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Axisymmetric melting of a long cylinder due to an infinite flux

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Abstract. By employing a new embedding technique, a short-time analytical solution for the axisymmetric melting of a long cylinder due to an infinite flux is presented in this paper. The sufficient condition for starting the instantaneous melting of the cylinder has been derived. The melt is removed as soon as it is formed. The method of solution is simple and straightforward and consists of assuming fictitious initial temperature for some fictitious extension of the actual region.

Keywords. Embedding technique; moving boundary problems; ablation; isotherm condition; heat-balance condition.

1. Introduction

The melting problem is essentially a two-phase heat transfer problem in which both liquid and solid phases are present. The problem considered in this paper is a one-phase problem in which the melt is removed as soon as it is formed. This problem is known as the ablation problem and may arise in many physical situations such as ablation of heat shields during re-entry of spacecraft into the earth's atmosphere, laser heating, oxygen diffusion with absorption etc.

Melting problems can be put under a more general class of problems known as moving boundary problems (MBPs). The solidification problem is mathematically analogous to the melting problem and there is no need to discuss the solidification problem separately and any reference to melting problems includes solidification problems as well. A typical feature of the MBPs is that, apart from the fixed boundary of the region under consideration, there is also a boundary inside the region which is unknown and is moving with time. Some boundary conditions are to be satisfied on this moving boundary. Depending on the portion of the fixed boundary on which the melting initiates, Sikarskie and Boley [20] have divided these melting problems into three classes. In class I problems, the melting starts simultaneously at all points on the fixed boundary. In class II and III problems, the melting starts over a portion of the fixed boundary and at a single point on the boundary respectively. This paper is concerned with a class I problem.

The unknown nature of the moving boundary and the boundary conditions to be satisfied on this moving boundary make the problems extremely complicated and their exact analytical solutions valid for total melting time and varied shapes are rather unconceivable. Class I problems are easier to tackle as compared to class II and III problems as in the latter case the problem has to be at least two-dimensional and the boundary conditions on the fixed boundary are very difficult to manage.

For one-dimensional class I melting problems, some exact similarity solutions exist in the literature such as Neumann solution [5], the series solution by Tao [21], the solution by Ožisik and Užzell [16] etc. Some more references can be found in the above mentioned references. These solutions pertain to very simple situations and geometries. Perturbation methods [17], approximate methods [10] and numerical methods [6, 7] have also been employed to study the melting problems. Fairly good account of the studies on MBPs can be found in [2, 13, 15]. Except numerical methods, sufficient applications of other methods to multi-dimensional problems are not available. Some of the numerical methods can be found in [9]. The utility of short-time analytical solutions in numerical schemes has been amply demonstrated in a recent work by Schulze *et al* [19] in which a practical problem of ingot solidification was considered. Short-time solutions can be used for starting some of the numerical schemes such as Murray and Landis scheme [14] and enthalpy scheme [8] and can also be used for checking the validity of such numerical schemes in which modifications are attempted.

Short-time solutions require some special techniques. Boley *et al* [4] (which contains many references of the works of Boley and co-workers) have contributed substantially in developing short-time solutions of ablation and solidification problems pertaining to one-, two-, and three-dimensional regions. Recently Gupta and Lahiri [12] and Gupta [11] have studied short-time solutions of solidification in cylindrical mold (one-dimensional) and semi-infinite mold (two-dimensional), respectively. In [3], Boley developed an embedding technique which consists of prescribing fictitious flux in the case of ablation problems and fictitious boundary flux (or fictitious boundary temperature) together with suitable fictitious initial temperature in the case of solidification problems. The solid and liquid regions are embedded in the region originally occupied by the melt. The temperature solution in Boley's technique is written with the help of Duhamel's theorem [5] and the moving boundary and other unknowns are determined by comparing different powers of time variable on both sides of some integro-differential equations. In the present method, the temperature solution is written with the help of the source solution and the solid region is first embedded in the original region occupied by the solid and then the original solid region is extended fictitiously and some fictitious initial temperature is prescribed in the fictitious extension. The unknowns can be evaluated by differentiating the equations with respect to the time variable and taking appropriate limits. This yields considerable simplification and the method of solution becomes straightforward.

In the present melting problem, the prescribed flux is infinite at time $t = 0$ and some remarks about taking this type of singular flux will be in order. In many physical situations the melting takes place instantaneously i.e. there is no preheating of the solid before melting and the melting starts as soon as the flux is applied. If the boundary of the solid at which flux is applied is at a temperature lower than the melting temperature then for instantaneous melting it is necessary (but not sufficient) that the applied flux is infinite and has singularity of the type $t^{-1/2}$. This type of singularity in the context of heat transfer coefficient has been experimentally observed earlier by Ruddle [18] and Tiller [22].

2. Problem formulation

A long cylindrical solid occupies the region $0 \le r \le R_0$, $|z| < \infty$ at time $t = 0$ (axisymmetric problem in which r and z are cylindrical polar coordinates). This solid is heated and the prescribed flux $Q(z, t)$ on the surface $r = R_0$ of the cylinder is a known quantity which is infinite at $t = 0$. $f(r, z)$ is the known initial temperature of the solid. It will be assumed that the melting starts instantaneously all over the boundary, $R = R_0$ (which happens, provided a sufficient condition is satisfied which will be derived later) and melt is removed as soon as it is formed. With time, the melting will progress along the interior till the whole of the solid is melted. The equation of the melting front may be written in the form

$$r = s(z, t). \tag{1}$$

The quantities of interest in the present study are the melting front and the temperature in the solid. If the melting does not start at $t = 0$ but starts at the time $t = t_m$, $t_m > 0$, then the problem in general will be a mixed problem of class II and III type which cannot be solved by the present technique and for which absolutely no reference is available in the literature.

The following dimensionless differential equation, initial and boundary conditions are to be satisfied:

$$2\frac{\partial T}{\partial V} = V\left(\frac{\partial^2 T}{\partial R^2} + \frac{1}{R}\frac{\partial T}{\partial R} + \frac{\partial^2 T}{\partial Z^2}\right) \tag{2}$$

$$0 \le R < S(Z, V), \ |Z| < \infty, \ V > 0,$$

$$T(R, Z, 0) = f(R, Z); \quad \left.\frac{\partial T}{\partial R}\right|_{R=0} = 0, \tag{3a, 3b}$$

$$T(R, Z, V)|_{R=S(Z, V)} = 1, \tag{4}$$

$$\left\{1 + \left(\frac{\partial S}{\partial Z}\right)^2\right\} \left.\frac{\partial T}{\partial R}\right|_{R=S} = Q(Z, V) + \frac{2\lambda}{V}\frac{\partial S}{\partial V}, \tag{5}$$

$$S(Z, V)|_{V=0} = 1. \tag{6}$$

$T(R, Z, V)$ is the dimensionless temperature. All the temperatures have been made dimensionless with the help of the melting temperature T_m which is taken as unique. Equation (4) is the isotherm condition and (5) is the heat balance condition in which the resultant heat flux vector is along the direction of R. The following dimensionless variables have been used in writing equations (2)–(6).

$$Z = z/R_0,\ R = r/R_0,\ V = 2(kt/R_0^2)^{1/2}, \tag{7}$$

$$\lambda = l/CT_m,\ f(R,\ Z) = f(r,\ z)/T_m, \tag{8}$$

$$Q(Z,\ V) = Q(z,\ t).\ R_0/KT_m,\ S(Z,\ V) = s(z,\ t)/R_0, \tag{9}$$

R_0 is the radius of the cylinder, K is the thermal conductivity, k is the diffusivity, l is the latent heat and C is the specific heat of the solid. Thermal properties are taken to be constants.

3. Solution

The solution of (2) with (3) can be written as

$$T(R, Z, V) = \frac{2}{\pi^{1/2}V^3}\left[\int_{-\infty}^{\infty}\int_0^1 H(p,\ q)f_2(p,\ q)\ \mathrm{d}p\mathrm{d}q + \int_{-\infty}^{\infty}\int_1^{\infty} H(p,\ q)f_2(p,\ q)\ \mathrm{d}p\mathrm{d}q\right], \tag{10}$$

$$0 \leq R < \infty,\ |Z| < \infty,\ V > 0,$$

$$H(p,\ q) = p\exp[-\{R^2+p^2+(Z-q)^2\}/V^2]I_0(2Rp/V^2). \tag{11}$$

$I_0(x)$ is the modified Bessel function of the first kind of order zero.

It can be easily checked that $T(R,\ Z,\ V)$ in (10) satisfies (2) and (3). $f_2(R,\ Z)$ is the unknown initial temperature in the fictitious extension $1 \leq R < \infty$, $|Z| < \infty$ of the original region. Mathematically, for the determination of two unknowns, namely, $S(Z,\ V)$ and $f_2(R,\ Z)$, there are two conditions (4) and (5) to be satisfied. We shall first formally obtain the short time analytical solution.

For large values of the argument, the following asymptotic series expansion can be used for $I_0(x)$ [5]

$$I_0(x) = \frac{\exp(x)}{(2\pi x)^{1/2}}\left\{1 + \frac{1}{8x} + \frac{9}{128x^2} + \ldots\right\}. \tag{12}$$

Firstly in (10), $I_0(x)$ is replaced by the first two terms of the asymptotic series in (12) and then (10) is substituted in (4) and (5). After making suitable substitutions the following equations are obtained

$$\pi S^{3/2} = \int_{\infty}^{-\infty}\int_{(S/V)}^{(S-1)/V} D_1(S,\ V,\ p)f_1(S-Vp,\ Z-Vq) \times\exp\{-(p^2+q^2)\}\mathrm{d}p\mathrm{d}q + \int_{\infty}^{-\infty}\int_{(S-1)/V}^{-\infty}\{D_1(S,\ V,\ p)/ (S-Vp)^{1/2}\}f_2(S-Vp,\ Z-Vq)\exp\{-(p^2+q^2)\}\ \mathrm{d}p\mathrm{d}q, \tag{13}$$

$$\left\{1+\left(\frac{\partial S}{\partial Z}\right)^2\right\}\left[\int_{\infty}^{-\infty}\int_{(S/V)}^{(S-1)V} D_2(S,\ V,\ p)f_1(S-Vp,\ Z-Vq)\right.$$

$$\times \exp\{-(p^2+q^2)\}\,\mathrm{d}p\mathrm{d}q + \int_{\infty}^{-\infty}\int_{(S-1)/V}^{-\infty}\{D_2(S,\,V,\,p)/(S-Vp)^{1/2}\}f_2(S-Vp,\,Z-Vq)\exp\{-(p^2+q^2)\}\,\mathrm{d}p\mathrm{d}q]$$

$$= \pi VQ(Z,\,V) + 2\pi\lambda\,\frac{\partial S}{\partial V}, \tag{14}$$

where

$$D_1 = S(S-Vp) + V^2/16, \tag{15}$$

$$D_2 = 2p\{S^2(S-Vp)\} + V^2S/16\} + VS(S-Vp)/2 + 3V^3/32, \tag{16}$$

$$f_1(R,\,Z) = f(R,\,Z)/R^{1/2}. \tag{17}$$

The following series expansions for the known and unknown functions will be assumed.

$$f_1(R,\,Z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(R-1)^m Z^n}{\lfloor m \,\lfloor n}\;\frac{\partial^{m+n}(f_2)}{\partial R^m \partial Z^n}\Bigg|_{\substack{R=1\\Z=0}},\quad 0 < R \le 1,\, |Z| < \infty, \tag{18}$$

$$f_2(R,\,Z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(R-1)^m Z^n}{\lfloor m \,\lfloor n}\;\frac{\partial^{m+n}(f_2)}{\partial R^m \partial Z^n}\Bigg|_{\substack{R=1\\Z=0}},\quad 1 < R < \infty,\ |Z| < \infty, \tag{19}$$

$$Q(Z,\,V) = \sum_{n=0}^{\infty} Q_n(Z)V^{n-1},\ V > 0,\ |Z| < \infty, \tag{20}$$

$$S(Z,\,V) = 1 + \sum_{n=1}^{\infty} A_n(Z)V^n,\ V > 0,\ |Z| < \infty. \tag{21}$$

In (18) and (19), the functions have been expanded in powers of $(R-1)$ as the melting initiates at $R=1$ and in short time solutions we are interested in the behaviour of these functions around $R=1$. In (20), the flux is infinite at $V=0$ and mathematically the solution can be determined only if singularity at $V=0$ is of the type $1/V$. Equation (20) is motivated by the Neumann solution. In order to obtain the unknowns, the series expansions given above are substituted in (13) and (14) and then the limits $V \to 0+$ of these equations (13) and (14) are taken. Two equations in the two unknowns $A_1(Z)$ and $f_2(1,\,Z)$ are obtained which when solved give a unique solution. Equations (13) and (14) are then differentiated once with respect to V and limits $V \to 0+$ of these equations are taken. Once again two equations in the two unknowns $\partial f_2/\partial R\,|_{R=1}$ and $A_2(Z)$ are obtained which when solved give a unique solution. This process of higher order differentiations and limits $V \to 0+$ can be continued till the desired coefficient in the moving boundary is obtained. However, after few differentiations the algebra becomes extremely lengthy as with each higher order differentiation the number of terms goes on increasing. It may be remarked here that the limits of integrations as well as integrands are functions of V and this should be taken care while differentiating the integrals and taking limits $V \to 0+$. Equation (13) determines the derivatives of the unknown function $f_2(R, Z)$ and (14) determines the unknown coefficients of the moving boundary.

Two relevant points may be pointed out here. Firstly, it has been assumed a priori that the unknown functions are sufficiently smooth so that operations of differentiations and limits $V \to 0+$ are valid. This cannot be proved as none of the unknown series can be determined completely. This sort of situation arises in many physical problems and what is done in such situations is to check whether the final outcome is correct or not and this has been done in the present work with the help of some analytical and numerical checks. Secondly, if the unknowns were not determined uniquely, then the method of solution would have failed.

The coefficients $A_1(Z)$, $A_2(Z)$ and $A_3(Z)$ in the moving boundary are given below. $A_1(Z)$ is the root of the following transcendental equation

$$2(\pi)^{1/2}\lambda A_1(Z)+\pi^{1/2}Q_0(Z)=\exp(-A_1^2)\,\{f_2(1, Z)-f_1(1, Z)\}, \tag{22}$$

$$f_2(1, Z)=\{2-\mathrm{erfc}(A_1)f_1(1, Z)\}/(1+\mathrm{erf}\,A_1). \tag{23}$$

If

$$(\pi)^{1/2}Q_0(Z)>2\{1-f_1(1, Z)\} \tag{24}$$

then $A_1(Z)<0$ for all Z, which implies that melting starts instantaneously and simultaneously all over the surface $R=1$ at $V=0$.

$$\begin{aligned}16\pi^{1/2}\lambda A_2(Z)= &-4\pi^{1/2}Q_1(Z)-20\pi^{1/2}A_1^2\\ &+2\pi^{1/2}\left.\frac{\partial f_1}{\partial R}\right|_{R=1}\mathrm{erfc}\,A_1-\{8A_1\exp(-A_1^2)\\ &-\pi^{1/2}\mathrm{erfc}\,A_1\}f_1(1, Z)+8A_1f_2(1, Z)\exp(-A_1^2)\\ &+2\pi^{1/2}(1+\mathrm{erf}\,A_1)\left.\frac{\partial f_2}{\partial R}\right|_{R=1},\end{aligned} \tag{25}$$

$$\begin{aligned}&\{2\exp(-A_1^2)+\pi^{1/2}A_1(1+\mathrm{erf}\,A_1)\}\left.\frac{\partial f_2}{\partial R}\right|_{R=1}\\ &\qquad=6\pi^{1/2}A_1+2\pi^{1/2}\mathrm{ierfc}(A_1)\left.\frac{\partial f_1}{\partial R}\right|_{R=1}\\ &+2(1+2A_2)\exp(-A_1^2)\,\{f_1(1, Z)-f_2(1, Z)\}+\{\exp(-A_1^2)\\ &-3\pi^{1/2}A_1(1+\mathrm{erf}\,A_1)\}f_2(1, Z)-4\pi^{1/2}A_1\,\mathrm{erfc}(A_1)f_1(1, Z),\end{aligned} \tag{26}$$

$$\begin{aligned}&96\pi^{1/2}\lambda A_3(Z)=-16\pi^{1/2}Q_2(Z)-240\pi^{1/2}\lambda A_1A_2\\ &-30\pi^{1/2}\lambda A_1^3+\{-16A_1'^2+32A_1A_3+8A_2+16A_2^2\\ &+64A_1^2A_2-32A_1^2A_2^2\}\exp(-A_1^2)\end{aligned}$$

$$\times\{f_1(1, Z)-f_2(1, Z)\}+8\{\pi^{1/2}A_1 \operatorname{erfc} A_1$$

$$+(3-6A_2-2A_1^2)\exp(-A_1^2)\}f_1(1, Z)+4\{7\pi^{1/2}A_1\operatorname{erfc} A_1$$

$$-(3+4A_2)\exp(-A_1^2)\}\left.\frac{\partial f_1}{\partial R}\right|_{R=1} - 8\pi^{1/2}\left.\frac{\partial^2 f_1}{\partial R^2}\right|_{R=1}\operatorname{ierfc} A_1$$

$$+(20A_2+16A_1^2-3)f_2(1, Z)\exp(-A_1^2)+4(1+4A_2)\exp(-A_1^2)$$

$$+5\pi^{1/2}A_1(1+\operatorname{erf} A_1)\left.\frac{\partial f_2}{\partial R}\right|_{R=1}+8\{\exp(-A_1^2)+\pi^{1/2}A_1$$

$$\times(1+\operatorname{erf} A_1)\}\left.\frac{\partial^2 f_2}{\partial R^2}\right|_{R=1}-4\exp(-A_1^2)\left.\frac{\partial^2 f_1}{\partial Z^2}\right|_{R=1}$$

$$+4\exp(-A_1^2)\left.\frac{\partial^2 f_2}{\partial Z^2}\right|_{R=1}, \tag{27}$$

$$A'(Z)=\mathrm{d}A_1/\mathrm{d}Z,$$

$$\{2A_1\exp(-A_1^2)+\pi^{1/2}(1+2A_1^2)(1+\operatorname{erf} A_1)\}\left.\frac{\partial^2 f_2}{\partial R^2}\right|_{R=1}$$

$$=\{2A_1\exp(-A_1^2)-\pi^{1/2}(1+2A_1^2)\operatorname{erfc} A_1\}\left.\frac{\partial^2 f_1}{\partial R^2}\right|_{R=1}$$

$$+2A_3\exp(-A_1^2)\{f_1(1, Z)-f_2(1, Z)\}$$

$$+2\{4A_1\exp(-A_1^2)-\pi^{1/2}(1+2A_2+4A_1^2)\operatorname{erfc} A_1\}\left.\frac{\partial f_1}{\partial R}\right|_{R=1}$$

$$+3\pi^{1/2}A_1^2+12\pi^{1/2}A_2+\{4A_1\exp(A_1^2)$$

$$-\pi^{1/2}(8A_2+4A_1^2+1/4)\operatorname{erfc} A_1\}f_1(1, Z)$$

$$-\{6A_1\exp(-A_1^2)+\pi^{1/2}(1+4A_2+6A_1^2)$$

$$\times(1+\operatorname{erf} A_1)\}\left.\frac{\partial f_2}{\partial R}\right|_{R=1}-3\{A_1\exp(-A_1^2)$$

$$+\pi^{1/2}(A_1^2+4A_2)(1+\operatorname{erf} A_1)\}f_2(1, Z)/2$$

$$+8A_1A_2(1-A_2)\exp(-A_1^2)\{f_1(1, Z)-f_2(1, Z)\}$$

$$+\pi^{1/2}\operatorname{erfc}(A_1)\left.\frac{\partial^2 f_1}{\partial Z^2}\right|_{R=1}$$

$$-\pi^{1/2}(1+\operatorname{erf} A_1)\left.\frac{\partial^2 f_2}{\partial Z^2}\right|_{R=1}. \tag{28}$$

In principle, other coefficients A_4, A_5 etc can also be determined, but the algebra becomes extremely complicated. Along with the unknowns in the moving boundary, the unknowns in the temperature solution in (10) are also determined.

We replace $I_0(x)$ in (10) by the first two terms of the asymptotic series in (12), use series expansions for f_1 and f_2 given in (18) and (19) and integrate term by term. $T(R, Z, V)$ is given by

$$T(R, Z, V) = -\tfrac{1}{2}\left[R^{1/2}\left\{f_1(1, Z) + (R-1)\left.\frac{\partial f_1}{\partial R}\right|_{R=1} + \frac{(R-1)^2}{2}\left.\frac{\partial^2 f_1}{\partial R^2}\right|_{R=1}\right\} + \frac{V^2 f_1(1, Z)}{16R^{3/2}} + \frac{R^{1/2}V^2}{4}\left.\frac{\partial^2 f_1}{\partial Z^2}\right|_{R=1}\right]$$

$$\times\left[\operatorname{erf}\frac{(R-1)}{V} - 1\right] + \frac{1}{2\pi^{1/2}}\left[-VR^{1/2}\left.\frac{\partial f_1}{\partial R}\right|_{R=1}\right.$$

$$-(R-1)\,VR^{1/2}\left.\frac{\partial^2 f_1}{\partial R^2}\right|_{R=1} - VR^{-1/2}f_1(1, Z) - VR^{-1/2}$$

$$\left.\times(R-1)\left.\frac{\partial f_1}{\partial R}\right|_{R=1}\right]\exp\left\{-\frac{(R-1)^2}{V^2}\right\} + \left(\frac{R^{1/2}V^2}{2\pi^{1/2}}\left.\frac{\partial^2 f_1}{\partial R^2}\right|_{R=1}\right.$$

$$\left.+R^{-1/2}V^2\left.\frac{\partial f_1}{\partial R}\right|_{R=1}\right)\left[-\frac{(R-1)}{2V}\exp\left\{-\frac{(R-1)^2}{V^2}\right\}\right.$$

$$\left.+\frac{\pi^{1/2}}{4}\operatorname{erfc}\left\{\frac{(R-1)}{V}\right\}\right] + \frac{1}{2}\left\{f_2(1, Z) + \frac{V^2}{16R^2}f_2(1, Z)\right.$$

$$+\frac{(R-1)^2}{2}\left.\frac{\partial^2 f_2}{\partial R^2}\right|_{R=1} + (R-1)\left.\frac{\partial f_2}{\partial R}\right|_{R=1} + \frac{V^2}{4}$$

$$\left.\times\frac{\partial^2 f_2}{\partial Z^2}\right|_{R=1}\Bigg\}\left[1 + \operatorname{erf}\left\{\frac{(R-1)}{V}\right\}\right]$$

$$-\frac{\pi^{1/2}}{2}\left\{-\frac{V}{2R}f_2(1, Z) - \frac{(R-1)V}{2R}\left.\frac{\partial f_2}{\partial R}\right|_{R=1} - V\left.\frac{\partial f_2}{\partial R}\right|_{R=1}\right.$$

$$\left.-V(R-1)\left.\frac{\partial^2 f_2}{\partial R^2}\right|_{R=1}\right\}\exp\left\{-\frac{(R-1)^2}{V^2}\right\}$$

$$-\frac{1}{4}\left\{-\frac{V^2}{8R^2}f_2(1, Z) + \frac{V^2}{2R}\left.\frac{\partial f_2}{\partial R}\right|_{R=1} + \frac{V^2}{2}\left.\frac{\partial^2 f_2}{\partial R^2}\right|_{R=1}\right\}.$$

$$\left[\frac{2(R-1)}{\pi^{1/2}V}\exp\left\{-\frac{(R-1)^2}{V^2}\right\} - \frac{\pi^{1/2}}{4} - \frac{\pi^{1/2}}{4}\operatorname{erf}\left\{\frac{(R-1)}{V}\right\}\right]$$

$$+\text{terms of the form } (R-1)^m V^n,\ \text{where } m+n>2. \qquad (29)$$

The above temperature solution is valid for small values of V, $|R-1|$ and $|Z|$. In obtaining (29), whenever the limit of integration is R/V, it has been taken as infinity and this is justified as the integrals or error function integrals [1].

If only two terms of the asymptotic series in (12) are considered and $f_1(R, Z)$ is defined by (17) then the integrands in (13) and (14) do not have any singularity. If more than two terms of (12) are to be considered then $f_1(R, Z)$ is to be suitably redefined such that the integrands in (13) and (14) do not have any singularity. However it may be noted that the asymptotic series in (12) does not converge.

4. Heat conduction without phase change

The short time temperature solution of pure heat conduction problem with $f(R, Z)$ as the initial temperature of the solid is given by (10) in which the unknown $f_2(R, Z)$ can be determined with the help of the boundary condition prescribed at $R = 1$ which could be temperature prescribed or flux prescribed or radiation type. The temperature is still given by (29) with $f_2(R, Z)$ determined with the help of the prescribed boundary condition.

5. Some analytical checks

If $Q_0(Z) = 0$ in (22) and the melting commences at $V = 0$ then $f_{1'}(1, Z)$ should be equal to unity for all Z. In this case, from (23), we get

$$f_1(1, Z) = f_2(1, Z) = 1. \tag{30}$$

Equation (30) implies that $T(1, Z, 0) = 1$ which it should be as this is the requirement of physics of the problem. It can be checked from (22) and (26) that (30) together with $Q_0(Z) = 0$ implies that $A_1(Z) = 0$ and $\partial f/\partial R|_{R=1} = \partial f_2/\partial R|_{R=1}$. For $V \ll 1$, $A_2(Z)$ will be the leading term in the moving boundary and it should be negative if melting has commenced at $V = 0$. From (25) it can be seen that if

$$Q_1(Z) > \partial f/\partial R|_{R=1} \tag{31}$$

then $A_2(Z)$ will be negative. Equation (31) is the requirement of the physics of the problem for the starting of melting and it can be written down independently without doing any mathematical calculations. This provides a useful check on the method of solution and the algebra involved.

Some numerical checks will also be given below.

6. Numerical results and discussion

There does not seem to be any systematic way to determine analytically the radius of convergence of the series for the moving boundary as firstly, the series cannot be determined completely; secondly, even the rough estimates for the coefficients A_1, A_2, A_3 etc are difficult to obtain and thirdly the asymptotic expansion of $I_0(x)$ used earlier in determining these coefficients does not converge. In such a situation, a pertinent question can be asked: What is the range of time for which this short time solution is valid. The criterion for the validity of the range of time for which the moving boundary solution is valid is simple. If the coefficients in the moving boundary are decreasing in absolute value (for fixed Z and given parameter values) then by calculating $|A_n(Z)\, V^n|$ (where A_n is the last term calculated in the moving boundary) it can be easily determined whether it makes any significant contribution to $\sum_{n=1}^{3} A_n(Z) V^n$ or not. If it does not make such a contribution then the moving boundary solution is valid atleast for this particular V. If the coefficients are not decreasing even then the solution is valid but the length of the time interval has to be extremely small. This criterion has been checked with the help of numerical schemes in [23], in which the analytical results for one-dimensional radially symmetric solidification problems were checked with the help of numerical schemes and were found in very good agreement. Numerical schemes for two-dimensional problems are fairly complicated and this infinite flux problem has not been tackled earlier by numerical schemes. It is hoped that the results presented in this paper can be used for starting and checking the numerical schemes.

Temperature solution given in (29) can be used for $V<1/2$ (as in the error function integrals the limit of integration R/V has been taken as ∞), $|R-1|$ small and $|Z|<1$.

In table 1 and figures 1 and 2, the data are as follows:

$$\lambda = 0.6928,\ f_1(R,\ Z) = 0.98 - 0.5(R-1)^2 \exp(-Z^2),$$

$$Q(Z,\ V) = 0.5\{1+\exp(-Z^2)\}\{1/V + 0.5 + 0.25V\}.$$

In table 1, the isotherm condition (5) has been checked numerically. For given values of V and Z, a value of R can be obtained from the relation

$$S = 1 + \sum_{n=1}^{3} A_n(Z) V^n \tag{32}$$

and this value of R was substituted in (29) together with given values of V and Z. The resulting temperature values are given below.

It can be easily seen that the temperature is almost unity at the moving boundary (the error is significant). This checking of isotherm condition provides a good check on the method of solution and the algebra involved.

Table 1. Checking the isotherm condition.

Z/V	S	Temperature	$A_1(Z)$	$A_2(Z)$	$A_3(Z)$
0.0/0.05	0.9647	1.0008	−0.6909	−0.2997	−0.0532
0.0/0.10	0.9278	0.9992	−0.6909	−0.2997	−0.0532
0.6/0.05	0.9703	0.9997	−0.5843	−0.1652	−0.0101
0.6/0.10	0.9399	1.0007	−0.5843	−0.1652	−0.0101

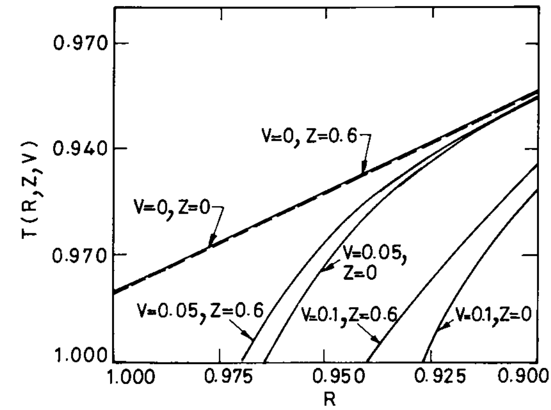

Figure 1. Temperature in the solid vs R for different values of V and Z.
$\lambda = 0.6928$, $f_1(R, Z) = 0.98 - 0.5(R-1)^2 \exp(-Z^2)$,
$Q(Z, V) = 0.5(1/V + 0.5 + 0.25V)\exp(-Z^2)$.

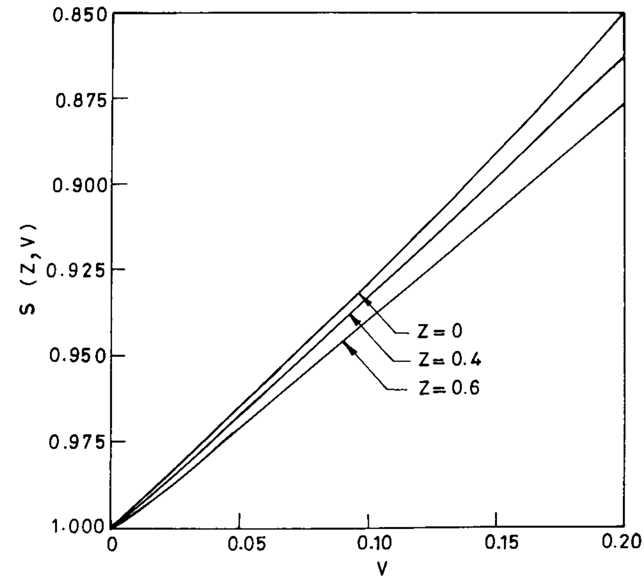

Figure 2. Melting front vs V for different values of Z. Data as in figure 1.

In figure 1, the temperature in the solid vs R has been plotted for different V and Z and this has been compared with the initial temperature of the solid. In figure 2 the melting front vs V has been plotted for $Z = 0$, $Z = 0.4$ and $Z = 0.6$. The trend of the graphs in figures 1 and 2 suggests that the solution is valid atleast for the values of Z, R and V reported in these figures. The melted portion of the solid is quite substantial for use of direct applications or checking the numerical schemes.

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