d) and the recurring relations (19a–d).

Using the boundary condition (12) after considering the solutions in the above forms and eliminating A_1, A_2 etc, we get a determinantal equation denoted by

$$Q(M, B_1, s, k, k_z) = 0, \tag{20}$$

where Q is a 5×5 determinant which reduces to a 4×4 determinant due to the particular form of the solution 5 when $D_m = E_m^+ = E_m^- = F_m = 0$ for $m \geq 1$. It may be pointed out that the same 4×4 determinant could have been obtained if we had confined to only equations (8)–(10). In the above 4×4 determinant every element is an infinite series of terms dependent upon the variables indicated in (20).

The critical value M_c of M for a particular value of k is given by the minimum of all the roots of M in (20) for all values of k_z with k kept fixed when $s = 0$. Since equations (8)–(11) can be transformed into the forms in which k_z appears only along with Mk_z^2 and not independently, to obtain the minimum value of M we set $k_z = k$ in (20).

We have calculated the values of M_c for $0 < k \leq 2.4$ then $B_1 = 10^{-2}$, 10^{-3} and 10^{-4}. The plots of M_c vs k have been presented in figure 1.

For smaller values of $k (k < 0.4)$ the changes in the roots of M_c in (20) are large for changes in the values of k and the roots of M_c have been located after fixing up M and observing the changes in the signs of the real and imaginary parts of Q as given in (20) for successive values of k from a set of values of k. The accuracy has been improved by taking large number of terms in each element of the determinant Q. For larger values of $k (k > 0.4)$ the changes in the roots of M_c in (20) are found to be small. To locate the roots of M_c we have repeated the above procedure by

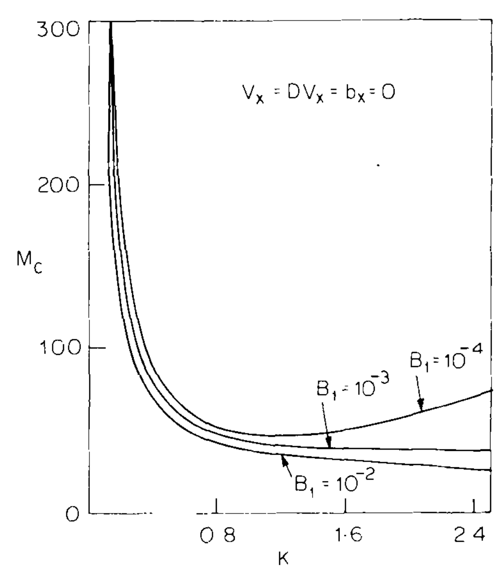

Fig. 1.

fixing up k and considering the changes in signs of the real and imaginary parts of Q for successive values of M from a set of values of M.

4. Discussion

From figure 1 we find that for $B_1 = 10^{-2}$, 10^{-3} and 10^{-4} the values of M_c remain nearly the same for $k < 0.4$ (approximately) whereas for larger values of k there is a slow increase in the values of M_c for $B_1 = 10^{-4}$ and slow decrease in the values of M_c for $B_1 = 10^{-3}$ and 10^{-2} and hence the additional uniform field has destabilizing effect.

It may be concluded that the observed undulation of the liquid metal air interface in an alternating longitudinal magnetic field may be partly caused by the geomagnetic field or any other stray field in the laboratory. In particular, for a liquid thick layer of mercury at 20°C when the kinematic viscosity, the magnetic permeability and the density are taken as 10^{-7} m^2/s, $4\pi\, 10^{-7}$ henry/m and 10^7 gm/m^3 respectively and Earth's magnetic field is taken as 0.7 Gauss [6], we find that for a longitudinal alternating field with amplitude 1.5 Gauss and angular frequency 5 kHz, the value of M is obtained as 352 (approximately) whereas the value of $B_1 = 0.46$ (approximately). Thus according to the results presented by the plots of M_c vs k we find the layer to be unstable since as B_1 increases from 10^{-4} the critical value of M decreases and is less than 40. But according to the results obtained by McHale and Melcher [5], the layer is supposed to remain stable.

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